MAT 21b: Homework 1
Due Wednesday, October 15th, 2014

For this homework assignment, you may work with other students, however the writeups must be entirely your own. I suggest that you think about the problems on your own first, and then discuss your thoughts with your classmates. You’ll get the most out of your time this way. The homework will be graded both on the content of your answers as well as on how well your thoughts are explained. Try to write your answers in a way that one of your classmates could easily follow your thoughts.

In this homework assignment we’ll attempt to show that the function:

$$f(x) = \sin \frac{1}{x}$$

is integrable over [0,1]. To be really pedantic, let’s define $f(x)$ to be:

$$f(x) = \begin{cases} \sin \frac{1}{x} & x > 0, \\ 0 & x = 0. \end{cases}$$

For this problem, the value of $f$ at $x = 0$ really doesn’t matter, but we might as well define it. We’ll be allowed to use two theorems, stated below, but otherwise we’ll be relying on the definition, as well as on our own trickiness. Part of the goal of this exercise is to force you to think a little bit more like a mathematician, and hopefully to also give you some intuition into what the definition of the definite integral says.

**Theorem 1.** Suppose that $f(x)$ is continuous on the closed interval $[a,b]$. Then $f$ is integrable on $[a,b]$.

**Theorem 2.** A function $f$ is integrable over $[a,b]$ if and only if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $P$ with $\|P\| < \delta$, then $U_P - L_P < \varepsilon$. Here $U_P$ is the upper sum for the partition $P$ and $L_P$ is the lower sum for the partition $P$.

Roughly, what Theorem 2 says is that a function is integrable if and only if the lower sums and upper sums are close together for fine partitions.

1. (2 points) Sketch a graph of $f(x) = \sin \frac{1}{x}$ on the interval $[0, 1]$. You may use a graphing calculator or computer (www.wolframalpha.com is particularly easy to use) to help you.

![Graph of f(x) = sin(1/x) on [0, 1] with some annotation points]
2. (2 points) Explain in words (full sentences!) why it seems that $f$ might not integrable over $[0, 1]$.

Since $f$ oscillates rapidly near $x = 0$, it would be difficult to assign an appropriate value to the height of any rectangle used to approximate the integral near 0.

3. (2 points) Explain in words why we cannot conclude, directly from Theorem 1, that $f(x)$ is integrable on $[0, 1]$.

Thm 1 requires that $f$ be continuous on $[0, 1]$ to conclude that it is integrable on $[0, 1]$. Unfortunately, $f$ is not continuous at $x = 0$.

4. (3 points) Let $\delta_1$ be some small number, $0 < \delta_1 < 1$. Using a partition $P$ that is only one interval, compute the upper Riemann sum and the lower Riemann sum of $f$ on $[0, \delta_1]$. You should find that $U_P - L_P = 2\delta_1$.

On $[0, \delta_1]$, the maximum value of $f$ is 1, and the minimum value of $f$ is $-1$.

Thus letting $P = \{0, \delta_1\}$,

$U_P = (1)(\delta_1) = \delta_1$

$L_P = (-1)(\delta_1) = -\delta_1$

$\Rightarrow U_P - L_P = \delta_1 - (-\delta_1) = 2\delta_1$.

It happens to be true that for any partition $P$ of $[0, \delta_1]$, we have that $U_P - L_P < 2\delta_1$. You will need to use this later (you don’t have to show it, but you should try to figure out why it is true, for your own understanding).
We will now begin to show that \( f \) is integrable. Let some small number \( \varepsilon > 0 \) be given. Choose \( \delta_1 = \frac{\varepsilon}{4} \).

5. (3 points) Use Theorem 1 and Theorem 2 to conclude that there is some \( \delta_2 > 0 \) such that for any partition \( P_2 \) of \([\delta_1, 1]\) with \( \|P_2\| < \delta_2 \), then \( U_{P_2} - L_{P_2} < \frac{\varepsilon}{2} \).

\[
\text{Since } f \text{ is continuous on } [\delta_1, 1], \text{ by thm 1, } f \text{ is integrable on } [\delta_1, 1]. \text{ Then, by thm 2, for any small number (which I'll call } \frac{\varepsilon}{2} \text{ there exists another small number (which I'll call } \delta_2 \text{) such that if } P_2 \text{ is any partition with } \|P_2\| < \delta_2, \text{ then } U_{P_2} - L_{P_2} < \frac{\varepsilon}{2}.\]

Notice that what you've done here is to control the difference between the upper sum and the lower sum on the interval \([\delta_1, 1]\).

Choose \( \delta = \min\{\delta_1, \delta_2\} \), and let \( P \) be any partition of \([0, 1]\) such that \( \|P\| < \delta \). You may assume that \( P \) contains the point \( \delta_1 \), and thus you can break \( P \) into two partitions:

\[
P_1 = \{x_0 = 0, x_1, \ldots, x_n = \delta_1\},
\]
\[
P_2 = \{x_0 = \delta_1, x_1, \ldots, x_n = 1\}.
\]

It doesn't really change things if you don't assume that \( P \) contains the point \( \delta_1 \), but it add some steps to what is already a long problem.

6. (3 points) Using the result from problem (4), conclude that \( U_{P_1} - L_{P_1} \leq \frac{\varepsilon}{2} \).

\[
\text{Since } P_1 \text{ is a partition of } [0, \delta_1], \text{ by part (4), we know } U_{P_1} - L_{P_1} \leq 2\delta_1 = 2 \left(\frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2}.
\]

Notice that what you've done here is to control the difference between the upper sum and the lower sum on the interval \([0, \delta_1]\).
7. (3 points) Using rules of sums and inequalities, and the results from problem (5) and problem (6), conclude that $U_p - L_p < \varepsilon$.

In (5), we showed that $U_{p_2} - L_{p_2} < \frac{\varepsilon}{2}$, and in (6), we showed that $U_{p_1} - L_{p_1} \leq \frac{\varepsilon}{2}$. Adding these,

$(U_{p_1} + U_{p_2}) - (L_{p_1} + L_{p_2}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Since $U_p = U_{p_1} + U_{p_2} + L_p = L_{p_1} + L_{p_2}$, we find:

$U_p - L_p < \varepsilon$.

8. (2 points) Explain why you can now conclude that $f$ is integrable on $[0, 1]$.

Parts 5-8 allow us to conclude that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $||P|| < \delta$, then $U_p - L_p < \varepsilon$. Thus by theorem 2, $f$ is integrable.