Incidence Matrices, Geometrical Bases, Combinatorial Prebases and Matroids

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Introduction.

This paper deals with the notion of a relation $A \subseteq (C \times D)$ between two finite sets C and D. Many different questions can be studied in terms of a relation. In this paper we will consider the following relations: the incidence relation between the "extended circuits" and the bases of a matroid, and the relation of inclusion of chambers in simplices in an affine point configuration. We also consider (Section 4) a class of relations whose incidence matrices satisfy certain conditions (matrices with the T-property).

A relation A can be equivalently presented with the help of its incidence matrix $\hat{A}$. We can then consider two linear spaces: the linear space V_D generated by the columns of $\hat{A}$ and the linear space V_C generated by the rows of $\hat{A}$ over some field. The study of bases in these linear spaces is useful in different combinatorial problems. We can mention, for example, the construction of bases related to the Kostant partition function [4] or further studies of it [1].

From our point of view, the notion of linear dependency is not quite a combinatorial notion: it depends on the field. First, a basis in V_D (or in V_C) may consist not only of elements $d \in D$ but of their linear combinations. If

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we consider a basis of the column matroid of the matrix $\hat{A}$ it will consist of columns (and not of their linear combinations), however, we still need to use linear combinations to express a column of $\hat{A}$ in terms of a basis.

The present paper naturally splits into two parts: Sections 1-4 and Sections 5-7. In Section 1 for a relation $A \subseteq (C \times D)$ and an ordering σ of C we construct a matroid $M(\sigma)$ on the set D and the dual matroid $M^*(\sigma)$. Bases of these matroids are constructed combinatorially and no linear dependencies of the columns of the matrix $\hat{A}$ are used.

Some interesting questions arise: What are the conditions on the relation A so that these matroids $M(\sigma)$ do not depend on the ordering σ? Can we define a unique matroid for a relation using the matroids $M(\sigma)$? These questions will be discussed thoughout Sections 1-4 (see also Conjecture 3.2 in Section 3).

In Section 2 we describe dual matroids $M^*(\sigma)$ (from Section 1) using the notion of a "nil-matrix". This section can be omitted in the first reading.

In Section 3 we consider a matroid $M = (E, B)$ (where B is the set of bases of M) and the incidence relation between, what we call, the extended circuits (see Definition 3.1) of the matroid M and the bases of M. Applying the results from Section 1 to this relation, we obtain the matroids $M(\sigma)$ on the set B of bases of the matroid M. For this relation an ordering σ is an ordering of the set of all extended circuits of the matroid M. We state the following

Conjecture. Let $M = (E, B)$ be a matroid on the set E with the set of bases B and X the set of all extended circuits of M. Let σ be an ordering of X and $M(\sigma) = (B, P_\sigma)$ the matroid constructed (according to Section 1) for the relation $A \subseteq (X \times B)$ and the ordering σ. Let r_σ be the rank of $M(\sigma)$. Denote $\bar{r} = \max_\sigma r_\sigma$ and

$$P = \{p : \ p \in \bigcup_\sigma P_\sigma \text{ and } |p| = \bar{r}\},$$

where $|p|$ is the cardinality of p. Then a pair (B, P) is a matroid.

In Section 4 we consider a relation A whose incidence matrix $\hat{A}$ satisfies the following property (the T-property): $\hat{A}$ does not contain submatrices of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For example, the incidence matrix (considered in Section 3) between the extended circuits and the bases of a matroid M has the T-property. We also introduce (Definition 4.3) a geometric interpretation ("graphoid") of a relation with the T-property. Thus, using the technique from Section 1 we construct the matroid $M^*(\sigma)$ for a graphoid and an ordering σ of its vertices. In a particular case when a graphoid is a graph (without isolated vertices and multiple edges) instead of matroids $M^*(\sigma)$ we obtain a unique matroid which coincides with the usual spanning trees matroid of a graph.

Roughly speaking, in terms of a graphoid the above Conjecture turns into the following: for each graphoid (or a matrix with the T-property) the technique from Section 1 yields a unique matroid.

In Sections 5-6 we again consider an arbitrary relation $A \subseteq (C \times D)$ with the incidence matrix $\hat{A}$. In section 5 we give a definition (Definition 5.2) of a geometrical basis[1] of a relation A with respect to F, where F is a subset of the set of all circuits of the column matroid on $\hat{A}$. Geometrical bases are certain bases of this column matroid.

