MAT 16C: Quiz 1
Friday, June 24

Write legibly and neatly. Show all your work for full credit.

1. (10 points)

Verify that the function:

\[ y = x + 2x^3 \]

Is a solution to the differential equation:

\[ x^2y'' - 3xy' + 3y = 0 \]

We begin by calculating the first and second derivatives of \( y \):

\[ y' = 1 + 6x^2 \]
\[ y'' = 12x \]

And then we plug them into the given differential equation and simplify:

\[ x^2y'' - 3xy' + 3y = 0 \]
\[ x^2(12x) - 3x(1 + 6x^2) + 3(x + 2x^3) = 0 \]
\[ 12x^3 - 3x - 18x^3 + 3x + 6x^3 = 0 \]
\[ (3x - 3x) + (12x^3 - 18x^3 + 6x^3) = 0 \]
\[ 0 = 0 \]

So the function \( y = x + 2x^3 \) satisfies the given differential equation.

2. (5 points) We’ve studied two models of population growth in class - exponential and logistic. Another common model of population growth is called the Gompertz model. In this model, it is assumed that the rate of change of the population is proportional to the product of the population and the log of \( K/P \), where \( P \) is the population and \( K \) is the “Carrying Capacity”. Write a differential equation that models this system. DO NOT SOLVE THE EQUATION.

\[ \frac{dP}{dt} = kP \log \left( \frac{K}{P} \right) \]
(note that the constant of proportionality \( k \) is different than the carrying capacity, \( K \))
3. (10 points)

a. (5pts) Find the general solution of the following differential equation (solve for \( y \)).

Note: you may assume that \( y \geq 0 \).

\[
\frac{dy}{dx} = x^2(1 + y)
\]

We begin by separating variables:

\[
\frac{dy}{1 + y} = x^2 \, dx
\]

And then integrating:

\[
\int \frac{dy}{1 + y} = \int x^2 \, dx
\]

\[
\log(1 + y) = \frac{x^3}{3} + C
\]

And then we just solve for \( y \):

\[
1 + y = e^{x^3/3 + C}
\]

\[
y = Ce^{x^3/3} - 1
\]

b. (5pts) Find the particular solution given that \( y = 3 \) when \( x = 0 \).

We just set \( x = 0 \) and \( y = 3 \) in our general solution, and then solve for \( C \):

\[
3 = Ce^0 - 1
\]

\[
C = 4
\]

Then the particular solution is:

\[
y = 4e^{x^3/3} - 1
\]