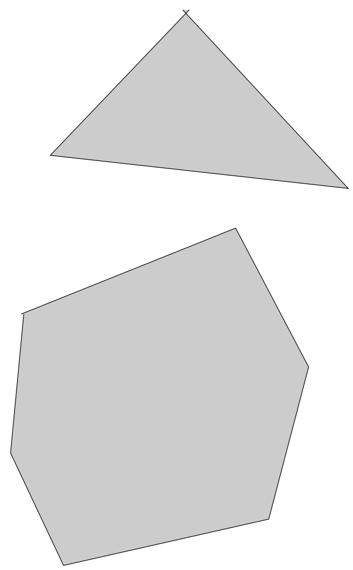
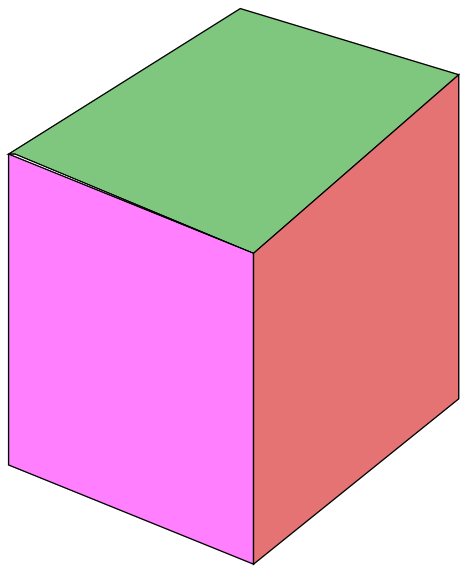
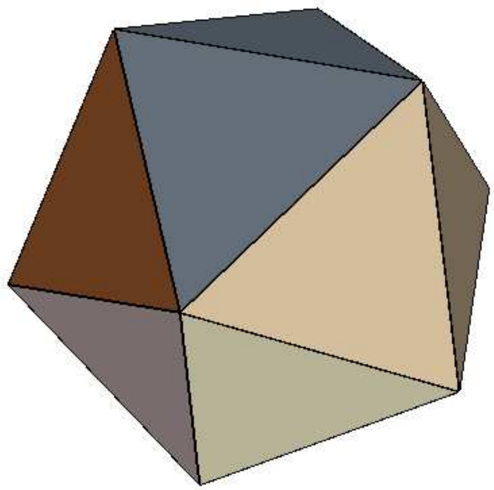
Easy-to-Explain but Hard-to-Solve Problems About Convex Polytopes

Jesús Antonio De Loera
Department of Mathematics
Univ. of California, Davis
http://www.math.ucdavis.edu/~deloera/



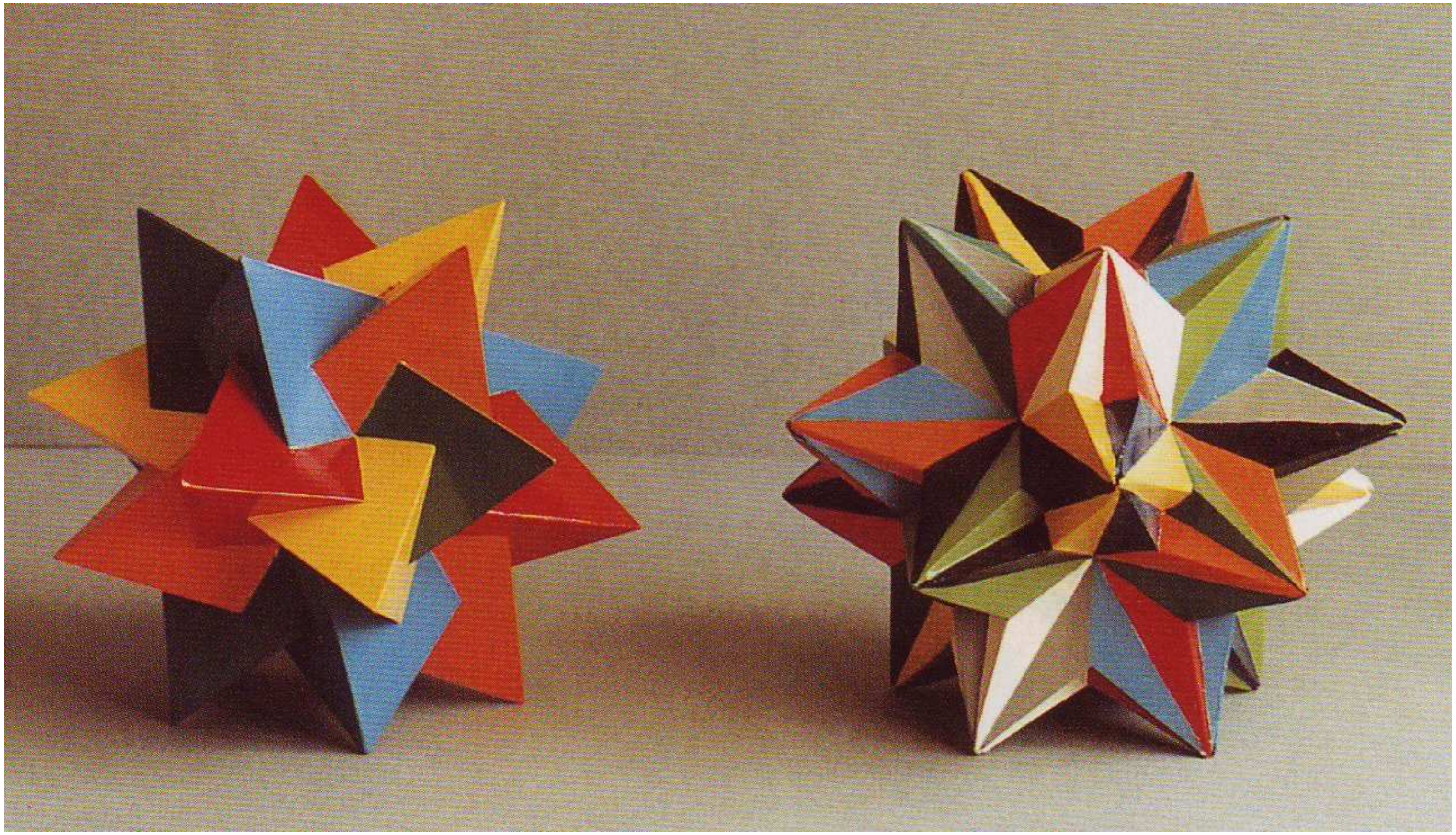




What is a Polytope?

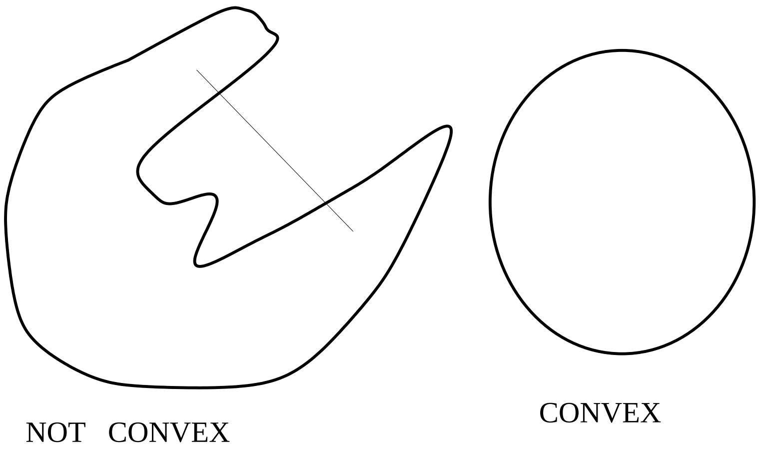
Well,something like this:

But NOT like this!



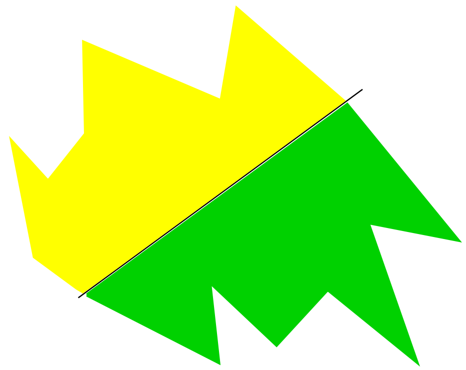
A definition PLEASE!

The word CONVEX stands for sets that contain any line segment joining two of its points:



Definition: A POLYTOPE is the convex hull of a finitely many points inside Euclidean space.

a (hyper)plane divides spaces into two *half-spaces*. Half-spaces are convex sets! Intersection of convex sets is a convex set!



Formally a half-space is a *linear inequality*:

$$a_1x_1 + a_2x_2 + \ldots + a_dx_d \leq b$$

Lemma: A polytope is a bounded subset of Euclidean space that results as the intersection of finitely many half-spaces.

An algebraic formulation for polyhedra

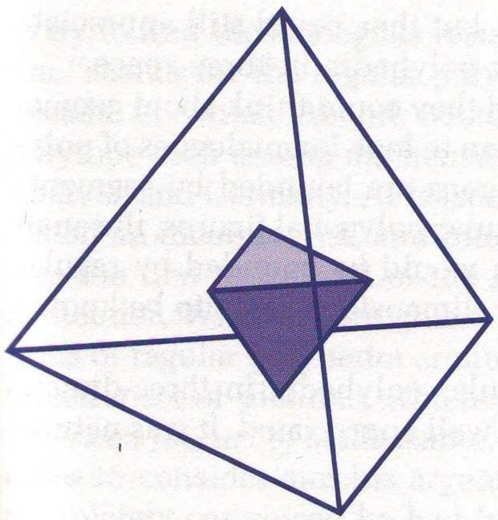
A polytope has also an algebraic representation as the set of solutions of a system of linear inequalities:

$$
\begin{gathered}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d \leq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d \leq b_2 \\
\vdots \\
a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d \leq b_k
\end{gathered}
$$

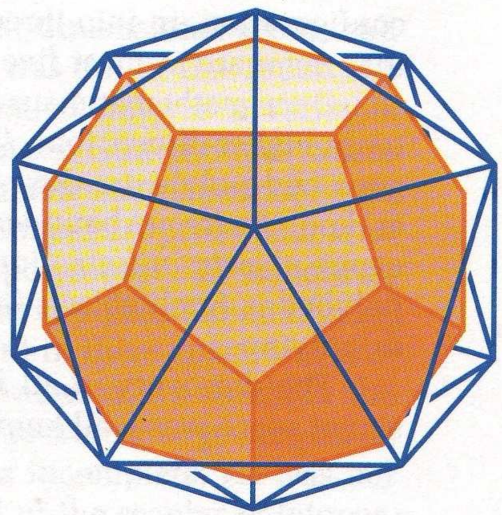
Note: This includes the possibility of using some linear equalities as well as inequalities!! Polytopes represented by sets of the form $\{x|Ax = b,\ x \geq 0\}$, for suitable matrix A, and vector b.