In section 6 we establish connections (Theorems 6.1 and 6.3) between the bases of matroids $M(\sigma)$ (constructed in Section 1) and geometrical bases of the relation A with respect to F. These Theorems give a combinatorial way of constructing bases of a column matroid on $\hat{A}$ using a subset of its circuits and a combinatorial way to express an element in terms of a basis.

In section 7 we consider the inclusion relation of chambers in simplices in an affine point configuration and prove (Theorem 7.4) that the bases of simplices constructed in [2] are geometrical. In [2] there is also an explicit construction of geometrical bases of chambers. These examples show that the study of geometrical bases might be of independent interest.

The results of this paper can be applied to a covering of a finite set by a system of subsets where subsets may overlap. Indeed, in this case we can naturally construct a relation: consider a finite set D and a covering $c_1, \ldots, c_k$, i.e. $c_i \subset D$ are such that $c_1 \cup \ldots \cup c_k = D$. Let $C = \{c_1, \ldots, c_k\}$. We define the relation $A \subseteq (C \times D)$ as follows: $(d, c_i) \in A$ iff $d \in c_i$.

Examples of coverings with overlapping are: 1) a covering of the convex hull of a finite set of points in an affine point configuration by overlapping

[1] Geometrical bases were introduced in [2] for the inclusion relation of chambers in simplices in an affine point configuration and were called there "combinatorial bases".

simplices[2]; 2) Grothendieck topology; 3) covering of a finite group G by double cosets $g_1 H g_2$, where H is a subgroup of the group G; for example, when G is a Coxeter group and H is a parabolic subgroup.

Since the construction of the matroid $M(\sigma)$ can be held for various relations (for a fixed ordering σ) it seems to us that the question about the conditions on the relation which entail the existence of a unique matroid (independent of an ordering σ) might be a decisive question and may serve as a characterization of the class of relations with good combinatorial properties.

1 Construction of matroids for a relation

Partition $D = \bigcup D_i$. Consider a relation $A \subseteq (C \times D)$, where C and D are two finite sets. Let $\hat{A} = \| a_{c,d} \|$, $c \in C$, $d \in D$ be the incidence matrix of this relation, i.e. $\hat{A}$ is a rectangular $m \times l$ matrix, where

$$a_{c,d} = 1, \quad if \quad (c,d) \in A, \; a_{c,d} = 0, \quad if \quad (c,d) \notin A, \tag{1}$$

and $m = |C|$, $l = |D|$ (where $|C|$ is the cardinality of C).

We will use the same letter D to denote both the set D in a relation $A \subseteq (C \times D)$ and the set of the columns of the matrix $\hat{A}$.

Let us assume that the matrix $\hat{A}$ has no zero-columns. Let $\sigma = (c_1, c_2, \ldots)$ be some fixed ordering of C (i.e. some ordering of the rows of the matrix $\hat{A}$.) Consider the following partition

$$D = \bigcup_{i=0} D_i, \quad D_i \cap D_j = \emptyset, \; for \; i \neq j, \tag{2}$$

where D_i, $i > 0$ is the set of the columns of $\hat{A}$ that have "0" in the first $i-1$ rows and "1" in the i-th row; D_0 is the set of the columns of $\hat{A}$ that have all "0" in the first l rows. (If $m \leq l$, we have $D_0 = \emptyset$ since $\hat{A}$ has no zero-columns.)

Of course, the partition $D = \bigcup D_i$ depends on the ordering σ. Note also that some of the sets D_i can be empty sets.

Matroids $M(\sigma)$ for a relation. Consider the following subset $p \subset D$:

$$p = \bigcup (D_i \setminus d_i), \; where \; d_i \in D_i \tag{3}$$

[2]This relation is considered in Section 7.

Thus, for each choice of $d_i \in D_i$ we obtain a set p. Let us denote by P_σ the set of all these sets p. The sets $p \in P_\sigma$ will be called *combinatorial prebases* of the relation A for the ordering σ of C.

Theorem 1.1 *Let A be a relation with some ordering σ of C. Then the pair (D, P_σ) is a matroid on the set D with the set of bases P_σ. The rank r_σ of this matroid $M(\sigma)$ is equal to*

$$r_\sigma = \sum_{D_i \neq \emptyset} (|D_i| - 1) = |D| - k_\sigma,$$

where k_σ is the number of nonempty subsets D_i in the partition (2).

