MAT 21b: Practice Midterm Exam
Friday, November 7th, 2014

Write legibly and neatly. Show all your work for full credit. This exam is substantially longer than the exam I will actually give.

1. Find the indefinite integral:

\[ \int \sqrt{x} \sin^2(x^{3/2} - 1) \, dx. \]

let \( u = x^{3/2} - 1 \)
\( du = \frac{3}{2} x^{1/2} \, dx \)

\[ \int \sqrt{x} \sin^2(x^{3/2} - 1) \, dx = \frac{2}{3} \int \sin^2 u \left( \frac{3}{2} \sqrt{x} \right) \, dx \]
\[ = \frac{2}{3} \int \sin^2 u \, du \]
\[ = \frac{2}{3} \int \frac{1 - \cos(2u)}{2} \, du \]
\[ = \frac{1}{3} \int (1 - \cos(2u)) \, du \]

\vdots
2. Find the definite integral:

\[
\int_{1}^{e} \frac{\sqrt{\ln x}}{x} \, dx
\]

Let \( u = \ln x \)

\[
\frac{du}{dx} = \frac{1}{x}
\]

So

\[
\int_{1}^{e} \frac{\sqrt{\ln x}}{x} \, dx = \int_{0}^{1} u^{\frac{1}{2}} \, du
\]

\[
= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \bigg|_{0}^{1}
\]

\[
= \frac{2}{3} \left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)
\]

\[
= \frac{2}{3}
\]
3. Find the indefinite integral:

\[
\int \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}} \, dx
\]

let \( u = 1 + 7 \tan x \)

\( du = 7 \sec^2 x \, dx \)

\[
\int \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}} \, dx = \frac{1}{7} \int u^{-2/3} \cdot 7 \sec^2 x \, dx
\]

\( = \frac{1}{7} \int u^{-2/3} \, du \)

\[
\therefore
\]
4. Find the area between the graphs of the functions $y = x^3$ and $y = x$, from $x = -1$ to $x = 2$. Be sure to draw a sketch of the functions, and shade in the region that you are finding the area of.

\[ A = \int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x - x^3) \, dx + \int_{1}^{2} (x^3 - x) \, dx \]
5. Find the area between the x axis and the graph of the function \( y = \sin 3x \), between \( x = \pi \) and \( x = 3\pi \). Be sure to draw a sketch of the functions, and shade in the region that you are finding the area of.

So

\[
A = 6 \cdot \text{area of one lobe}
\]

\[
= 6 \int_{\pi/3}^{\pi} \sin(3x) \, dx
\]

let \( u = 3x \)

\[
du = dx
\]

\[
= 6 \int_{\pi/3}^{\pi} \sin u \, du
\]

\[
= \boxed{0}
\]
6. Calculate the average value of the function \( y = x\sqrt{1 + x} \) from \( x = 3 \) to \( x = 8 \).

\[
\text{Avg} = \frac{1}{8-3} \int_{3}^{8} x\sqrt{1 + x} \, dx
\]

\[
= \frac{1}{5} \int_{3}^{8} x\sqrt{1 + x} \, dx = \frac{1}{5} \int_{4}^{9} (u-1) u^{\frac{3}{2}} \, du
\]

Let \( u = 1 + x \)

\( du = dx \)

\( x = u - 1 \)
7. Set up a Riemann sum, using the midpoint rule, with \( n \) equally spaced points, that estimates the area underneath the curve \( y = 2e^y \), from \( y = 2 \) to \( y = 6 \).

Try writing down a left sum first, then shift each part of the partition to the right by \( h/2 \).

\[
\frac{4}{n} \sum_{i=0}^{n-1} 2e^{2 + \frac{1}{2} + ih}
\]
8. Find the derivative, \( \frac{df}{dx} \), of the function:

\[
f(x) = \int x^2 \tan(t^2+t) \, dt.
\]

Whoops. \( \tan(t^2 + t) \)

\[
f(x) = \int_0^x \tan(t^2+t) \, dt + \int_x^0 \tan(t^2+t) \, dt
\]

\[
= -\int_0^x \tan(t^2+t) \, dt + \int_x^0 \tan(t^2+t) \, dt
\]

So \( \frac{df}{dx}(x) = -\tan((x^2)^2 + x^2) \cdot (x^2)' + \tan(x^2 + x) \)

\[
= -2x \tan(x^4 + x^2) + \tan(x^2 + x)
\]

\[ V = \int_0^5 s(h) \, dh \]

Taking a sluice right down the middle, we get:

by similar triangles,

\[ \frac{h}{5} = \frac{5/2}{3/2} = \frac{5}{3} \]

So

\[ s = \frac{3}{5} h \]
10. Find the volume of the solid generated by revolving, about the y axis, the region in the first quadrant bounded above by the curve \( y = x^{1/3} \), and below by the curve \( y = x^4 \).

