MAT 21b: Practice Final
Wednesday, December 12th, 2014

Write legibly and neatly. Show all your work for full credit. This practice test is longer than the real exam will be.

1. Let \( f(x) = x + x^2 \) on the interval \([0, 3]\).
   
   (a) Find a formula for the Riemann sum obtained by dividing the given interval into \( n \) equal subintervals and use the right-hand endpoint for each \( c_k \).

   (b) Take the limit of your formula from part (a) to compute the area under the curve \( y = f(x) \) on the given interval. It may be useful for you to use the formula:

   \[
   \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}
   \]

   (c) Compute the integral using the fundamental theorem of calculus and show that these agree.

   \[\begin{align*}
   a) \quad R_{SN} &= \sum_{k=1}^{n} \left( \frac{3-0}{n} \right) f\left( \frac{k(3-0)}{n} \right) \quad = \sum_{k=1}^{n} \frac{3}{n} \left( \frac{3k}{n} + \left( \frac{3k}{n} \right)^2 \right) \\
   &\text{b) } \lim_{N \to \infty} R_{SN} = \left( \frac{a}{n^2} \sum_{k=1}^{n} k + \frac{27}{n^3} \sum_{k=1}^{n} k^2 \right) \\
   &= \lim_{N \to \infty} \left( \frac{a}{n^2} \frac{n(n+1)}{2} + \frac{27}{n^3} \sum_{k=1}^{n} \frac{n(n+1)(2n+1)}{6} \right) \\
   &= \frac{a}{2} + \frac{27}{3} \\
   &= \frac{27}{2} \\
   &\text{c) } \int_{0}^{3} (x + x^2) \, dx = \left. \frac{x^2}{2} + \frac{x^3}{3} \right|_{0}^{3} = \frac{9}{2} + \frac{27}{3} = \frac{27}{2}
   \end{align*}\]
2. The inequality $\sec x \geq 1 + \frac{x^2}{2}$ holds on $(-\pi/2, \pi/2)$. Use this inequality to find an upper bound for the value of $\int_0^1 \sec x \, dx$.

\[
\int_0^1 \sec x \, dx > \int_0^1 \left(1 + \frac{x^2}{2}\right) \, dx
\]

\[
= x \bigg|_0^1 + \frac{x^3}{6} \bigg|_0^1
\]

\[
= 1 + \frac{1}{6}
\]

\[
= \frac{7}{6}
\]

Thus, $\int_0^1 \sec x \, dx \geq \frac{7}{6}$.
3. Find the derivative \( \frac{dy}{dx} \) of the function:

\[
y = x \int_2^{x^2} \sin(t^3) dt
\]

Applying the product rule:

\[
\frac{dy}{dx} = \int_2^{x^2} \sin(t^3) dt + x \frac{d}{dx} \int_2^{x^2} \sin(t^3) dt
\]

\[
= \int_2^{x^2} \sin(t^3) dt + x \left( \sin((x^2)^3) \frac{d}{dx} (x^2) \right)
\]

\[
= \int_2^{x^2} \sin(t^3) dt + 2x^2 \sin(x^6)
\]
Let $f$ be any continuous function on $[0,a]$. Compute:

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx$$

**Hint:** Let $I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx$. Now make the substitution $u = a - x$. You'll get a new integral. Since it's a definite integral, after you've completely done the substitution, you can replace the symbol 'u' with the symbol 'x'. Now add the second integral to the first integral, and you'll get $2I = \text{some other integral which should simplify nicely}$.

Let $I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx = \int_0^a \frac{f(a-u)}{f(a-u) + f(u)} \, (-du)$

Let $u = a - x$

$du = - \, dx$

$x = a - u$

$$I = \int_0^a \frac{f(a - x)}{f(a-x) + f(x)} \, dx$$

Note: in this last step, I renamed the symbol 'u' to be 'x'.

One has to exercise care when doing this to be sure the variable we are renaming is truly a dummy variable.