In general, matroids $M(\sigma)$ constructed for different orderings σ of C are different matroids, even have different rank.

Proof. Let an ordering σ of C be fixed and $D = \bigcup D_i$ be the partition (2) of the set D for the ordering σ. We have to prove that the sets $p \in P_\sigma$ satisfy the exchange axiom for bases of a matroid, i.e.

if $p \neq p'$, $p, p' \in P_\sigma$ and $d' \in (p' \setminus p)$ then there exists $d \in (p \setminus p')$ such that $(p \setminus d) \cup d' \in P_\sigma$.

Let $d' \in (p' \setminus p)$. Due to (2) there exists i such that $d' \in D_i$. Since $d' \in p'$, therefore, there exists a column $d \neq d'$, $d \in D_i$ such that $d \notin p'$ (according to formula (3)). We also have $d \in p$. (Indeed, according to formula (3) there can be only one column from D_i that does not belong to p, but $d' \notin p$). Clearly, the set of columns $(p \setminus d) \cup d'$ satisfies formula (3) and, therefore, $(p \setminus d) \cup d' \in P_\sigma$.

It is also easy to see that $P_\sigma \neq \emptyset$, and thus (D, P_σ) is a matroid. □

Note that we have not actually used that we have a particular partition $D = \bigcup D_i$.

Dual matroids. Recall that if $M = (E, B)$ is a matroid with the set B of bases then the pair (E, B'), where B' is the collection of the sets b' such that $b' = E \setminus b$ for any $b \in B$, is also a matroid; it is called the matroid dual to M.

Let $A \subseteq (C \times D)$ be a relation with an ordering σ of C and $p \in P_\sigma$ be a combinatorial prebasis of the relation A. Consider the set $q = D \setminus p$, where $p \in P_\sigma$ and denote by Q_σ the set of all such q. The following theorem follows immediately from the definition of a dual matroid.

Theorem 1.2 *The pair $M^*(\sigma) = (D, Q_\sigma)$ is a matroid dual to the matroid $M(\sigma) = (D, P_\sigma)$ and has the rank k_σ.*

2 Nil-matrices of an incidence matrix.

2.1 Connection between nil-matrices and combinatorial prebases

A useful technique in the study of the matroids $M(\sigma)$ constructed in Section 1 for a relation $A \subseteq (C \times D)$ is based on the notion of a *nil-matrix*. This connection is established in Propositions 2.2 and 2.3.

Definition 2.1 *A rectangular $m \times l$, $m \geq l$ incidence matrix $\|a_{i,k}\|$ is called a nil-matrix if by permutations of its rows and columns it can be transformed to a matrix such that*

$$a_{i,i} = 1 \text{ and } a_{i,k} = 0 \text{ for } i < k,\ i = 1, \ldots, m \quad (4)$$

It is also convenient to say that a rectangular $m \times l$, $m \geq l$ incidence matrix $\|a_{i,k}\|$ that satisfy the condition (4) has the *N-property*.

Let $\hat{A}$ be an incidence matrix of order $m \times n$ with the set of columns D and the set of rows C. Let us introduce some notations:
$\hat{N}$ is a submatrix of $\hat{A}$ of order $m \times l$, $1 \leq l \leq n$ such that it is a nil-matrix;
N is the set of columns of the matrix $\hat{N}$;
$\mathcal{N}(A)$ is the set of all nil-matrices $\hat{N}$ of the matrix $\hat{A}$;
$r(\hat{N})$ is the rank of $\hat{N}$;
$\bar{r} = \max\ r(\hat{N})$ over all $\hat{N} \in \mathcal{N}(A)$;
$\mathcal{N}_{\bar{r}}(A)$ is the set of all nil-matrices of A for which $r(\hat{N}) = \bar{r}$.

Proposition 2.2 *Let $A \subseteq (C \times D)$ be a relation with the incidence matrix $\hat{A}$. Let σ be an ordering of C such that in the partition (2) $D_0 = \emptyset$. Let $p \in P_\sigma$ be a combinatorial prebasis of this relation , i.e. $p = \bigcup(D_i \setminus d_i)$, where $d_i \in D_i$. Then the submatrix $\hat{R}$ of the matrix $\hat{A}$ consisting of the columns $\{d_i\}$ is a nil-matrix.*

Proof. If $D_i \neq \emptyset$ for $i = 1, \ldots, k$, the matrix $\hat{R}$ has already the N-property. If for some i we have $D_i = \emptyset$, it is easy to see, that by permutations of rows of $\hat{R}$ it can be transformed to a matrix with the N-property. □

The following proposition can be easily proved.