Faces of Polytopes

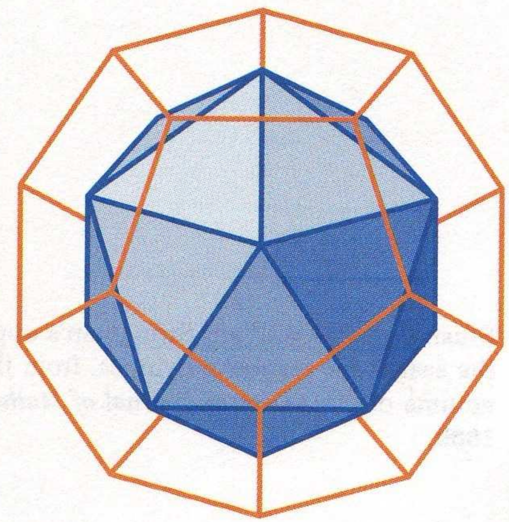
Duality



The self-dual tetrahedron.



The dodecahedron is dual to the icosahedron.



The icosahedron is dual to the dodecahedron.

Some Numeric Properties of Polyhedra

- **Euler's formula** $V - E + F = 2$, relates the number of vertices V, edges E, and facets F of a 3-dimensional polytope.

Definition: Given a convex polytope P, denote by $f_i(P)$ the number of i-dimensional faces. The vector $(f_0(P), f_1(P), \ldots, f_d(P))$ is the f-**vector** of P.

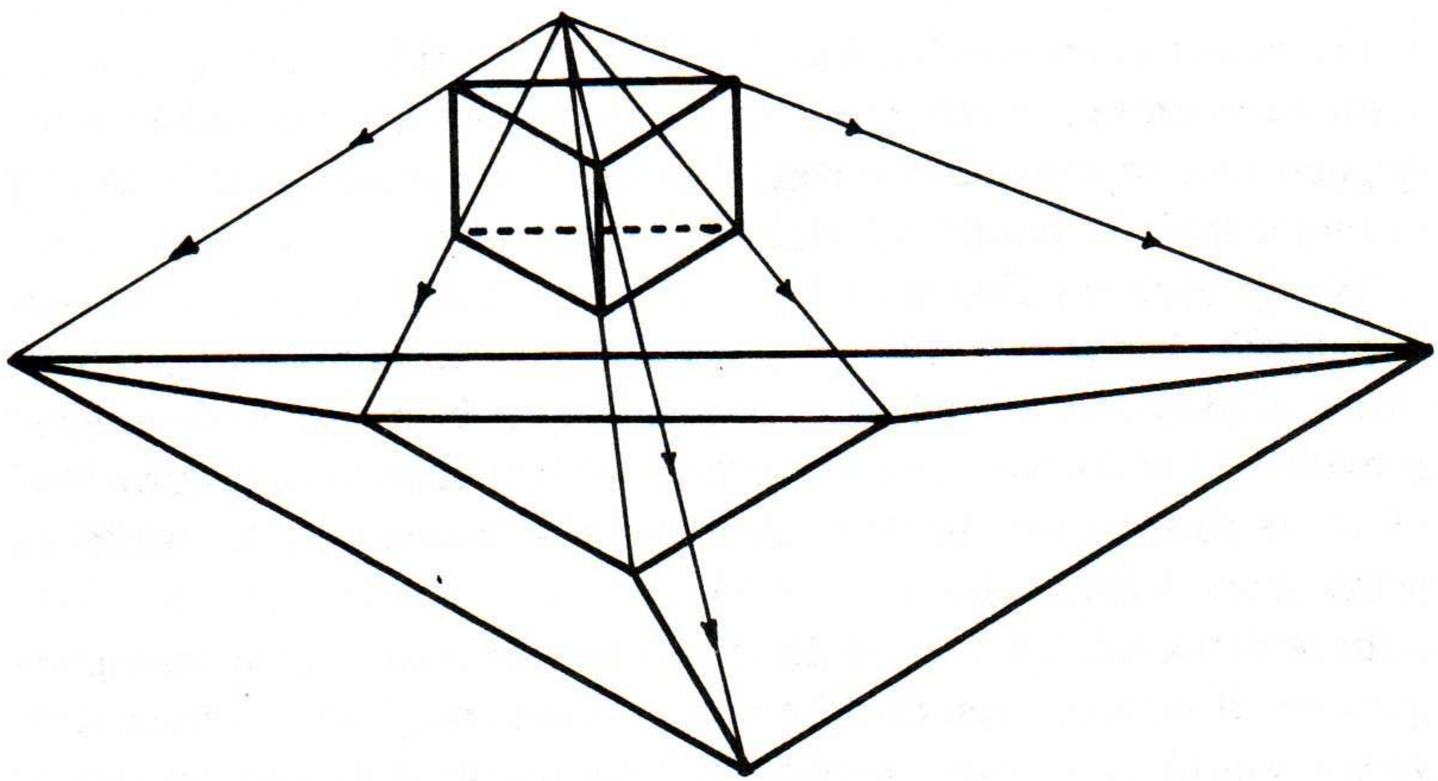
- A very active research area is to characterize the integer vectors $(f_0(P), f_1(P), \ldots, f_d(P))$, which are f-vectors!

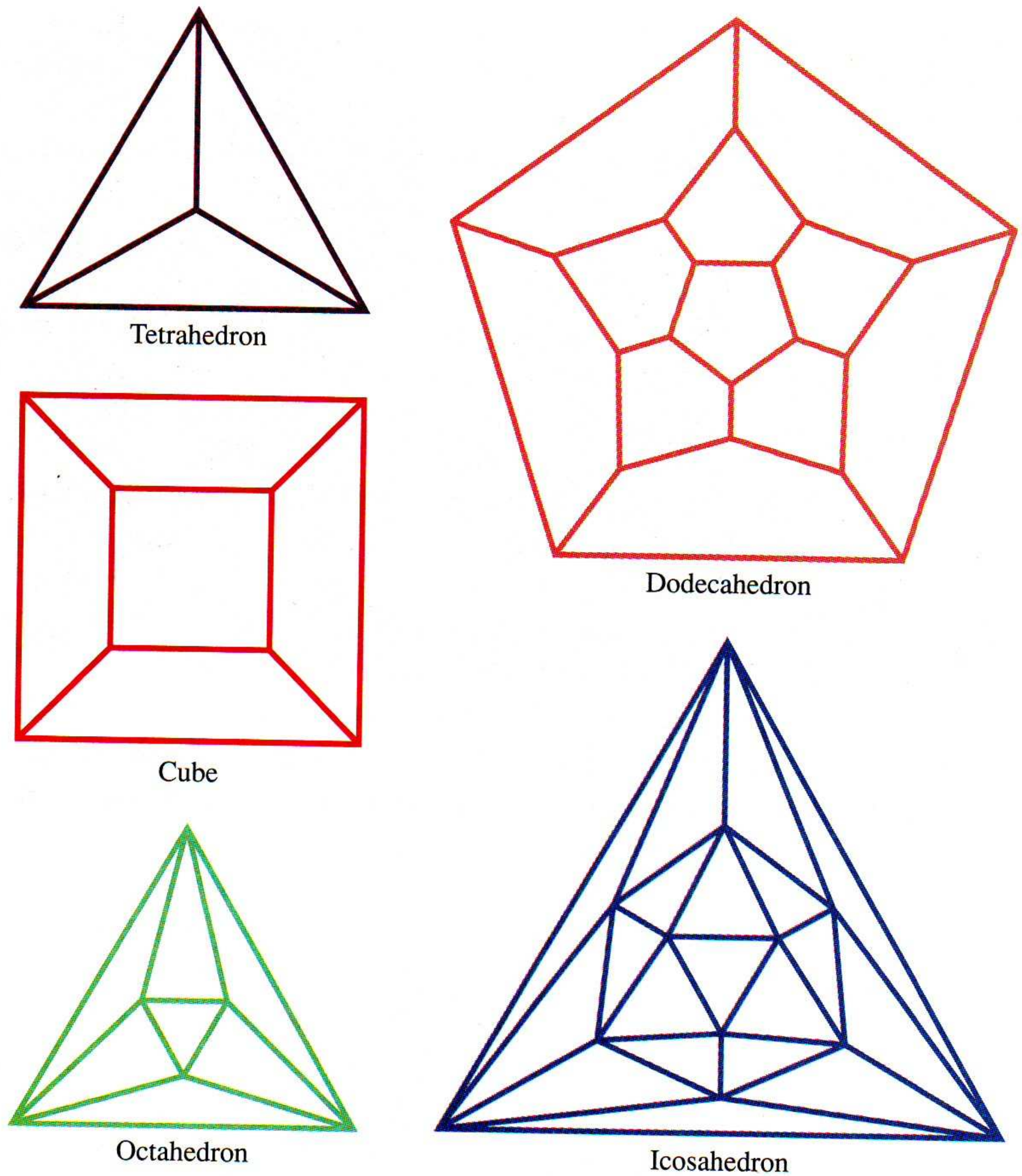
- SOME SAMPLE QUESTIONS:
  - Can one find all constraints characterizing f-vectors of 4-dimensional polytopes?
  - What is the largest number of facets possible in a d-dimensional polyhedron with n vertices? i.e., what is the largest value of $f_{d-1}(P)$?
  - For centrally-symmetric d-polytopes, how big can $\sum f_i(P)$ be? Believed to be at least 3^d.

