MAT 21b: Homework 2
Due Wednesday, October 22nd, 2014

For this homework assignment, you may work with other students, however the writeups must be entirely your own. I suggest that you think about the problems on your own first, and then discuss your thoughts with your classmates. You'll get the most out of your time this way. The homework will be graded both on the content of your answers as well as on how well your thoughts are explained. Try to write your answers in a way that one of your classmates could easily follow your thoughts.

Define \( g(x) \) and \( h(x) \) by the formulae:

\[
g(x) = \frac{e^{ix} + e^{-ix}}{2} \quad h(x) = \frac{e^{ix} - e^{-ix}}{2i}.
\]

Here \( i \) is the imaginary unit, defined as \( i = \sqrt{-1} \). Hence \( i^2 = -1 \).

1. **(2 points)** Show that \( \int g(x) \, dx = h(x) + C \).

\[
\int g(x) \, dx = \int \frac{e^{ix} + e^{-ix}}{2} \, dx = \frac{1}{2} \left( \int e^{ix} \, dx + \int e^{-ix} \, dx \right)
\]
\[
= \frac{1}{2} \left( \frac{e^{ix}}{i} - \frac{e^{-ix}}{-i} \right) + C
\]
\[
= \frac{e^{ix} - e^{-ix}}{2i} + C = h(x) + C
\]

2. **(2 points)** Show that \( \int h(x) \, dx = -g(x) + C \).

\[
\int h(x) \, dx = \int \frac{e^{ix} - e^{-ix}}{2i} \, dx = \frac{1}{2i} \left[ \int e^{ix} \, dx - \int e^{-ix} \, dx \right]
\]
\[
= \frac{1}{2i} \left( \frac{e^{ix}}{i} - \frac{e^{-ix}}{-i} \right) + C
\]
\[
= \frac{1}{2i^2} \left( e^{ix} + e^{-ix} \right) + C
\]
\[
= \frac{e^{ix} + e^{-ix}}{2} + C = -g(x) + C
\]
3. (3 points) Show that the following formula holds:

\[ h(a)h(b) = \frac{g(a-b) - g(a+b)}{2} \]

\[
h(a)h(b) = \left(\frac{e^{ia} - e^{-ia}}{2i}\right)\left(\frac{e^{ib} - e^{-ib}}{2i}\right) = \frac{e^{ia}e^{ib} - e^{-ia}e^{-ib} - e^{ia}e^{-ib} + e^{-ia}e^{ib}}{4}
\]

\[
= -\frac{1}{4} (e^{i(a+b)} - e^{i(a-b)} - e^{-i(a-b)} + e^{-i(a+b)})
\]

\[
= -\frac{1}{2} \left( e^{i(a+b)} + e^{i(a-b)} \right)
\]

\[
= -\frac{1}{2} \left( g(a+b) - g(a-b) \right)
\]

\[
= \frac{g(a-b) - g(a+b)}{2}
\]

4. (3 points) Use part (3) to compute the following indefinite integral:

\[ \int h(3x)h(x) \, dx \]

From part 3, we find that:

\[ h(3x)h(x) = \frac{g(3x-x) - g(3x+x)}{2} \]

\[
= \frac{1}{2} (g(2x) - g(4x))
\]

Thus:

\[ \int h(3x)h(x) \, dx = \frac{1}{2} \left[ \int g(2x) \, dx - \int g(4x) \, dx \right] \]

We will do an u-sub to compute \( \int g(ax) \, dx = \frac{1}{a} \int g(u) \, du \)

let \( u = ax \)

\[ du = a \, dx \]

from part (1) \( \frac{1}{a} h(u) + C \)

\[ = \frac{1}{a} h(ax) + C \]

Thus:

\[
\int h(3x)h(x) \, dx = \frac{1}{2} \left( \frac{1}{2} h(2x) - \frac{1}{4} h(4x) \right) + C
\]

\[
= \frac{1}{4} h(2x) - \frac{1}{8} h(4x) + C
\]