Thus

$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx$ and $I = \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} \, dx$

Adding these together, we find:

$2I = \int_0^a \frac{f(x) + f(a-x)}{f(x) + f(a-x)} \, dx$

$= \int_0^a \, dx$

$= a$

Thus

$$I = \frac{a}{2}$$
5. Find the total area between the graph of the function $y = x\sqrt{4-x^2}$ and the $x$-axis on the interval $[-2, 2]$.

$y = 0$ at $x = 0$ and $x = \pm 2$. So we only need to worry about the change of sign at $x = 0$. Testing a value at $x = 1$, we find $y = (-1)\sqrt{4-1} = -\sqrt{3} < 0$ at $x = 1$,

$y = 1\sqrt{4-1} = \sqrt{3} > 0$

So

$$A = \int_{-2}^{0} -x\sqrt{4-x^2} \, dx + \int_{0}^{2} x\sqrt{4-x^2} \, dx$$

$$A = 2 \int_{0}^{2} x\sqrt{4-x^2} \, dx$$

We could also deduce this from the fact that $x\sqrt{4-x^2}$ is an odd function.

Let $u = 4-x^2$

$du = -2x \, dx$

So

$$A = -\int_{4}^{0} u^{1/2} \, du$$

$$= \left. \frac{u^{3/2}}{3/2} \right|_{4}^{0}$$

$$= \frac{2}{3} (4^{3/2})$$

$$A = \frac{16}{3}$$

Find points of intersection:

\[ x^2 = 2 - x^2 \]
\[ 2x^2 = 2 \]
\[ x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \]

diameter as function of \( x \): \( D(x) = (2 - x^2) - x^2 = 2 - 2x^2 \)

radius as function of \( x \): \( r(x) = \frac{D}{2} = 1 - x^2 \)

area as function of \( x \): \[ A(x) = \pi r^2 = \pi (1 - x^2)^2 = \pi \left(1 - 2x^2 + x^4\right) \]

\[ V = \int_{-1}^{1} A(x) \, dx = \int_{-1}^{1} \pi (1 - 2x^2 + x^4) \, dx \]
\[ = 2\pi \int_{0}^{1} (1 - 2x^2 + x^4) \, dx \]
\[ = 2\pi \left( x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \bigg|_{0}^{1} \]
\[ = 2\pi \left( 1 - \frac{2}{3} + \frac{1}{5} \right) \]
\[ = 2\pi \left( \frac{15}{15} - \frac{10}{15} + \frac{3}{15} \right) \]
\[ = 2\pi \left( \frac{8}{15} \right) \]

\[ V = \frac{16}{15} \pi \]
\[ V = 2 \int_{0}^{\pi/2} \pi (R^2 - r^2) \, dx \]

\[ = 2\pi \left[ \frac{\pi}{2} - \int_{0}^{\pi/2} \cos x \, dx \right] \]

\[ = 2\pi \left[ \frac{\pi}{2} - \sin x \bigg|_{0}^{\pi/2} \right] \]

\[ = 2\pi \left[ \frac{\pi}{2} - (\sin (\pi/2) - \sin (0)) \right] \]

\[ = 2\pi \left( \frac{\pi}{2} - 1 \right) \]

\[ = \pi^2 - 2\pi \]

\[ V = \int_{0}^{\sqrt{3}} 2\pi rh \, dx \]

\[ = \int_{0}^{\sqrt{3}} 2\pi x \sqrt{x^2 + 1} \, dx \]

Let \( u = x^2 + 1 \)

\( du = 2x \, dx \)

\[ = \pi \int_{1}^{4} u^{\frac{1}{2}} \, du \]

\[ = \pi \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{1}^{4} \]

\[ = \frac{2\pi}{3} \left( 4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) \]

\[ = \frac{2\pi}{3} (8 - 1) \]

\[ V = \frac{14\pi}{3} \]
\[ S = \int_0^{15/4} 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

\[ x = 2\sqrt{4-y} \]

\[ \frac{dx}{dy} = 2 \left( \frac{1}{2} \right) (4-y)^{-1/2} (-1) = \frac{-1}{\sqrt{4-y}} \]

\[ S = \int_0^{15/4} 2\pi (2\sqrt{4-y}) \sqrt{1 + \frac{1}{4-y}} \, dy \]