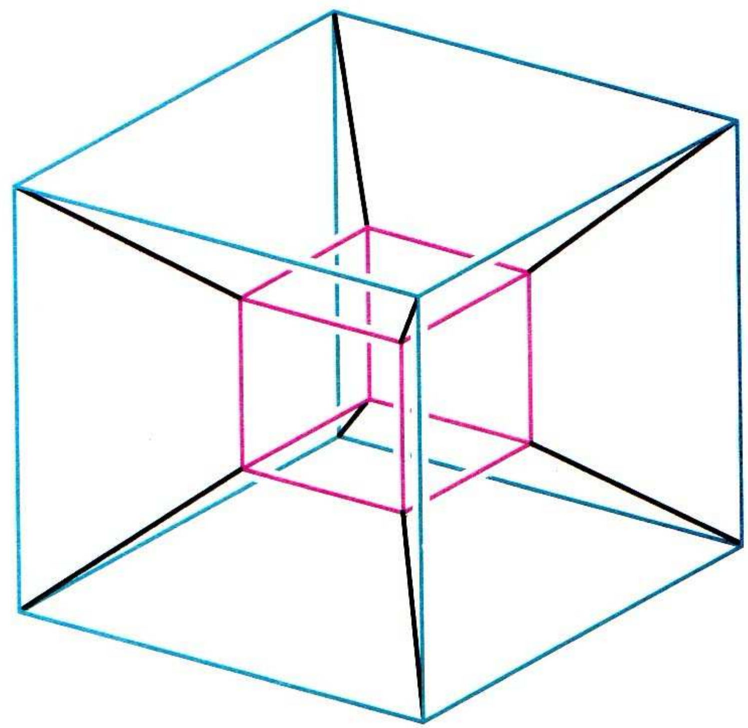
Problem 1:

Unfolding of Polytopes

Ways to Visualize Polytopes



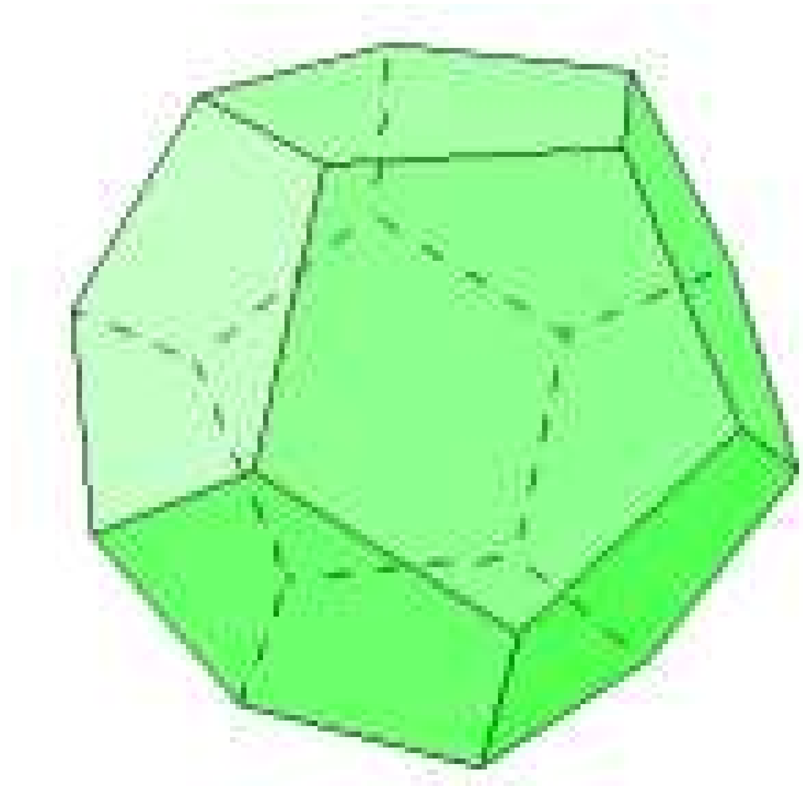

Tetrahedron
Dodecahedron
Cube
Octahedron
Icosahedron

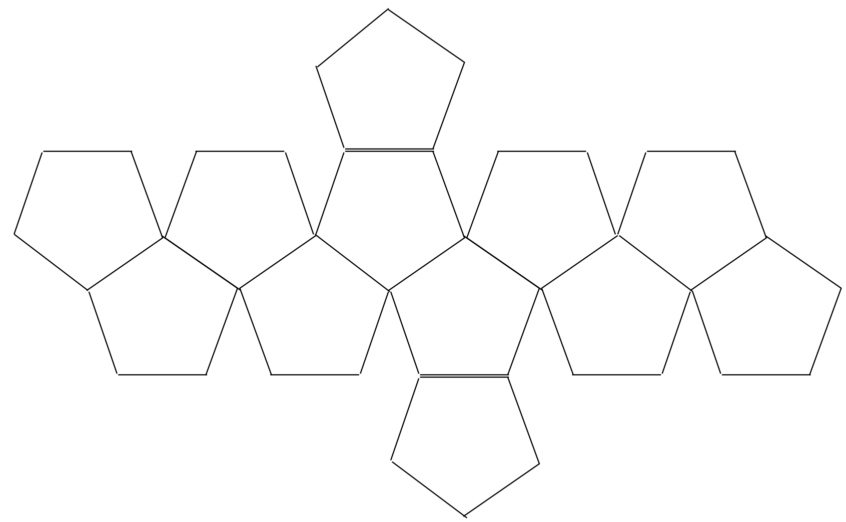


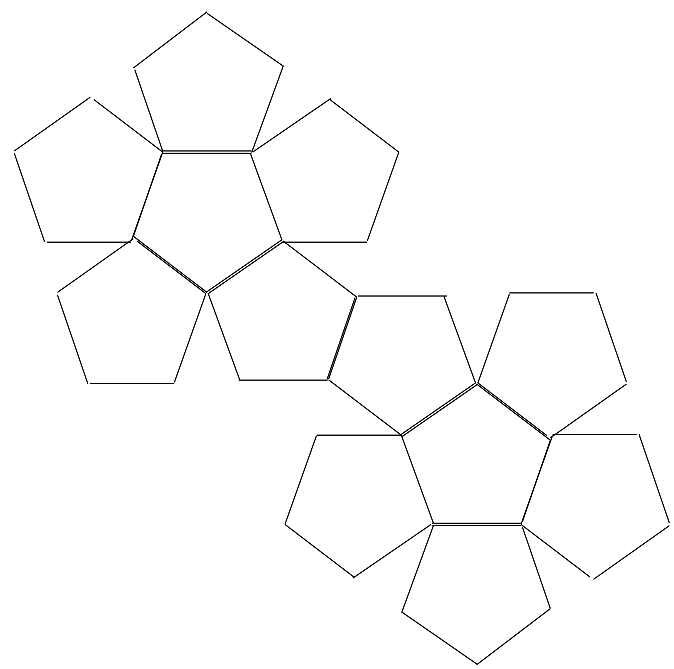
The central projection of a hypercube from four-space to three-space appears as a cube within a cube.