Proposition 2.3 *Let $\hat{A}$ be an incidence matrix of order $m \times n$ with the set of columns D and the set of rows C. Let $\hat{N} \in \mathcal{N}(A)$ be a nil-matrix of order $m \times \bar{r}$, where $\bar{r} = \max r(\hat{N})$ over all $\hat{N} \in \mathcal{N}(A)$. Then there exists an ordering σ of C and a combinatorial prebasis $p \in P_\sigma$ such that $p = D \setminus N$. For the ordering σ the number of nonempty sets D_i in the partition (2) is equal to $\bar{r}$.*

Note that in terms of the dual matroid $M^*_\sigma = (D, Q_\sigma)$ Propositions 2.2 and 2.3 mean the following:

1. Any basis $q \in Q_\sigma$ is a nil-matrix.
2. For any nil-matrix $\hat{N} \in \mathcal{N}_{\bar{r}}(A)$ there exists an ordering σ of C and a basis $q \in Q_\sigma$ such that $q = N$.

Using nil-matrices, the questions about the matroids $M(\sigma)$ which we asked in the introduction, can be reformulated in the following way.

Let $\hat{A}$ be an incidence matrix with the set of rows C and the set of columns D. Is the pair $(D, \mathcal{N}_{\bar{r}}(A))$ a matroid? What are the conditions on the matrix $\hat{A}$ when the pair $(D, \mathcal{N}_{\bar{r}}(A))$ is a matroid? Does in this case the set $\mathcal{N}(A)$ of all nil-matrices of the matrix $\hat{A}$ form the collection of all independent sets of this matroid?

Consider the following example – the matrix $\hat{A}$ with the columns d_1, d_2, d_3, d_4, or simply $1, 2, 3, 4$:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We have

$$\mathcal{N}(A) = \{1,\ 2,\ 3,\ 4,\ (1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4),\ (1,3,4),\ (2,3,4)\},$$

$\bar{r} = 3$ (note that $rank(\hat{A}) = 4$) and $\mathcal{N}_{\bar{r}}(A) = \{(1,3,4),\ (2,3,4)\}$.

One can check that in this example the pair $(D, \mathcal{N}_{\bar{r}}(A))$ is a matroid with the bases $(1,3,4)$ and $(2,3,4)$. However, the set of independent sets of this matroid does not contain the set $(1,2)$, which is a nil-matrix. In other words, the nil-matrix consisting of the columns $(1,2)$ cannot be extended to a nil-matrix of the maximal rank $\bar{r} = 3$. Therefore, even when $(D, \mathcal{N}_{\bar{r}}(A))$ is a matroid, the set $\mathcal{N}(A))$ is not the set of all independent sets of this matroid.

2.2 Some properties of nil-matrices.

Propositions 2.4 and 2.5 describe some useful properties of nil-matrices.

Proposition 2.4 *Let $\hat{N}$ be a nil-matrix and N its set of columns. Let $N' \subset N$. Then the matrix $\hat{N}'$ consisting of the columns N' is a nil-matrix.*

Proof. Obvious.

For example, consider the following nil-matrix and remove its second column.

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \\ \cdots & \cdots & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ * & 0 \\ * & 1 \\ \cdots & \cdots \end{pmatrix}.$$

After permuting the second and the third rows we obtain the matrix with the N-property:

$$\begin{pmatrix} 1 & 0 \\ * & 0 \\ * & 1 \\ \cdots & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ * & 1 \\ * & 0 \\ \cdots & \cdots \end{pmatrix}.$$

Proposition 2.5 *Let $\hat{N}$ be a nil-matrix of order $m \times l$, $m \geq l$ and d an arbitrary vector-column of length m consisting of "0" and "1". Then there exists a column $d' \in N$ such that the matrix consisting of the columns $(N \setminus d') \cup d$ is a nil-matrix.*

Proof. Let $\hat{N}$ be a nil-matrix and d a vector-column consisting of "0" and "1". Since $\hat{N}$ is a nil-matrix then by permutations of its rows and columns it can be transformed to a matrix with the N-property. Let us enumerate the columns of this matrix, i.e. the k-th column has the first "1" in the k-th place (from top to bottom). Let us permute the rows of d together with the rows of $\hat{N}$. Suppose that in the obtained column the first (from top to bottom) "1" is in the k-th row. Two cases are possible: 1) $k \leq l$; 2) $k > l$.