\[ = 4\pi \int_0^{15/4} \sqrt{4-y + \frac{4-y}{4-y}} \, dy \]

\[ = 4\pi \int_0^{15/4} \sqrt{5-y} \, dy \]

\[ u = 5-y \]

\[ du = -dy \]

\[ = -4\pi \int_0^{5-15/4} u^{1/2} \, du \]

\[ = -4\pi \frac{u^{3/2}}{3/2} \bigg|_0^{5-15/4} \]

\[ = 4\pi \frac{(5)^{3/2}}{3/2} \]

\[ = \frac{8\pi}{3} \left( 5^{3/2} - \left( \frac{5}{4} \right)^{3/2} \right) \]

\[ = \frac{8\pi}{3} 5^{3/2} \left( 1 - \left( \frac{1}{4} \right)^{3/2} \right) \]

\[ = \frac{8\pi}{3} 5^{3/2} \left( 1 - \frac{1}{8} \right) \]

\[ = \frac{8\pi}{3} 5^{3/2} \left( \frac{7}{8} \right) \]

\[ = \frac{7\pi}{3} 5^{3/2} \]

\[ = \frac{35\pi}{3} \sqrt{5} \]
10. (a) Compute the definite integral:
\[ \int_0^1 \theta \cos \pi \theta \, d\theta. \]

(b) Compute the indefinite integral:
\[ \int e^{2x} \cos 3x \, dx. \]

(a) \[ \int_0^1 \theta \cos (\pi \theta) \, d\theta \]

Let \( u = \theta \) \( \Rightarrow \) \( du = d\theta \)

\[ dv = \cos (\pi \theta) \, d\theta \]

\[ v = \frac{1}{\pi} \sin (\pi \theta) \]

So
\[ \int_0^1 \theta \cos (\pi \theta) \, d\theta = \frac{1}{\pi} \sin (\pi \theta) \bigg|_0^1 - \frac{1}{\pi} \int_0^1 \sin (\pi \theta) \, d\theta \]

\[ = -\frac{1}{\pi} \left[ \frac{-\cos (\pi \theta)}{\pi} \right]_0^1 \]

\[ = -\frac{1}{\pi^2} (-1 - 1) = \frac{2}{\pi^2} \]

(b) Let \( u = \cos 3x \) \( \Rightarrow \) \( du = -3 \sin 3x \, dx \)

\[ dv = e^{2x} \, dx \]

\[ v = \frac{1}{2} e^{2x} \]

So
\[ \int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int 3 \sin 3x \, e^{2x} \, dx \]

Let \( u = 3 \sin 3x \) \( \Rightarrow \) \( du = 3 \cos 3x \, dx \)

\[ dv = e^{2x} \, dx \]

\[ v = \frac{1}{2} e^{2x} \]

So
\[ \int e^{2x} \cos 3x \, dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \left[ \frac{1}{2} \sin 3x e^{2x} - \frac{3}{2} \int 2 e^{2x} \cos 3x \, dx \right] \]

\[ = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} \sin 3x e^{2x} - \frac{9}{4} \int e^{2x} \cos 3x \, dx + C \]

\[ \int e^{2x} \cos 3x \, dx (1 + \frac{9}{4}) = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} \sin 3x e^{2x} + C \]

\[ \int e^{2x} \cos 3x \, dx = \frac{9}{13} \left( \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x \right) + C \]
11. Compute the indefinite integral: \[ \int \sec^3 x \tan^3 x \, dx. \]

Let \( u = \sec x \)

\[ du = \sec x \tan x \, dx \]

\[ \sin^2 x + \cos^2 x = 1 \]
\[ \cos^2 x \]

\[ \tan^2 x + 1 = \sec^2 x \]

\[ \Rightarrow \tan^2 x = \sec^2 x - 1 \]

So \[ \int \sec^3 x \tan^3 x \, dx = \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx \]

\[ = \int u^2 (u^2 - 1) \, du \]

\[ = \int u^4 \, du - \int u^2 \, du \]

\[ = \frac{u^5}{5} - \frac{u^3}{3} + C \]

\[ = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C \]
12. Compute the average value of the function:

\[ f(x) = \frac{3x + 1}{x^2 - 2x - 15}, \]

from \( x = 0 \) to \( x = 2 \).

\[ f(x) = \frac{3x + 1}{(x - 5)(x + 3)} \]

Note \( f(x) \) is continuous on \([0, 2]\), thus the integral is not improper.