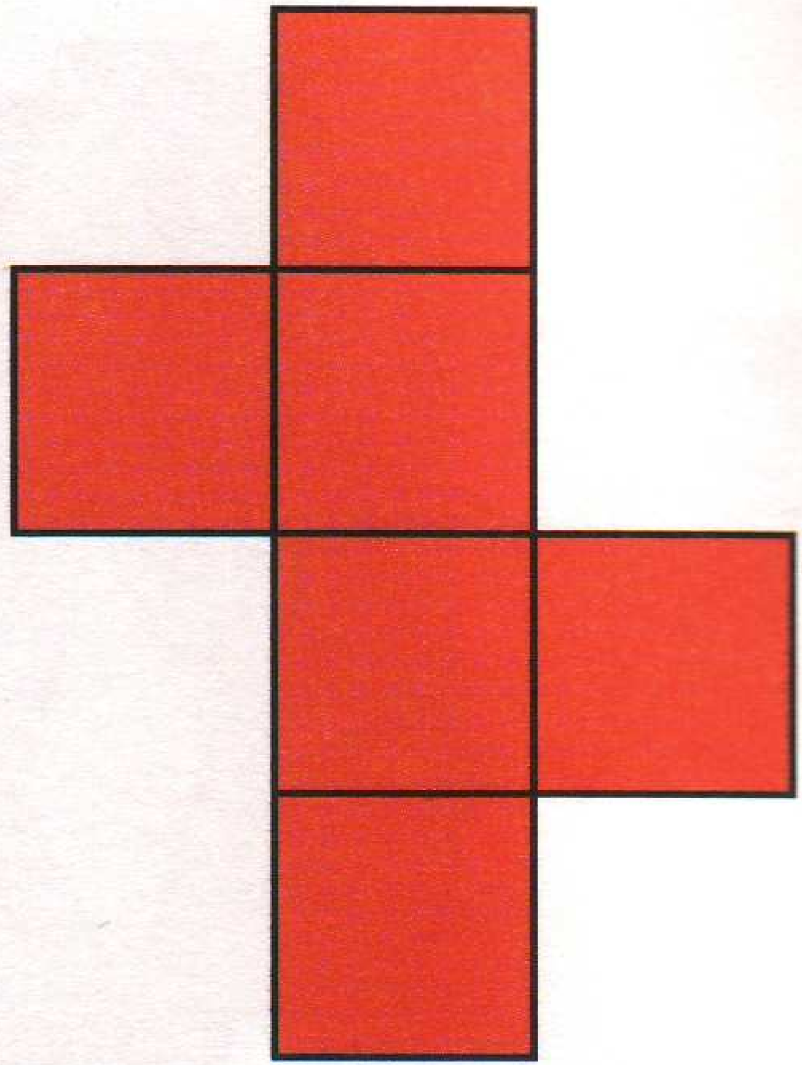
Unfolding Polyhedra

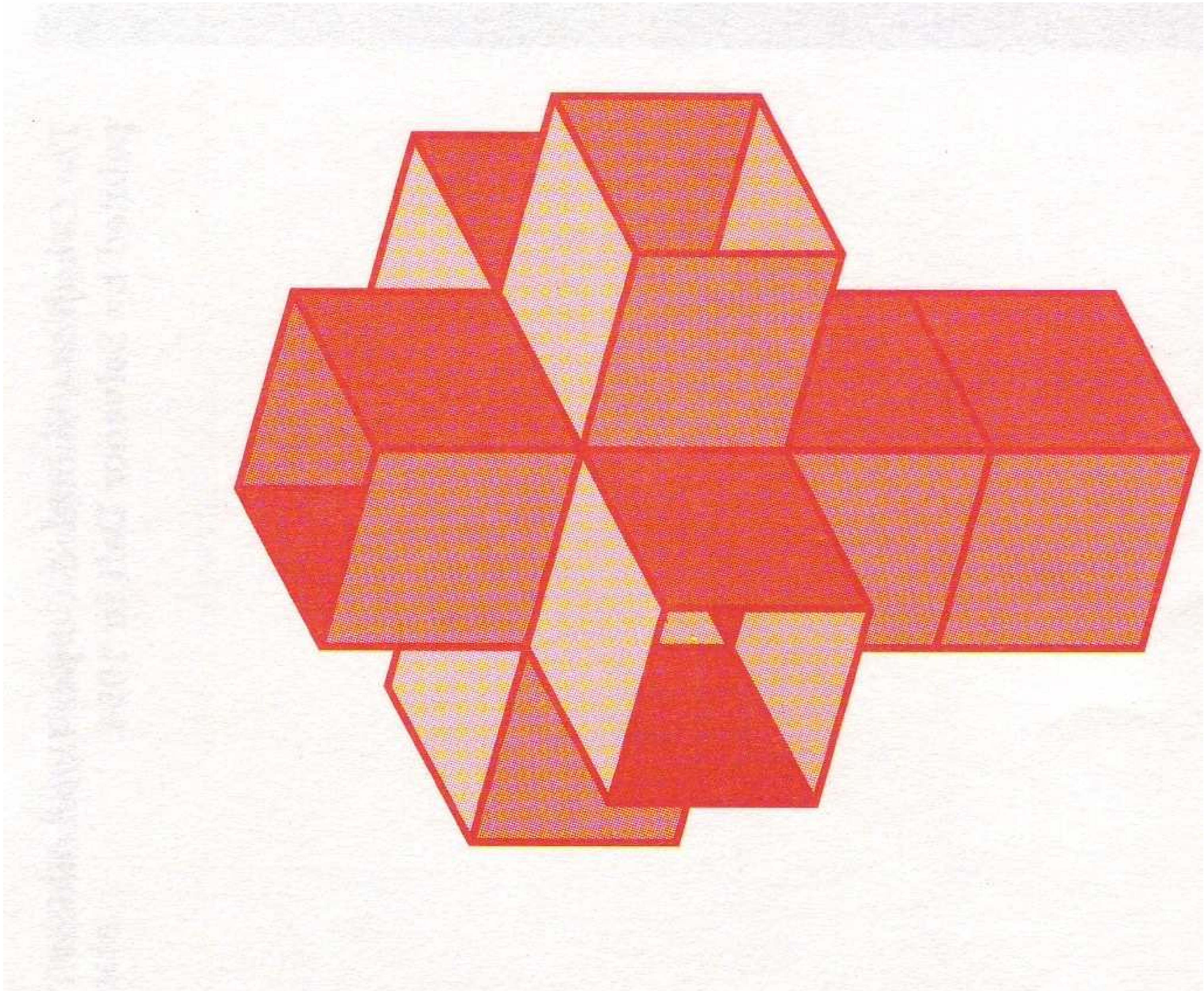
What happens if we use scissors and cut along the edges of a polyhedron? What happens to a dodecahedron?

