Once upon time, there were $k$ lattice points inside a polytope...

Jesús A. De Loera

University of California, Davis

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THE
SOUTHERN MOST POINT
OF CONTINENTAL ASIA
CLEAN LATTICE TETRAHEDRA

BRUCE REZNICK

Abstract. A clean lattice tetrahedron is a non-degenerate tetrahedron with the property that the only lattice points on its boundary are its vertices. We present some new proofs of old results and some new results on clean lattice tetrahedra, with an emphasis on counting the number of its interior lattice points and on computing its lattice width.

1. Introduction and Overview

Let $T = T(v_1, \ldots, v_n) = \text{conv}(v_1, \ldots, v_n)$ be a non-degenerate simplex with vertices $v_j \in \mathbb{Z}^n$. We say that $T$ is clean if there are no non-vertex lattice points on the boundary of $T$. Let $i(T) = \#(\text{int}(T) \cap \mathbb{Z}^n)$ denote the number of lattice points in the interior of a clean lattice simplex $T$. If $i(T) = k$, then $T$ is called a $k$-point lattice simplex. If $i(T) = 0$, then $T$ is called empty. This paper is mainly concerned with clean tetrahedra.
A \( k \)-lattice polytope is a lattice polytope containing exactly \( k \) lattice points in its interior. Here are all 2-lattice gons and 3-lattice gons.
A question around Bruce’s flavor

Given $m$ Linear Inequalities with rational coefficients, they define a polytope.

\[
\begin{align*}
  a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_n & \leq b_1 \\
  a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_n & \leq b_2 \\
  \vdots \\
  a_{m,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_n & \leq b_m 
\end{align*}
\]

If we want exactly $k$-lattice points inside, what is the smallest number of constraints $m$ one really needs?

Let us start with $k = 0$. First, I want not a single lattice point inside! Clearly if $m = 1, 2$ that is not enough!! Polyhedra with no interior integral points are called lattice-free. The interest in lattice-free polyhedra is motivated by applications in mixed-integer optimization.
Jean-Paul Doignon, David E. Bell & Herbert Scarf (1970’s)

They found the answer...
Theorem of Doignon-Bell-Scarf

**Theorem** Let $A$ be a $m \times n$ matrix and $b$ a vector of $\mathbb{Q}^m$. If the problem $P_A(b) = \{x : Ax \leq b, x \in \mathbb{Z}^n\}$ has no integer solution, then there is a subset $S$ of the $m$ rows of $A$ of cardinality no more than $2^n$, so that the smaller system has no integer solution either.
By contradiction

Suppose \( E \) polytope \( P \) without lattice pts

and more than \( 2^n \) hyperplanes but if we remove any hyperplane... new polytope has a lattice pts
We can "push" the hyperplanes until we have at least one lattice point on each hyperplane!

![Diagram with lattice points and hyperplanes]

We have more than $2^n$ lattice points on the boundary of $P$. None in interior!
Each lattice point has a pattern of parities
(Even, odd, odd, even, even, odd)

⇒ A vector of parities is repeated!!

⇒ The midpoint of those two red points
  is a lattice point
  in the interior of P
  a contradiction!
Next case...

If one wants to have exactly \( k \geq 1 \) integral points inside the polytope, how many hyperplanes does one need to enclose them??
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how many hyperplanes does one need to enclose them ??
Theorem (Iskander Aliev, JDL, Quentin Louveaux)

Let $A$ be a $m \times n$ matrix and $b$ a vector of $\mathbb{Q}^m$. and Let $n$, $k$ be non-negative integers.

There exists a magic number $c(k, n)$, depending only on $k$ and $n$, such that if the polytope $P_A(b) = \{x : Ax \leq b\}$ has exactly $k$ integral solutions, then a subset of the inequalities of $Ax \leq b$, of cardinality no more than $c(k, n)$, has exactly the same $k$ integer solutions as $P_A(b)$.

Original Doignon-Bell-Scarf is case of $k = 0$. 
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MAIN THEOREM:

Theorem (Iskander Aliev, JDL, Quentin Louveaux)

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Corollary
For \( n, k \) non-negative integers, there exists a magic number \( c(k, n) \), determined by \( k \) and \( n \), such that

- For any system of inequalities \( \{x : Ax \leq b\} \) in \( \mathbb{R}^n \), if every subset of the constraints of cardinality \( c(k, n) \) has at least \( k \) integer solutions, then the entire system of inequalities must have at least \( k \) integral solutions.
Corollary
For \( n, k \) non-negative integers, there exists a magic number \( c(k, n) \), determined by \( k \) and \( n \), such that

- For any system of inequalities \( \{x : Ax \leq b\} \) in \( \mathbb{R}^n \), if every subset of the constraints of cardinality \( c(k, n) \) has at least \( k \) integer solutions, then the entire system of inequalities must have at least \( k \) integral solutions.

- Let \( (X_i)_{i \in \Lambda} \) be a collection of convex sets in \( \mathbb{R}^n \), where at least one of these sets is compact.
  If exactly \( k \) integer points are in \( \bigcap_{i \in \Lambda} X_i \), then there is a subcollection of size less than or equal to \( c(n, k) \) with exactly the same integer points in their intersection.
**Corollary**
For $n, k$ non-negative integers, there exists a **magic number** $c(k, n)$, determined by $k$ and $n$, such that

- For any system of inequalities $\{x : Ax \leq b\}$ in $\mathbb{R}^n$, if every subset of the constraints of cardinality $c(k, n)$ has at least $k$ integer solutions, then the entire system of inequalities must have at least $k$ integral solutions.

- Let $(X_i)_{i \in \Lambda}$ be a collection of convex sets in $\mathbb{R}^n$, where at least one of these sets is compact.
  If exactly $k$ integer points are in $\bigcap_{i \in \Lambda} X_i$, then there is a subcollection of size less than or equal to $c(n, k)$ with exactly the same integer points in their intersection.

But, **WHAT IS THE MAGIC NUMBER** $c(n, k)$?

Well, when $k = 0$ we knew $c(0, n) = 2^n$. 
THEOREM 2: Bound for $c(n, k)$

For $n, k$ non-negative integers

$$c(k, n) \leq \lceil \frac{2(k + 1)}{3} \rceil 2^n - 2 \lceil \frac{2(k + 1)}{3} \rceil + 2$$

and the bound is tight for $c(0, n)$ and $c(1, n)$. 

Example

For $c(1, 2) = 6$ and $c(1, 3) = 14$, but $c(3, 2) = 6$ (but 8 is our bound!)

OPEN PROBLEM Find the exact value of the $c(k, n)$ for $k \geq 2$. 
MAIN THEOREM 2: Bound for $c(n, k)$

- **Theorem 2** For $n, k$ non-negative integers
  \[
  c(k, n) \leq \left\lceil \frac{2(k + 1)}{3} \right\rceil 2^n - 2 \left\lceil \frac{2(k + 1)}{3} \right\rceil + 2
  \]
  and the bound is tight for $c(0, n)$ and $c(1, n)$.

- **Example** For $c(1, 2) = 6$ and $c(1, 3) = 14$, but $c(3, 2) = 6$ (but 8 is our bound!)

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![Graph showing data points and lines representing the bounds and the exact values for different values of $c(k, n)$.]