So \( \text{Avg} = \frac{1}{2-0} \int_{0}^{2} f(x) \, dx \)

\[ = \frac{1}{2} \int_{0}^{2} \frac{3x + 1}{(x - 5)(x + 3)} \, dx \]

Now \( \frac{3x + 1}{(x - 5)(x + 3)} = \frac{A}{x - 5} + \frac{B}{x + 3} = \frac{A(x + 3) + B(x - 5)}{(x - 5)(x + 3)} \)

\[ \Rightarrow 3x + 1 = x(A + B) + (3A - 5B) \]

So \( \begin{cases} A + B = 3 \\ 3A - 5B = 1 \end{cases} \Rightarrow \begin{cases} 5A + 5B = 15 \\ 3A - 5B = 1 \end{cases} \)

Thus \( 8A = 16 \) \( \Rightarrow A = 2 \) \( \Rightarrow B = 1 \)

So \( \text{Avg} = \frac{1}{2} \left[ \int_{0}^{2} \frac{2}{x - 5} \, dx + \int_{0}^{2} \frac{1}{x + 3} \, dx \right] \)

\[ = \ln |x - 5|_{0}^{2} + \frac{1}{2} \ln |x + 3|_{0}^{2} \]

\[ = \ln 3 - \ln 5 + \frac{1}{2} (\ln 5 - \ln 3) \]

\[ = \frac{1}{2} \ln 3 - \frac{1}{2} \ln 5 \]

\[ \text{Avg} = \frac{1}{2} \ln \frac{3}{5} \]
13. Compute the indefinite integral:

\[ \int \frac{x^4}{(1-x^2)^{3/2}} \, dx. \]

Note: You may assume that \(-1 < x < 1\).

Let \( x = \sin \theta \)

\( dx = \cos \theta \, d\theta \)

\[ 1-x^2 = 1-\sin^2 \theta = \cos^2 \theta \]

so \( (1-x^2)^{1/2} = \cos \theta \)

\[ \Rightarrow \int \frac{x^4}{(1-x^2)^{3/2}} \, dx = \int \frac{\sin^4 \theta}{\cos^3 \theta} \cos \theta \, d\theta \]

\[ = \int \tan^4 \theta \sec^2 \theta \, d\theta \]

Let \( u = \tan \theta \)

\( du = \sec^2 \theta \, d\theta \)

\[ = \int u^4 \, du \]

\[ = \frac{u^5}{5} + C \]

\[ = \frac{\tan^5 \theta}{5} + C \]

Since \( x = \sin \theta \)

\[ \tan \theta = \frac{x}{\sqrt{1-x^2}} \]

So \( \tan \theta = \frac{x}{\sqrt{1-x^2}} \)
14. (a) Use the trapezoidal rule and 6 equal length subintervals to compute an estimate to:

\[ \int_0^3 e^x \, dx \]

You will have to use a calculator to do this (of course, no calculators will be allowed on the real exam!)

(b) Compute a bound for the error.

(c) Compute the exact integral, and find the error between your approximation and the exact integral. Do you satisfy the bound that you found?

(d) Compute a bound for the error had you used Simpson's rule, instead.

\[ a) \quad \int_0^3 e^x \, dx \approx \frac{3}{6} \left( e^0 + e^{1/2} + e + e^{3/2} + e^2 + e^{5/2} + e^3 \right) = 19.4815... \]

\[ b) \quad f'' = e^x \quad \text{so} \quad |\text{Err}| \leq \frac{(b-a)^3 M}{12 n^2} = \frac{(3)^3 e^3}{12(6)^2} = 1.255... \]

\[ c) \quad \int_0^3 e^x \, dx = e^x \bigg|_0^3 = e^3 - e^0 = e^3 - 1 = 19.0855... \]