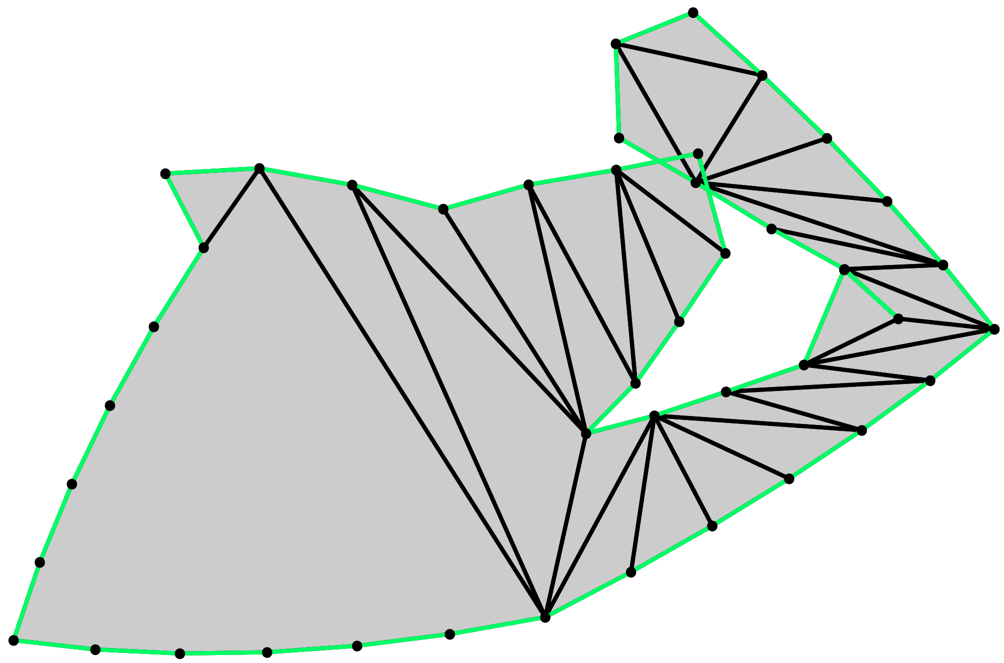


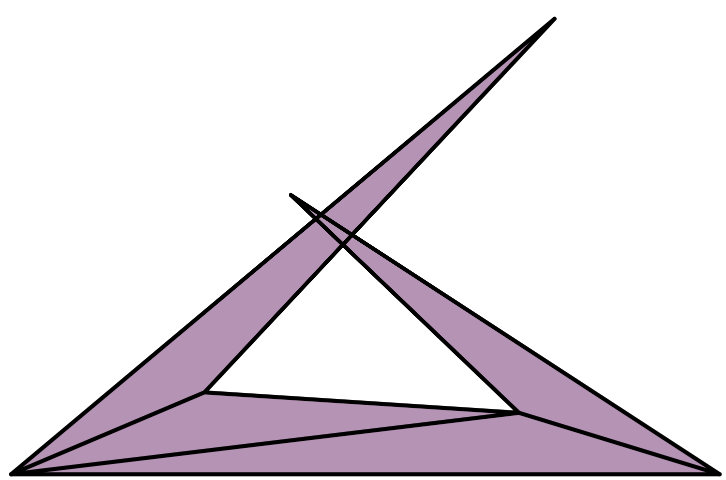










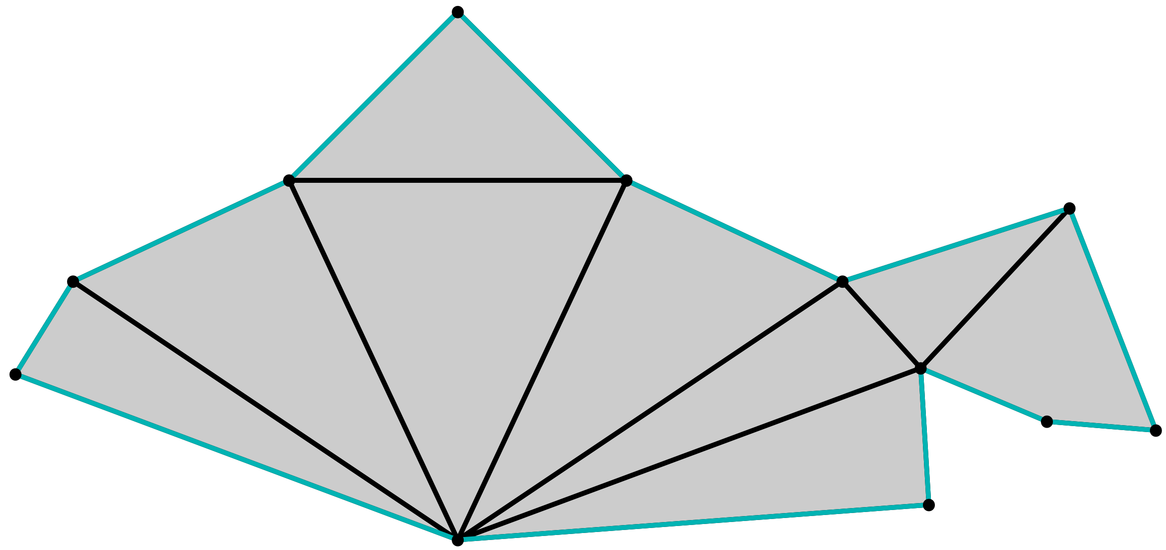


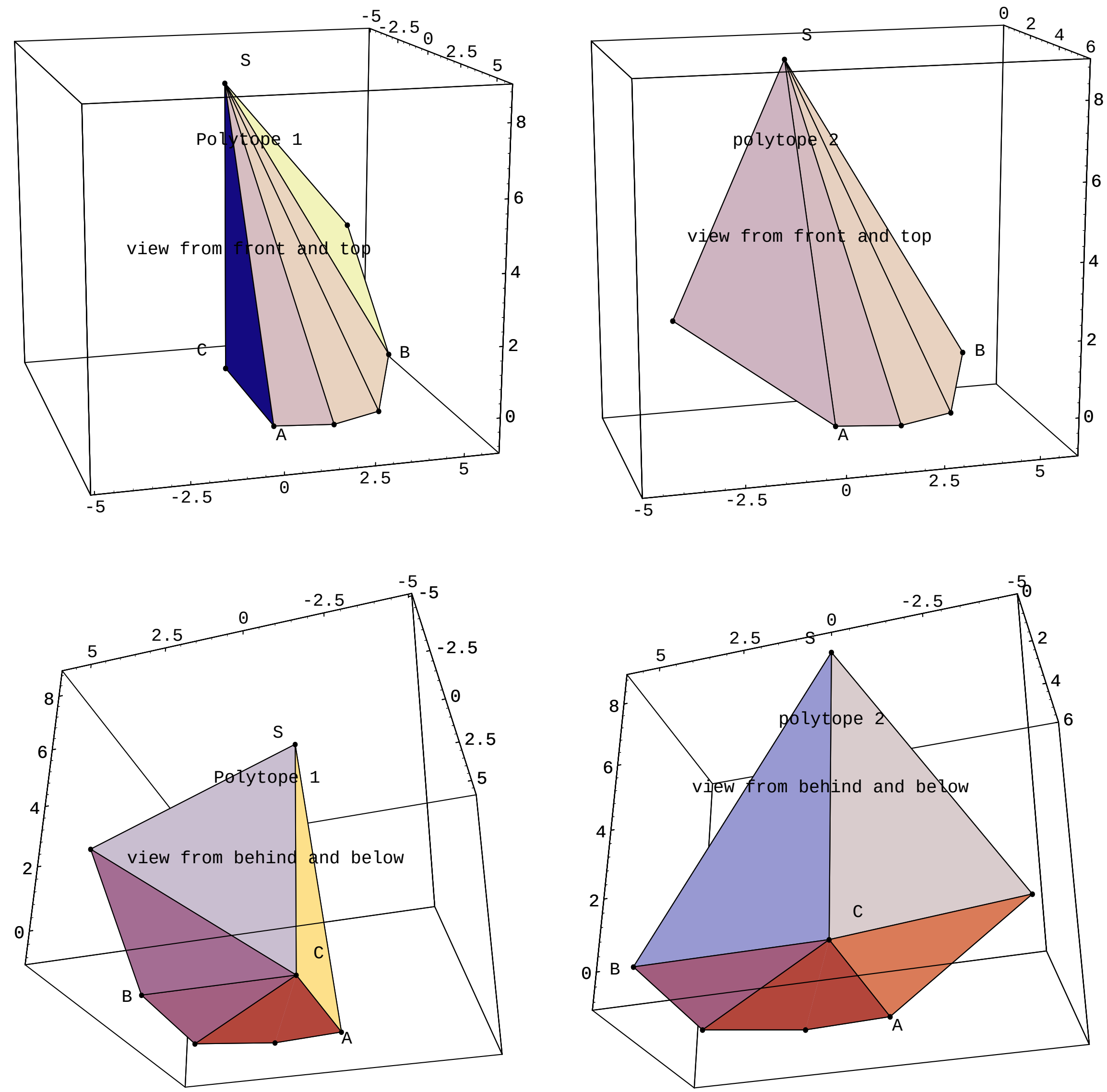
Some Questions

- Question: Can one always find an unfolding that has no self-overlappings?

- Question: What is the largest number of distinct unfoldings that one can have for a polyhedron with n facets? For example, there 11 unfoldings for the 3-cube!

- Question: Is there always a single way to glue together an unfolding to reconstruct a polyhedron?

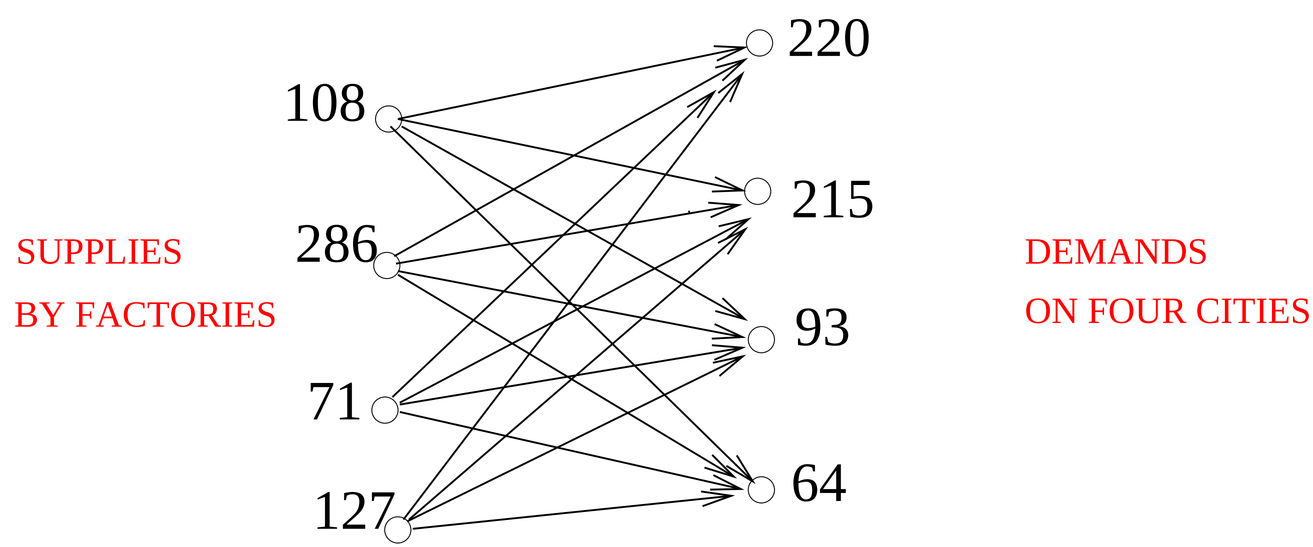



Polytope 1
view from front and top
S
C
B
A
polytope 2
view from front and top
S
B
A
Polytope 1
view from behind and below
S
B
C
A
polytope 2
view from behind and below
S
C
B
A

Problem 2

Graphs of Polyhedra

A Transportation Problems:



We need to transport laptops from factories to consumers. There is a cost C_{ij} associated with transporting one laptop from factory i to city j. We wish to minimizes the total cost.

?	?	?	?	220
?	?	?	?	215
?	?	?	?	93
?	?	?	?	64
108	286	71	127	

The set of all possible solutions are matrices whose row/column sums equal the given supply/demand data.

We can describe it by linear constraints!

The possible tables are non-negative integer solutions of the system of equations: Four equations, one for each row sum and column sum. For example,

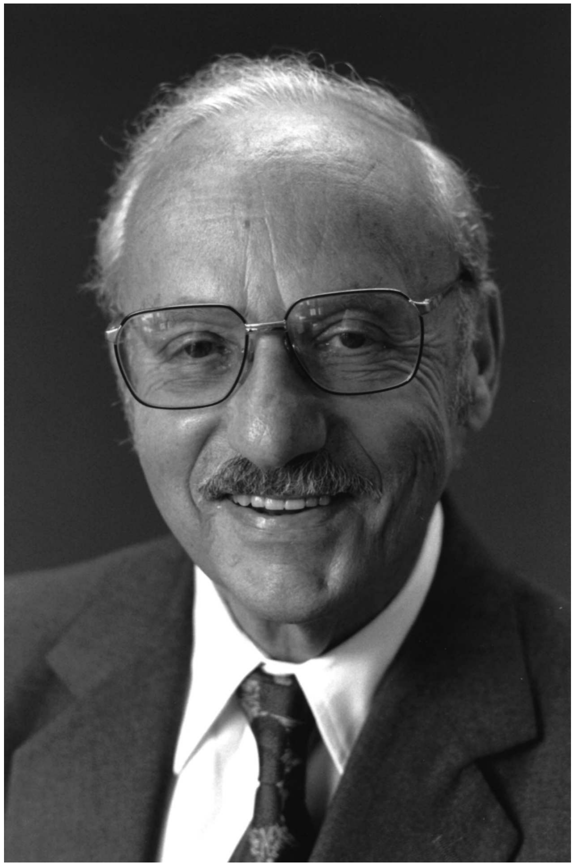
$x_{11} + x_{12} + x_{13} + x_{14} = 220,$ **first row**
$x_{13} + x_{23} + x_{33} + x_{43} = 71,$ **third column**, and of course $x_{ij} \geq 0$

We need a method to solve the Linear Programming Problem

maximize a linear function $\sum\sum c_{i,j}x_{ij}$ subject to the above linear constraints!

The Simplex Method

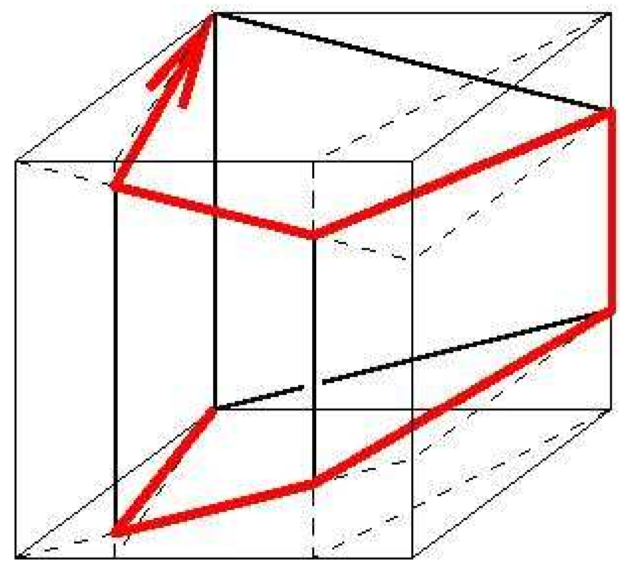
George Dantzig, inventor of the simplex algorithm



The simplex method

- **Lemma:** A vertex of the polytope is always an optimal solution for a linear program. We need to find a special vertex of the polytope!

- The simplex method **walks** along the graph of the polytope, each time moving to a better and better cost!



Hirsch Conjecture

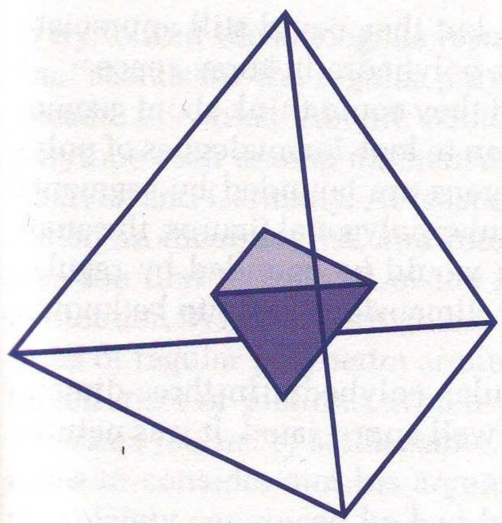
- Performance of the simplex method depends on the diameter of the graph of the polytope: largest distance between any pair of nodes.

- Open Conjecture: The diameter of a polytope P is at most $\#$ of facets$(P) - dim(P)$. It has been open for 40 years now!

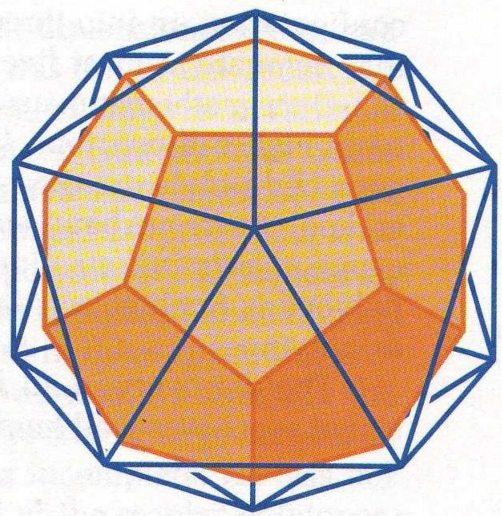
- It is known to be true in many instances, e.g. for polytopes with $0/1$ vertices.