\[
y = x^{1/3}, \quad x = y^{1/4}
\]

by washer method:

\[
V = \int_0^1 \left[ \pi \left( y^{1/4} \right)^2 - \pi \left( y^{3} \right)^2 \right] \, dy
\]

by shell method

\[
V = \int_0^1 2\pi x \left( x^{1/3} - x^4 \right) \, dx
\]

\[ V = \int_{0}^{3} 2\pi r h \, dx \]

\[ = \int_{0}^{3} 2\pi x \frac{9x}{\sqrt{x^3 + 9}} \, dx \]

\[ = 6\pi \int_{0}^{3} 3x^2 \left( x^3 + 9 \right)^{-\frac{1}{2}} \, dx \]

Let \( u = x^3 + 9 \)

\[ du = 3x^2 \, dx \]
12. Find the arc length of the curve \( y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5 \), from \( x = 1 \) to \( x = 8 \).

\[
L = \int_1^8 \sqrt{1 + y'^2} \, dx
\]

\[
y' = x^{1/3} - \frac{1}{4}x^{-1/3}
\]

\[
y'^2 = x^{2/3} - \frac{1}{2} + \frac{1}{16}x^{-2/3}
\]

\[
L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{1}{16}x^{-2/3}} \, dx
\]

\[
= \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{1}{16}x^{-2/3}} \, dx
\]

\[
= \int_1^8 \sqrt{(x^{1/3} + \frac{1}{4}x^{-1/3})^2} \, dx
\]
\[ x = \left( e^y + e^{-y} \right) / 2 \quad 0 \leq y \leq \ln 2 \]

\[ SA = \int_0^{\ln 2} 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

\[ \frac{dx}{dy} = \frac{1}{2} (e^y - e^{-y}) \]

\[ \left( \frac{dx}{dy} \right)^2 = \frac{1}{4} (e^{2y} - 2 + e^{-2y}) \]

\[ SA = \int_0^{\ln 2} 2\pi \frac{1}{2} (e^y + e^{-y}) \sqrt{1 + \frac{1}{4} e^{2y} - \frac{1}{2} + \frac{1}{4} e^{-2y}} \, dy \]

\[ = \int_0^{\ln 2} \pi (e^y + e^{-y}) \sqrt{\frac{1}{4} e^{2y} + \frac{1}{2} + \frac{1}{4} e^{-2y}} \, dy \]

\[ = \pi \int_0^{\ln 2} (e^y + e^{-y}) \sqrt{\left( \frac{e^y + e^{-y}}{2} \right)^2} \, dy \]
14. (a) Suppose that:

\[ \sum_{k=1}^{5} a_k = 7, \quad \sum_{k=6}^{10} a_k = 4, \quad \text{and} \quad \sum_{k=1}^{10} b_k = 12. \]

Evaluate:

\[ \sum_{k=1}^{10} (2a_k + b_k). \]

(b) Write the sum 4 + 9 + 14 + 19 + 24 + 29 using Sigma notation.

\[ \sum_{k=1}^{6} (5k + 4), \quad \sum_{k=0}^{5} (5k + 4) \]
15. (a) Let \( f(x) = x^3 \). Using a partition with five equally spaced points, compute an estimate of \( \int_1^5 f(x) \, dx \) using the trapezoidal rule. What is the error in your estimate?

(b) If you used a partition with 50 equally spaced points to estimate the same integral, approximately what would be the error in your estimate?

\( \int_1^5 f(x) \, dx \approx h \left( \frac{1}{2} f(1) + f(2) + f(3) + f(4) + \frac{1}{2} f(5) \right) \)

\[ h = \frac{5 - 1}{4} = \frac{4}{4} \]

\[ \text{\# of subintervals} \neq \text{number of pairs.} \]

So

\[ = 1 \left( \frac{1}{2} 1^3 + 2^3 + 3^3 + 4^3 + \frac{1}{2} 5^3 \right) \]

\[ = 162 \]

\[ \int_1^5 x^3 \, dx = \frac{x^4}{4} \bigg|_1^5 = 156 \]

So the error is 6.

(b) with 50 equally spaced points, \( n = 49 \).

So we would expect the error to be

\[ \frac{6}{(49/4)^2} \approx 0.04 \]
16. Let \( f(x) = x^3 + 7 \), and define the numbers \( F \) and \( G \) by:

\[
F = \int_0^2 f(x) \, dx, \\
G = \int_2^6 f(x) \, dx.
\]

Let \( F_n \) be the estimate of \( \int_0^2 f(x) \, dx \) given by using Simpsons rule with \( n \) points, and let \( G_n \) be the estimate of \( \int_2^6 f(x) \, dx \) using Simpsons rule with \( n \) points. Define the errors:

\[
E_F = |F - F_n|, \\
E_G = |G - G_n|.
\]

Do you expect \( E_F \) or \( E_G \) to be bigger? Explain your choice.

\[
E_G \text{ would be larger. (or at least, we would expect it to be)}
\]

The error estimate is:

\[
Err < \frac{M \, (b - a)^5}{180 \, n^4}
\]

\[
= \frac{M}{180 \, (b - a)^2 \, h^4}
\]

for \( G \), \( M \), \( (b - a) \) and \( h \) are all larger.

So we expect the error to be larger.