Yes, \( |\text{Err}| = |19.4815... - 19.0855...| = 0.3959... < 1.255... \)

\[ d) \quad f''' = e^x \quad \text{so} \quad |\text{Err}| \leq \frac{M(b-a)^5}{180 n^4} = \frac{e^{125/3}}{180 6^4} = 0.02092... \]
15. The intensity $L(x)$ of light $x$ feet beneath the surface of the ocean satisfies the differential equation:

$$\frac{dL}{dx} = -kL.$$  

As a diver, you know from experience that diving to 18 feet in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-sixteenth of the surface value. About how deep can you expect to work without artificial light?

$$\frac{dL}{L} = -k dx \quad \text{so} \quad \int \frac{dL}{L} = -\int k dx$$

\[ \ln |L| = -kx + C \]

\[ L = Ce^{-kx} \]

Call the intensity of light at $x=0$, $L_0$.

Then

\[ L = L_0 e^{-kx} \]

At 18 ft,

\[ \frac{1}{2} L_0 = L_0 e^{-k \times 18} \]

\[ \ln \frac{1}{2} = -k \times 18 \]

\[ k = \frac{\ln 2}{18} \]

To have the light intensity be $\frac{1}{16} L_0$,

\[ \frac{1}{16} L_0 = L_0 e^{-\frac{\ln 2}{18} x} \]

\[ 16 = e^{\frac{\ln 2}{18} x} \]

\[ \ln 16 = \frac{\ln 2}{18} x \]

\[ 4 \ln 2 = \frac{\ln 2}{18} x \]

\[ x = 72 \text{ feet}. \]
16. Solve the initial value problem:

\[
\sqrt{x} \frac{dy}{dx} = e^{x + \sqrt{x}},
\]
\[
y(0) = 1.
\]

\[
x^{1/2} \frac{dy}{dx} = e^y \sqrt{x}
\]
\[
\int e^{-y} dy = \int e^{\sqrt{x}} x^{-1/2} dx
\]
\[
- e^{-y} = \int e^{\sqrt{x}} x^{-1/2} dx
\]
\[
\text{let } u = x^{1/2}
\]
\[
du = \frac{1}{2} x^{-1/2}
\]
\[
- e^{-y} = 2 \int e^u du
\]
\[
- e^{-y} = 2e^u + C
\]
\[
e^{-y} = -2e^{\sqrt{x}} + C
\]
\[
a + x = 0, \quad y = 1
\]
\[
\Rightarrow e^{-1} = -2 + C
\]
\[
C = \frac{1}{e} + 2
\]

So
\[
e^{-y} = -2e^{\sqrt{x}} + \frac{1}{e} + 2
\]
\[
y = \ln \left( -2e^{\sqrt{x}} + \frac{1}{e} + 2 \right)
\]
\[
y = -\ln \left( -2e^{\sqrt{x}} + \frac{1}{e} + 2 \right)
\]
17. Let $\frac{dx}{dt}$ be given by:

$$\frac{dx}{dt} = \sin x$$

Determine all equilibrium values - note there are infinitely many! Draw a phase line for this differential equation. Make sure to label all equilibrium values, and denote whether they are stable or unstable. Draw arrows indicating how $x$ changes when not at an equilibrium value. Of course, the phase line is infinitely long, so you can’t draw all of it that is relevant. Draw a large enough portion to ensure that you really know what is going on.

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**equilibrium values when** $\sin x = 0$  
so at $\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$  
or at $n\pi$ for $n$ any integer

---

$$\sin x$$
18. Does the following integral converge or diverge? If it converges, find its value.

\[ \int_0^1 x \ln x \, dx. \]

\[ \int_0^1 x \ln x \, dx = \lim_{a \to 0^+} \int_a^1 x \ln x \, dx. \]

Let \( u = \ln x \), \( du = \frac{1}{x} \, dx \), \( dv = x \, dx \), \( v = \frac{x^2}{2} \).