- It is best possible bound for polytopes of dimension 8 or higher. Best known general bound is $\frac{2^{d-2}(n-d+5/2)}{3}$.

Duality



The self-dual tetrahedron.



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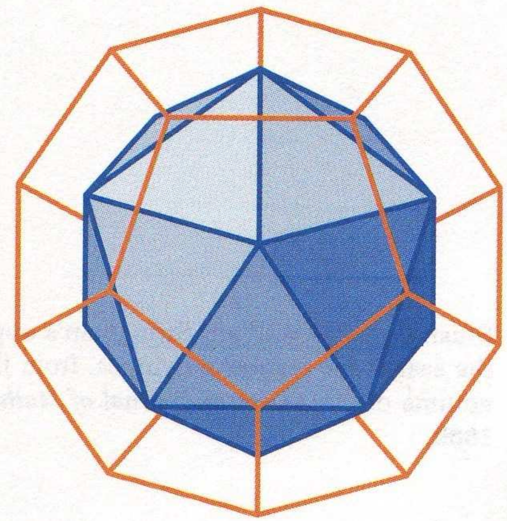
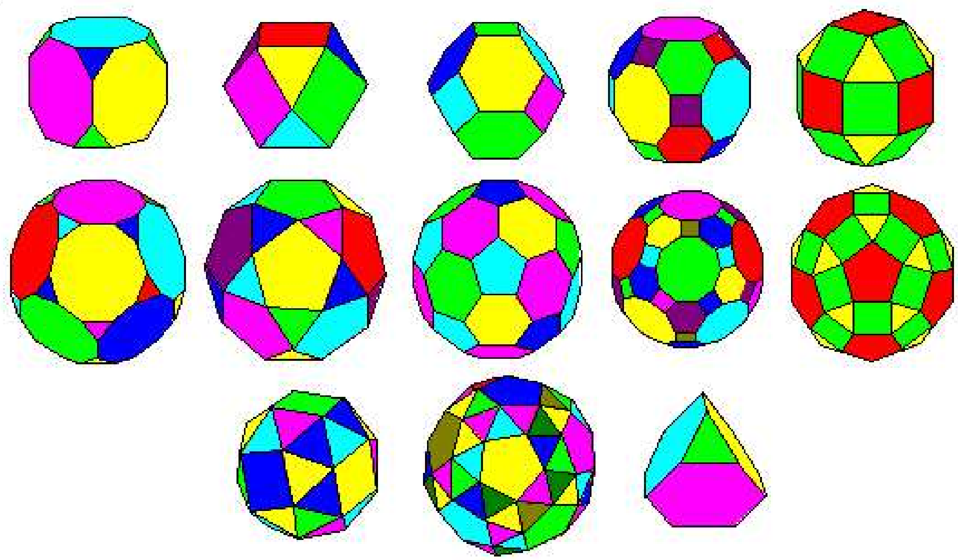


The icosahedron is dual to the dodecahedron.

Problems about faces can also be rephrased as problems about vertices!

Coloring Faces/Vertices

Given a 3-dimensional polyhedron we want to color its faces (or vertices), with the minimum number of colors possible, in such a way that two adjacent elements have different colors.

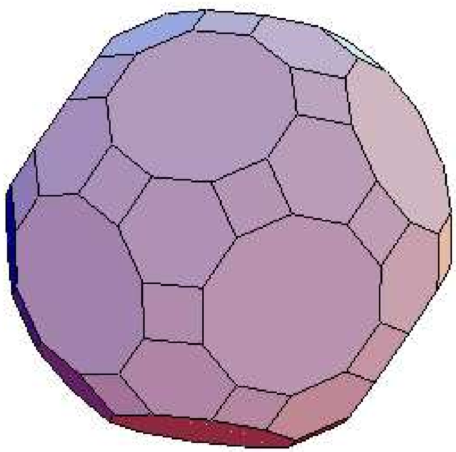


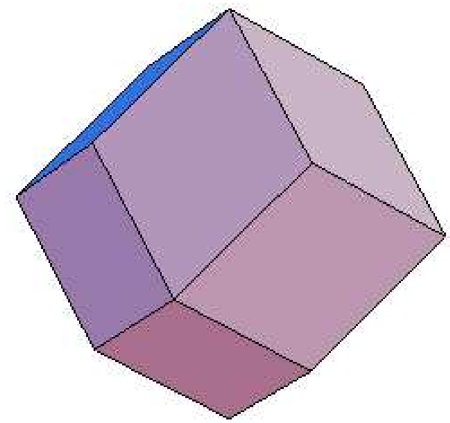
Theorem[The four-color theorem] Four colors always suffice!

Zonotopes

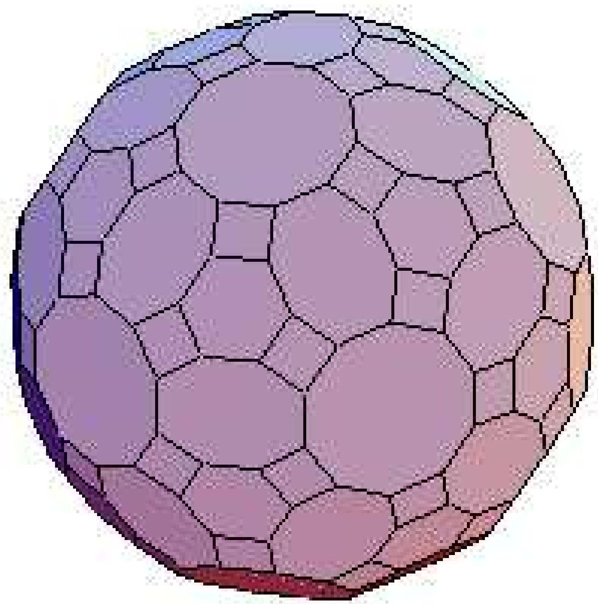
Question: Are there special families of 3-colorable 3-polytopes?

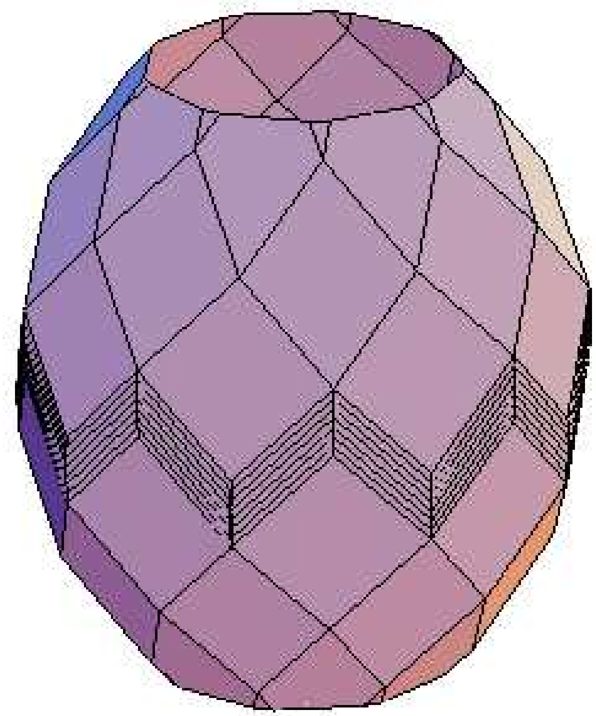
A zonotope is the linear projection of a k-dimensional cube.



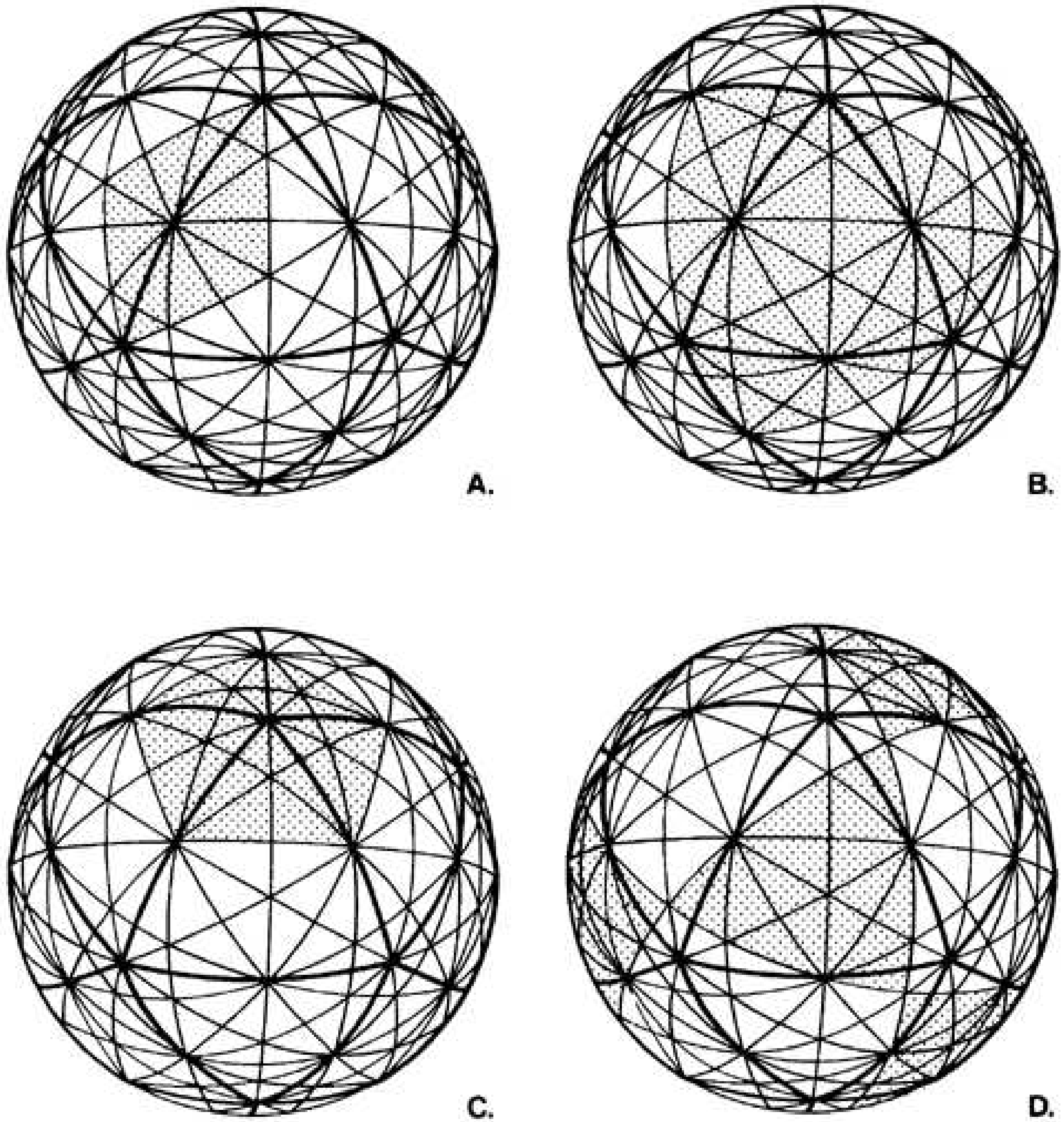


Conjecture The vertices of the graph of 3-zonotopes are 3-colorable.





