Please think about these questions for a few minutes, and write down a few thoughts. In a few minutes, we’ll discuss them in small groups, and then as a class.

1. Let \( f(x) = e^x \). Find the line that passes through the point \((0, f(0))\) and \((h, f(h))\). What is the slope of that line? How does this relate to the definition of the derivative (at \( x = 0 \))? 

2. Compute the slope of this line for \( h = 0.1, \ h = 0.01, \) and \( h = 0.001 \). What is the true value of the derivative? Compute the error (i.e. error = |estimate − truth|.) Everytime you decrease \( h \) by a factor of ten, by what factor does the error decrease?
What we did in parts (1) and (2) was to find an estimate of the derivative at $x = 0$ using two points, namely those at $x = 0$ and $x = h$. We did this by finding a line - name some function $L(x) = Ax + B$ such that $L(x)$ passed through the two points $(0, f(0))$ and $(f, f(h))$. We estimated the derivative of $f(0)$ by finding the derivative of the line $L(x)$, which is easy to compute since it is just it’s slope. Now let’s use three points - those at $x = 0$, $x = h$, and $x = 2h$. Since we have three points, we can fit a quadratic function through the points. I.e. let $P(x) = Ax^2 + Bx + C$. If $A$, $B$, and $C$ are such that the following three equations are satisfied:

$$A(0)^2 + B(0) + C = f(0) \quad (1a)$$
$$A(h)^2 + B(h) + C = f(h) \quad (1b)$$
$$A(2h)^2 + B(2h) + C = f(2h) \quad (1c)$$

Then $P(x)$ passes through the three points $(0, f(0))$, $(h, f(h))$, and $(2h, f(2h))$.

3. Find the derivative of $P(x) = Ax^2 + Bx + C$ at the point $0$, in terms of $A$, $B$, and $C$. Luckily, two of these seem to disappear. Which ones?

4. Solve the system of equations in 1 for $B$. 
5. From the previous page, we have that $f'(0) \approx P'(0) = B$. Using your solution for $B$, compute an approximation for $f'(0)$ using $h = 0.1$, $h = 0.01$, and $h = 0.001$. Compute the error of these approximations. Everytime you decrease $h$ by a factor of ten, by what factor does the error decrease?

6. If we fit a polynomial $P(x) = Ax^3 + Bx^2 + Cx + D$ to the four points $(0, f(0))$, $(h, f(h))$, $(2h, f(2h))$, and $(3h, f(3h))$ (I won’t make you solve that messy system), and approximated $f'(x) = P'(x)$, our approximation for the derivative becomes:

$$f'(0) \approx \frac{-11f(0) + 18f(h) - 9f(2h) + 2f(3h)}{6h}.$$  

Compute an approximation for $f'(0)$ using $h = 0.1$, $h = 0.01$, and $h = 0.001$. Compute the error of these approximations. Everytime you decrease $h$ by a factor of ten, by what factor does the error decrease?