\[ = \lim_{a \to 0^+} \left[ \frac{x^2}{2} \ln x \bigg|_a^1 - \int_a^1 \frac{x^2}{2} \, dx \right] \]

\[ = \lim_{a \to 0^+} \left[ -\frac{a^2}{2} \ln a - \frac{x^2}{4} \bigg|_a^1 \right] \]

\[ = \lim_{a \to 0^+} \left[ -\frac{a^2}{2} \ln a - \left( \frac{1}{4} - \frac{a^2}{4} \right) \right] \]

\[ = -\frac{1}{4} - \frac{1}{2} \lim_{a \to 0^+} a^2 \ln a \]

So what does \( \lim_{a \to 0^+} a^2 \ln a \) equal? Well, it's of the form 0\( \cdot \)\(-\infty \).

\[ \lim_{a \to 0^+} \frac{\ln a}{a^2} = \lim_{a \to 0^+} \frac{\frac{1}{a}}{2a} = \lim_{a \to 0^+} \frac{1}{2} a^2 = 0 \]

So

\[ \int_0^1 x \ln x \, dx = -\frac{1}{4}. \]
19. Does the following integral converge or diverge? You must justify your answer.

\[ \int_1^\infty \frac{1}{xe^x} \, dx. \]

It converges on \((1, \infty)\), since \(\frac{1}{xe^x} \leq \frac{1}{e^x}\)

\[ \int_1^\infty \frac{1}{xe^x} \, dx \leq \int_1^\infty \frac{1}{e^x} \, dx \]

converges.

(you can just integrate it to see...)

\[ \int_1^\infty \frac{1}{e^x} \, dx = \left[ -e^{-x} \right]_1^\infty = \lim_{b \to \infty} \left( -e^{-b} + e^{-1} \right) = 0 + e^{-1} = \frac{1}{e}. \]
20. (a) Find a parametrization for the circle of radius 2, centered at (2, 2).
    (b) Compute the arc length of the curve from part (a).
    (c) This can also be found using a standard geometric formula. Check that you got the right answer.

a) \[ x = 2 + 2\cos t \quad \text{for} \quad 0 \leq t \leq 2\pi \]
    \[ y = 2 + 2\sin t \]

b) \[ \frac{dx}{dt} = -2\sin t \]
    \[ \frac{dy}{dt} = 2\cos t \]

So \[ L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t} \, dt \]

\[ = 2 \int_0^{2\pi} \, dt \]

\[ = 4\pi \]

c) \[ C = 2\pi r = 2\pi (2) = 4\pi \]
21. Eliminate $t$ to find an equation for the curve given by:

$$x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$$

Sketch the graph of this curve.

$$y = (t^2)^3 - 2(t^2)^2$$

$$y = x^3 - 2x^2$$

$$y = x^2(x-2)$$

So, at $x = 0$ and $x = 2$.

$$\frac{dy}{dx} = 3x^2 - 4x = x(3x - 4)$$

Increasing on $-\infty < x < 0$

Decreasing on $0 < x < \frac{4}{3}$

Increasing on $\frac{4}{3} < x < \infty$
22. Find an equation for the line tangent to the curve:

\[ x = -\sqrt{t+1} \]
\[ y = \sqrt{3t} = \sqrt{3} \cdot t^{1/2} \]

At \( t = 3 \). What is the concavity at that point?

\[ \frac{dx}{dt} = \frac{-1}{2\sqrt{t+1}} \quad \frac{dy}{dt} = \frac{\sqrt{3}}{2\sqrt{t}} \]

So \[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{t+1}}}{\frac{-1}{2\sqrt{t+1}}} = -\frac{\sqrt{3}}{4} \cdot \frac{\sqrt{t+1}}{\sqrt{t}} \]

At \( t = 3 \),

\[ \frac{dy}{dx} = -\frac{\sqrt{3}}{4} \cdot \frac{\sqrt{4}}{\sqrt{3}} = -\frac{1}{2} \]

\[ x = -\sqrt{4} = -2 \]
\[ y = \sqrt{3 \cdot 3} = 3 \]