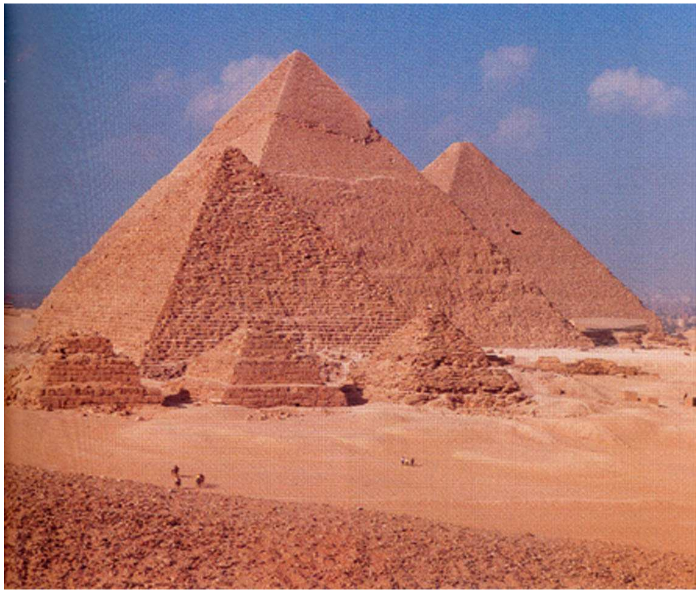
by duality, the same as: The regions of a great-circle arrangement can be colored with 3-colors!


A.
B.
C.
D.

Problem 3

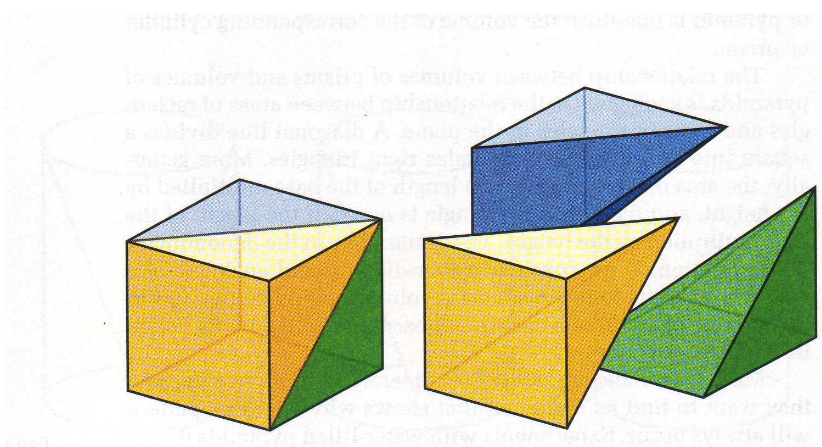
Volume and its relatives

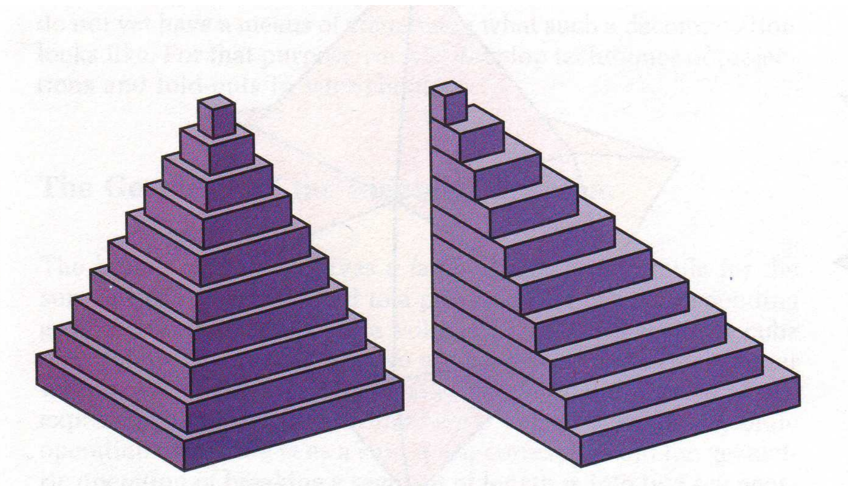
What is the volume of a Polytope?



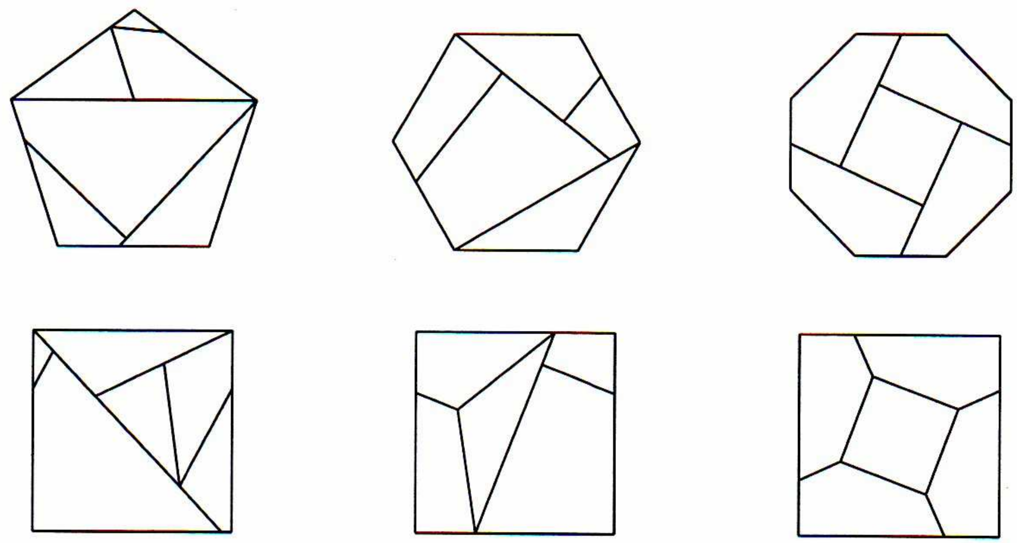
$$\text{volume of pyramid} = \frac{1}{3}(\text{area of base}) \times \text{height}$$

Proofs rely on an infinite process!





Hilbert's third Problem

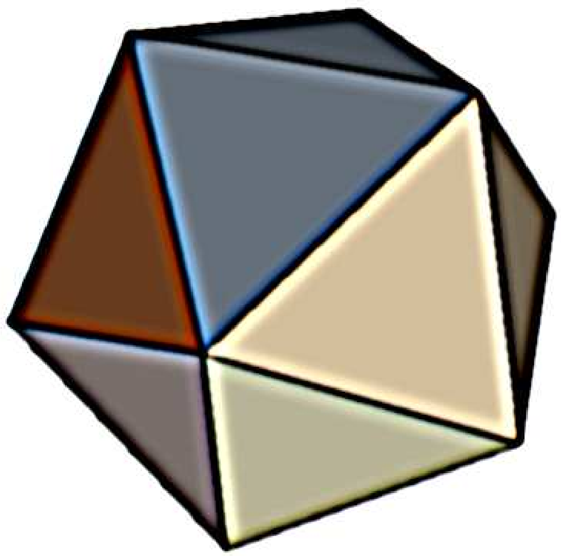


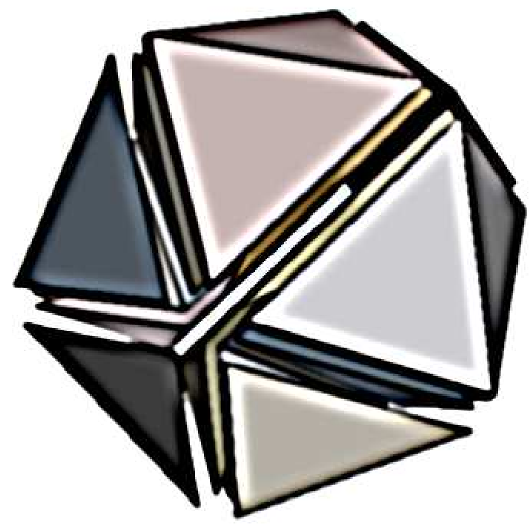
Polygons of the same area are equidecomposable, i.e., one can be partitioned into pieces that can be reassembled into the other.

Are any two convex 3-dimensional polytopes of the same volume equidecomposable? **NO!**

Enough to know how to do it for tetrahedra!

Another method to compute the volume of a polyhedron is to divide it as a disjoint union of simplices. We can triangulate the polyhedron in many different ways.

