Please think about these questions for a few minutes, and write down a few thoughts. In a few minutes, we’ll discuss them in small groups, and then as a class.

In this worksheet we’ll be working out one of the forms of The Fundamental Theorem of Calculus. There are actually a lot of these theorems, and they all in one way or another say that differentiation and integration undo one another. Today, we’ll try to show that:

\[ f(x) = \frac{d}{dx} \int_a^x f(t) \, dt \]  

To make graphing and visualization easier for us, let’s assume that \( f(x) > 0 \) for \( x \in [a,b] \), though it doesn’t really matter.

1. Geometrically, we know that the integral of \( f \) from \( a \) to \( b \) represents the area under the curve \( y = f(x) \) and above the \( x \)-axis. Using pictures, convince yourself that, for \( a < b < c \), the following result holds:

\[ \int_a^c f(t) \, dx - \int_a^b f(t) \, dx = \int_b^c f(t) \, dx \]  

2. Recall that the definition of the derivative is:

\[ \frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \].

Define \( F(x) \) by the quation:

\[ F(x) = \int_a^x f(t) \, dt \].

Write down an expression for \( \frac{dF}{dx} \). Simplify using equation 2.
3. Recall that the average value of a function $f$ over an interval $[a,b]$ is given by:

$$\bar{f} = \frac{\text{area}}{b-a} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

The expression we found in part (2) is quite close to this. If $f$ is smooth and $h$ is small, approximately what does this equal?

4. Notice that $\frac{dF}{dx}$, given how $F$ is defined, is precisely the right hand side of equation 1, thus establishing the claim made at the top of the worksheet. This computation, while instructive, is nevertheless opaque. Guided by our computation, can you construct a directly geometric argument that says ‘the change in the area under $f$ from $a$ to $x$ is given by $f(x)$’?