So a line tangent to the curve at \( t = 3 \) is:

\[ y - 3 = -\frac{1}{2} (x - (-2)) \]

\[ y = 3 - \frac{1}{2}x + 1 = y = 4 - \frac{1}{2}x \]

Concavity:

\[ \frac{d^2y}{dt^2} = \frac{\sqrt{3}}{8\sqrt{t+1}} \cdot \frac{1}{\sqrt{t}} - \frac{\sqrt{3}}{4} \cdot \frac{\sqrt{t+1}}{t^{3/2}} \cdot \frac{(-1)}{t} - t^{-3/2} \]

\[ = \frac{\sqrt{3}}{8} \left( \frac{1}{\sqrt{t+1}} + \frac{\sqrt{t+1}}{t^{3/2}} \right) \]

\[ \frac{dy}{dx^2} = \frac{d^2y/dt^2}{dx/dt} = \frac{\sqrt{3}}{4} \cdot \left( \frac{1}{\sqrt{t}} - \frac{t+1}{t^{3/2}} \right) \]

So \[ \frac{dy}{dx^2} = \frac{d^2y/dt^2}{dx/dt} = \frac{\sqrt{3}}{4} \left( \frac{1}{\sqrt{t}} - \frac{t+1}{t^{3/2}} \right) \]

At \( t = 3 \):

\[ \frac{\sqrt{3}}{4} \left( \frac{9-4}{2^{3/2}} \right) > 0 \]

Since \( \frac{dy}{dx} < 0 \) for \( x \leq -2 \) and \( \frac{d^2y}{dx^2} > 0 \) for \( x \leq -2 \), the tangent line is above the curve at \( t = 3 \) for \( x \leq -2 \).
23. Find the length of the curve:

\[ x = \cos t \]

\[ y = t + \sin t \]

From \(0 \leq t \leq \pi\).

\[
L = \int_0^\pi \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt
\]

\[
= \int_0^\pi \sqrt{(\cos t)^2 + ((1 + \cos t)^2} \, dt
\]

\[
= \int_0^\pi \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t} \, dt
\]

\[
= \int_0^\pi \sqrt{2 + 2\cos t} \, dt
\]

\[
\cos^2 t = \frac{1 + \cos 2t}{2}
\]

\[
2 \cos^2 \frac{t}{2} = 1 + \cos t
\]

\[
4 \cos^2 \frac{t}{2} = 2 + 2\cos t
\]

\[
= \int_0^\pi \sqrt{4 \cos^2 \frac{t}{2}} \, dt
\]

\[
= 2 \int_0^\pi \cos \frac{t}{2} \, dt
\]

\[
= 4 \left[ \sin \frac{t}{2} \right]_0^\pi
\]

\[
= 4 \left( \sin \frac{\pi}{2} - \sin 0 \right)
\]

\[
= 4
\]
24. Find an equation in cartesian coordinates for the graph given by:

\[ r = 4 \csc \theta \]

\[ r = \frac{4}{\sin \theta} \]

\[ r \sin \theta = 4 \]

\[ y = 4 \]
25. Sketch a graph of the polar curve given by:

\[ r = 2 + \cos \theta \]

\[ \theta = 0 \quad r = 3 \]
\[ \theta = \frac{\pi}{6} \quad r = 2 + \frac{\sqrt{3}}{2} \]
\[ \theta = \frac{\pi}{4} \quad r = 2 + \frac{\sqrt{2}}{2} \]
\[ \theta = \frac{\pi}{3} \quad r = 2 + 1 \]
\[ \theta = \frac{\pi}{2} \quad r = 2 \]
\[ \theta = \frac{2\pi}{3} \quad r = 2 - \frac{1}{2} \]
\[ \theta = \frac{3\pi}{4} \quad r = 2 - \frac{\sqrt{2}}{2} \]
\[ \theta = \frac{5\pi}{6} \quad r = 2 - \frac{\sqrt{3}}{2} \]
\[ \theta = \pi \quad r = 1 \]

\[ \cos (-\theta) = \cos \theta \]

So this reflects across the x-axis.
26. Find the length of the curve:

\[ r = \frac{6}{1 + \cos \theta} \]

for \( \theta \) in the interval \( 0 \leq \theta \leq \frac{\pi}{2} \).