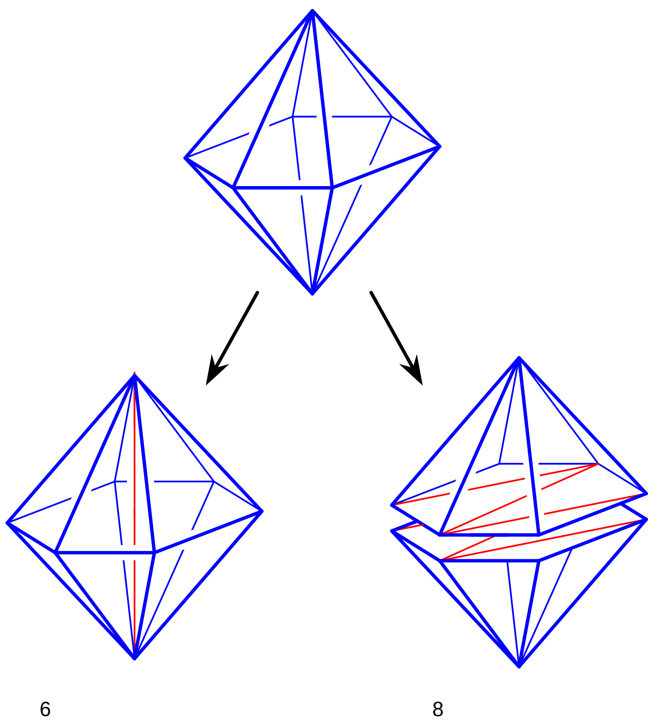


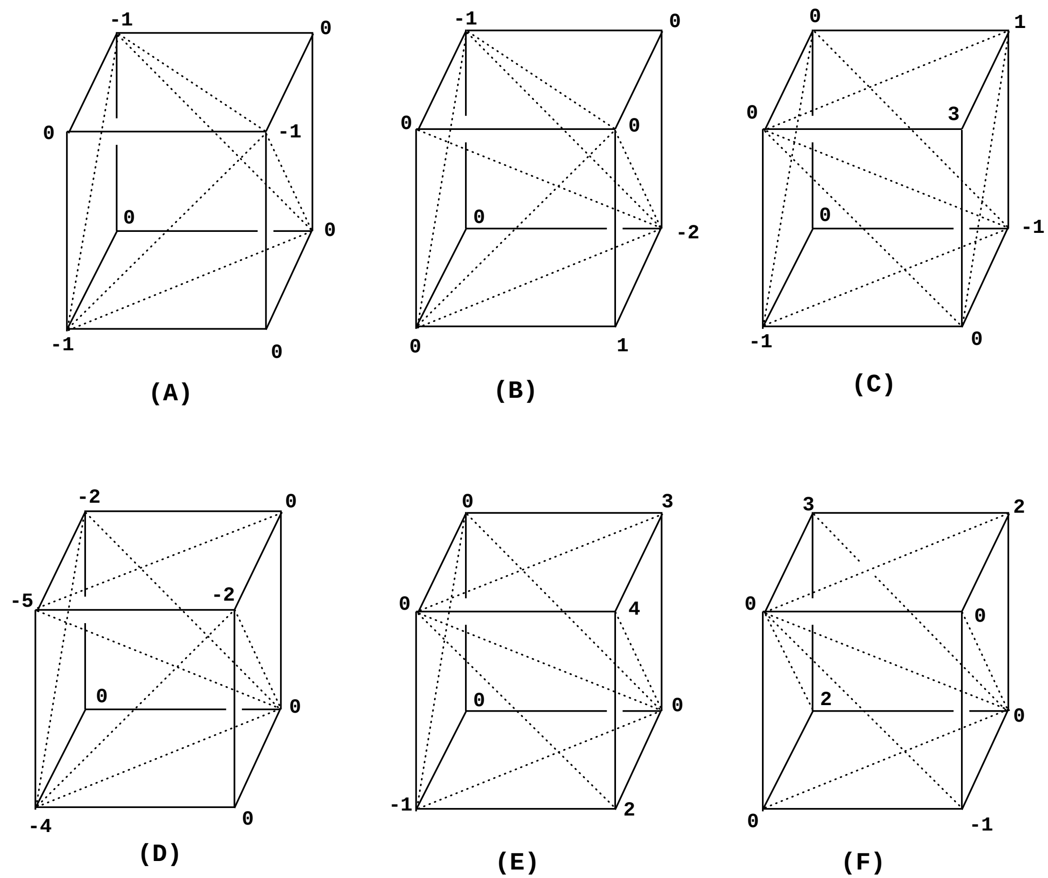


calculate volume for each tetrahedron and then add them up!

The size of a triangulation

There are many different triangulations of a convex polyhedron and they come in different sizes! i.e. the number of tetrahedra changes.





Open Question If for a 3-dimensional polyhedron P we know that there is triangulation of size k_1 and triangulations of size k_2, with $k_2 > k_1$ is there a triangulation of every size k, with $k_1 < k < k_2$?

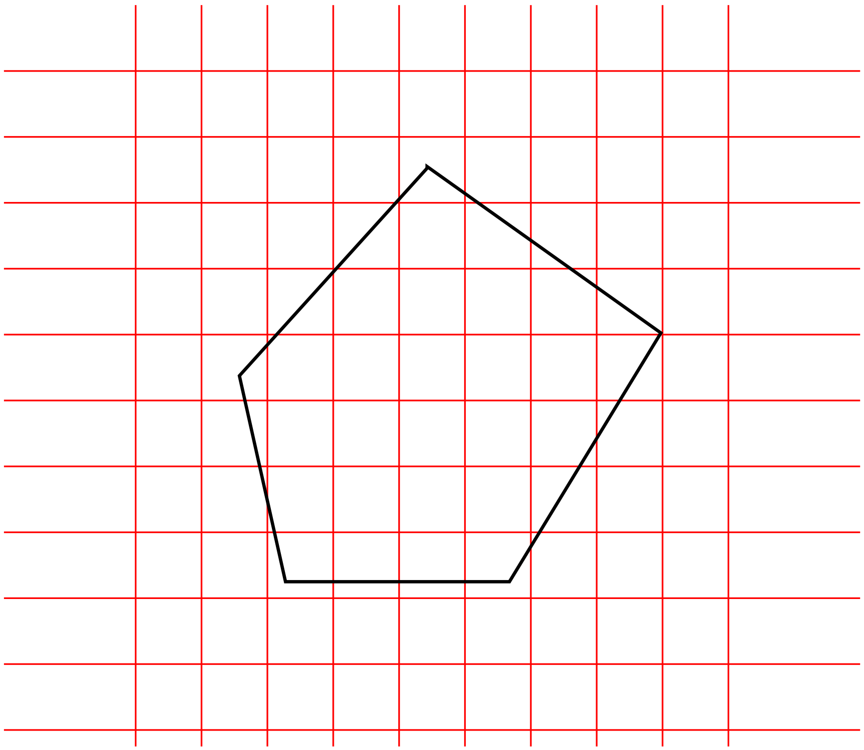
The connectivity of the triangulation

We can define the dual graph of a triangulation, the graph which has one vertex for each tetrahedron and an edge joining two such vertices if the tetrahedra are adjacent:

Open Question Is it true that every 3-dimensional polyhedron has a triangulation whose dual graph is Hamiltonian? In other words, we can visit by a walk all the vertices of the graph once without repeating vertices.

Counting lattice points

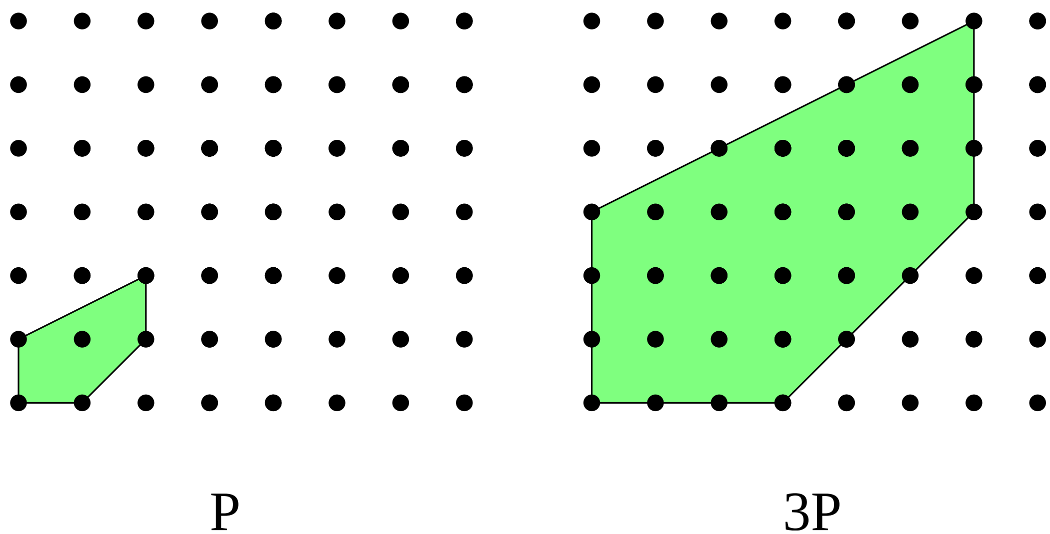
Lattice points are those points with integer coordinates: $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | x_i \text{ integer}\}$ We wish to count how many lie inside a given polytope!



We can approximate the volume!

Let P be a convex polytope in $\mathbb{R}^d$. For each integer $n \geq 1$, let

$$nP = \{nq | q \in P\}$$



Counting function approximates volume

For P a d-polytope, let

$$i(P,n) = \#(nP \cap \mathbb{Z}^d) = \#\{q \in P \mid nq \in \mathbb{Z}^d\}$$

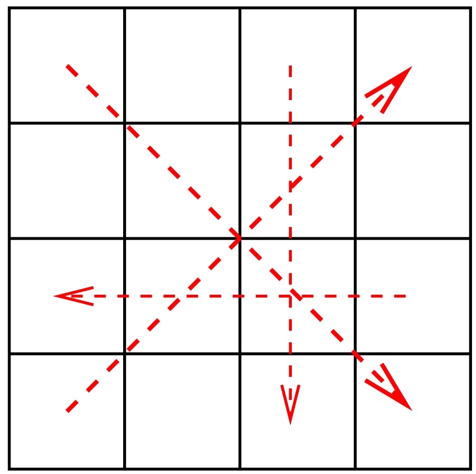
This is the number of lattice points in the dilation nP.

$$\text{Volume of } P = limit_{n\to\infty} \frac{i(P,n)}{n^d}$$

At each dilation we can approximate the volume by placing a small unit cube centered at each lattice point:

Combinatorics via Lattice points

Many objects can be counted as the lattice points in some polytope. For instance, Sudoku configurations, matchings on graphs, and **MAGIC squares:**



12	0	5	7
0	12	7	5
7	5	0	12
5	7	12	0

5

CHALLENGE:**HOW MANY** 4×4 **magic squares with sum n are there?** Same as counting the points with integer coordinates inside the n-th dilation of a "magic square" polytope!

OPEN PROBLEM: Figure out a formula for the volume of $n \times n$ magic squares or, more strongly, for the number of lattice points of each dilation.