MAT 21b: Homework 7
Due Wednesday, December 10th, 2014

For this homework assignment, you may work with other students, however the writeups must be entirely your own. I suggest that you think about the problems on your own first, and then discuss your thoughts with your classmates. You'll get the most out of your time this way. The homework will be graded both on the content of your answers as well as on how well your thoughts are explained. Try to write your answers in a way that one of your classmates could easily follow your thoughts.

In this homework assignment we will try to explain the data below:

Which shows a dramatic collapse of the Atlantic cod fishery. To do this, we will work with a modified version of the Logistic model that we talked about in class that includes a term that models an Allee effect:

\[
\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \left(\frac{P}{k} - 1\right)
\]

(1)

An Allee effect is effectively a minimum population size, below which the species will die out. Strong Allee effects are often observed in schooling fish - when there are too few fish to form schools large enough to protect the fish from predators, the fish are no longer able to survive.

1. (2 points) Find all equilibrium population values to eq. (1).

Equilibrium values are when \( \frac{dP}{dt} = 0 \).

So \( 0 = rP \left(1 - \frac{P}{K}\right) \left(\frac{P}{k} - 1\right) \)

\( P = 0, \ P = K, \ P = k \)
2. **(4 points)** Sketch a phase line for this model. When labeling equilibrium, be sure to denote whether the equilibrium value is *stable* or *unstable*. Make sure to add arrows to your phase line to show how the population will change when not at equilibrium.

3. **(2 points)** Sketch a graph of $\frac{dp}{dt}$ against $P$, for $r = 1$, $k = 20$ and $K = 100$. Restrict your graph to physical values of $P$ ($P \geq 0$). You may find it helpful to use a calculator, or Wolfram Alpha. Again, label the equilibrium values, and denote whether they are stable or unstable, and add arrows to show how the population will change when not at equilibrium.
4. (3 points) We will now add fishing to the model:

\[
\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right) \left( \frac{P}{k} - 1 \right) - EP
\]

Here \( E \) is a constant denoting the effort exerted by fishermen. Redraw your graph from question (3). Add a line to denote a small amount of fishing (that is, be sure there are still two equilibrium values!) Label the new lower equilibrium value \( k^* \) and the new upper equilibrium value \( K^* \). Has their stability changed? Be sure to label the stability of each equilibrium, and again draw arrows indicating how the population will change when not at an equilibrium value. How does \( k^* \) compare to \( k \)? Using words (complete sentences), explain why this equilibrium has changed in this way.

\[ k^* \text{ is greater than } k. \text{ We can think of } k \text{ as the minimum population size necessary for species survival in the absence of fishing, and } k^* \text{ as the minimum population size necessary for survival with fishing.} \]

If you put pressure on a population (e.g., through fishing), it makes sense that the minimum population size necessary for survival should increase.
5. (3 points) Redraw your graph from question (3). Add a line corresponding to a larger amount of fishing. To be specific, give the line a slope so that there is now precisely one equilibrium value for the population \( P \). Label this equilibrium value \( \bar{K} \). Is this equilibrium value stable or unstable? Again, be sure to draw arrows indicating how the population will change when not at an equilibrium.

\[ \bar{K} \text{ is a half-stable equilibrium — we didn't discuss this in class.} \]

This behavior is similar to \( \frac{dy}{dt} = y^2 \), and it is the simplest example of a catastrophe system. Try sketching these kinds of graphs for

\[ \frac{dy}{dt} = \varepsilon + y^2 \quad \frac{dy}{dt} = y^2 \quad \frac{dy}{dt} = -\varepsilon + y^2 \]

for any small value \( \varepsilon > 0 \). Notice that this small change makes a big difference in the qualitative behavior of the system!
6. (3 points) Assuming \( r = 1, k = 20, \) and \( K = 100, \) compute \( \tilde{K}. \) To do this, you'll need to find two equations that both contain \( P \) and \( E, \) and then solve for \( P. \) One of these equations comes directly from the equilibrium condition. To find the other equation, notice that in your answer to question (5), the line representing fishing is tangent to the curve \( rP(1 - P/K)(P/k - 1) \) at the point \( P = \tilde{K}. \) How does \( \tilde{K} \) compare to \( K? \) In particular, are fishermen still catching a lot of fish?

with these parameters, \[ \frac{dP}{dt} = P(1 - \frac{P}{100})(\frac{P}{20} - 1) - EP \]

one equation is the equilibrium condition:
\[ 0 = \tilde{K}(1 - \frac{\tilde{K}}{100})(\frac{\tilde{K}}{20} - 1) - E\tilde{K} \] \hspace{1cm} (1)

the other is the tangent condition:
\[ \frac{d}{dP} \left( P(1 - \frac{P}{100})(\frac{P}{20} - 1) \right) \bigg|_{P=\tilde{K}} = \frac{d}{dP} (EP) \bigg|_{P=\tilde{K}} \]

\[ \Rightarrow \left[ (1 - \frac{\tilde{P}}{100})(\frac{\tilde{P}}{20} - 1) - \frac{\tilde{P}}{100}(\frac{\tilde{P}}{20} - 1) + \frac{\tilde{P}}{20}(1 - \frac{\tilde{P}}{100}) \right] \bigg|_{P=\tilde{K}} = E \]

thus \[ E = (1 - \frac{\tilde{K}}{100})(\frac{\tilde{K}}{20} - 1) - \frac{\tilde{K}}{100}(\frac{\tilde{K}}{20} - 1) + \frac{\tilde{K}}{20}(1 - \frac{\tilde{K}}{100}) \] \hspace{1cm} (2)

Simplifying \( (1) \):
\[ E = (1 - \frac{\tilde{K}}{100})(\frac{\tilde{K}}{20} - 1) \]

and plugging in \( (2) \):
\[ (1 - \frac{\tilde{K}}{100})(\frac{\tilde{K}}{20} - 1) - \frac{\tilde{K}}{100}(\frac{\tilde{K}}{20} - 1) + \frac{\tilde{K}}{20}(1 - \frac{\tilde{K}}{100}) = (1 - \frac{\tilde{K}}{100})(\frac{\tilde{K}}{20} - 1) \]

\[ \Rightarrow \frac{\tilde{K}}{20} (1 - \frac{\tilde{K}}{100}) = \frac{2}{180} (\frac{\tilde{K}}{20} - 1) \]

\[ 5(1 - \frac{\tilde{K}}{100}) = \frac{2}{20} \tilde{K} - 1 \]

\[ 5 - \frac{\tilde{K}}{20} = \frac{2}{20} \tilde{K} - 1 \]

\[ 6 = \frac{2}{20} \tilde{K} \]

\[ \tilde{K} = 60 \]

Yes, \( 60 \) is \( 60\% \) of the carrying capacity. At \( K = 100, \) they still get lots of fish!
7. (3 points) Redraw your graph from question (3). Now add a line corresponding to a slightly larger amount of fishing than you drew in question (5). With this amount of fishing, how many equilibrium points are there in the system? What are their stability? Again, be sure to draw arrows indicating how the population will change when not at an equilibrium. Discuss briefly (in full sentences!) why this explains the data shown in the graph at the beginning of the assignment.

Equilibrium at $P = 0$, it is stable!

If the fisherman fish at the value $E = (1 - \frac{60}{100})(\frac{60}{20} - 1)$, then the system has an equilibrium at $P = 60 = k$, and the fishermen are still catching lots of fish. This is what we see on the graph—lots of fish being caught. However, if they even slightly increase $E$, the equilibrium disappears, and the population will rapidly decrease to $P = 0$, as we see on the graph, and in the data.