\[
\frac{dr}{d\theta} = \frac{-6}{(1 + \cos \theta)^2} (-\sin \theta)
\]

\[
L = \int_{0}^{\pi/2} \sqrt{\left( \frac{6}{1 + \cos \theta} \right)^2 + \left( \frac{6 \sin \theta}{(1 + \cos \theta)^2} \right)^2} \, d\theta
\]

\[
= \int_{0}^{\pi/2} \sqrt{\frac{36}{(1 + \cos \theta)^2} + \frac{36 \sin^2 \theta}{(1 + \cos \theta)^4}} \, d\theta
\]

\[
= 6 \int_{0}^{\pi/2} \sqrt{\frac{(1 + \cos \theta)^2 + \sin^2 \theta}{(1 + \cos \theta)^4}} \, d\theta
\]

\[
= 6 \int_{0}^{\pi/2} \frac{1}{(1 + \cos \theta)^2} \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} \, d\theta
\]

\[
= 6 \int_{0}^{\pi/2} \frac{1}{(1 + \cos \theta)^2} \sqrt{2 + 2 \cos \theta} \, d\theta
\]

\[
= 6 \int_{0}^{\pi/2} \sqrt{\frac{2 (1 + \cos \theta)}{(1 + \cos \theta)^4}} \, d\theta
\]

\[
= 6 \sqrt{2} \int_{0}^{\pi/2} (1 + \cos \theta)^{-3/2} \, d\theta
\]

\[
= 6 \sqrt{2} \int_{0}^{\pi/2} (2 \cos^2 \frac{\theta}{2})^{-3/2} \, d\theta
\]

\[
= 6 \sqrt{2} \int_{0}^{\pi/2} 2^{-3/2} (\cos \frac{\theta}{2})^{-3} \, d\theta
\]
This integral turns out to be hard. Let me manipulate it a bit.

\[
\frac{6\sqrt{2}}{2\sqrt{2}} \int_{0}^{\frac{\pi}{2}} \frac{(\cos^2 \theta)^{-3}}{2} \, d\theta = 6 \int_{0}^{\frac{\pi}{4}} \cos^{-3} u \, du \\
\text{let } u = \frac{\theta}{2} \\
\text{du} = \frac{1}{2} \, d\theta \\
= 6 \int_{0}^{\frac{\pi}{4}} \frac{\cos u}{\cos^3 u} \, du \\
= 6 \int_{0}^{\frac{\pi}{4}} \frac{\cos u}{(1 - \sin^2 u)^2} \, du \\
\text{let } v = \sin u \\
\text{dv} = \cos u \, du \\
= 6 \int_{0}^{\frac{\pi}{4}} \frac{1}{(1 - v^2)^2} \, dv
\]

Now do partial fractions:

\[
\frac{1}{(1 - v^2)^2} = \frac{A + Bv}{1 - v^2} + \frac{C + Dw}{(1 - v^2)^2} = \frac{(A + Bv)(1 - v^2) + C + Dw}{(1 - v^2)^2}
\]

and so on...

I didn't realize this problem was so difficult...
(too hard for a test question!)
\[ A = \int_{\pi/4}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2} (2 \sin \theta)^2 \, d\theta \\
= 2 \int_{\pi/4}^{\pi/2} \sin^2 \theta \, d\theta \\
= 2 \int_{\pi/4}^{\pi/2} \frac{1 - \cos 2\theta}{2} \, d\theta \\
= \int_{\pi/4}^{\pi/2} d\theta - \int_{\pi/4}^{\pi/2} \cos 2\theta \, d\theta \\
= \frac{\pi}{4} - \left[ \sin 2\theta \right]_{\pi/4}^{\pi/2} \\
= \frac{\pi}{4} - \frac{1}{2} \left( \sin \left( \frac{\pi}{2} \right) - \sin \left( \frac{\pi}{4} \right) \right) \\
= \frac{\pi}{4} + \frac{1}{2} \\
\]

\[ A = \frac{\pi}{4} + \frac{1}{2} \]