If $k \leq l$, then we exchange the column d with the k-th column of $\hat{N}$, i.e. we choose the k-th column of $\hat{N}$ as d'. Evidently, the matrix with the columns $(N \setminus d') \cup d$ has the N-property and, therefore, is a nil-matrix.

Let $k > l$. Then we exchange the column d with the last column of $\hat{N}$, i.e. we choose the l-th column of $\hat{N}$ as d'. After permuting the l-th and the k-th rows in the matrix with the columns $(N \setminus d') \cup d$ we obtain the matrix with the N-property. □

Proposition 2.5 gives the illusion that if $\hat{A}$ is an incidence matrix and $\mathcal{N}_{\bar{r}}(A)$ is the set of all its nil-matrices of order $m \times \bar{r}$, (where $\bar{r} = \max r(\hat{N})$ over all $\hat{N} \in \mathcal{N}(A)$) then $\mathcal{N}_{\bar{r}}(A)$ satisfies the exchange axiom for bases of a matroid. However, Proposition 2.5 differs from the exchange axiom for bases of a matroid in the following way. Indeed, let $\hat{N}, \hat{N}' \in \mathcal{N}_{\bar{r}}(A)$ and $d \in N' \setminus N$. Then by Proposition 2.5 there exists a column $d' \in N$ such that the matrix consisting of the columns $(N \setminus d') \cup d$ is a nil-matrix. However, we have not required that $d' \in N \setminus N'$.

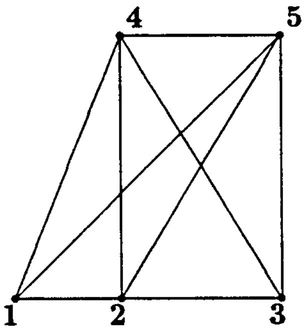
3 Relation between bases and extended circuits of a matroid

Let $M = (E, B)$ be a matroid of rank r on the set E with the set B of bases. Let C be the set of all circuits of M.

Definition 3.1 *A subset $x \subset E$ is an extended circuit of a matroid M if there exists a circuit $c \in C$ such that $x \supseteq c$ and for any $e \in c$, $(x \setminus e) \in B$.*

Denote by X the set of all extended circuits of a matroid M. For any $x \in X$ we have $|x| = r + 1$.

Example. Let $M = (E, C)$ be the matroid below, i.e. $E = \{1, \ldots, 5\}$, $C = \{123, 1245, 1345, 2345\}$.



Then the set of all extended circuits of M is $X = \{1234, 1235, 1245, 1345, 2345\}$.

The relation $A \subseteq (X \times B)$ is defined as follows: $(x, b) \in A$, iff $b \subset x$ for $x \in X$ and $b \in B$. The incidence matrix $\hat{A} = ||a_{x,b}||$ takes the form

$$a_{x,b} = 1, \; if \; b \subset x, \; and \; a_{x,b} = 0, \; if \; b \not\subset x \tag{5}$$

For the relation $A \subseteq (X \times B)$ with a fixed ordering σ of X we can construct (according to Section 1) the matroid $M(\sigma) = (B, P_\sigma)$ on the set B of bases of the matroid M. Let r_σ be the rank of $M(\sigma)$ and $\bar{r} = \max_\sigma r_\sigma$. We have the following

Conjecure 3.2 *The pair (B, P), where*

$$P = \{p : \; p \in \bigcup_\sigma P_\sigma \; and \; |p| = r\}$$

is a matroid.

In the interesting paper [3] there is a definition of a matroid on the set of bases of the initial matroid. Connection between this matroid and the matroids $M(\sigma)$ from Section 3 will be studied separately.

4 Matroids for a relation with the T-property

4.1 Matrices with the T-property and graphoids

In section 3 we have considered a relation between the extended circuits and the bases of a matroid M. Such relation satisfy the following property.

Definition 4.1 *We will say that an incidence matrix $\hat{A} = \|a_{i,k}\|$ has the T-property if it does not have any second order submatrix*

$$\begin{pmatrix} a_{ik} & a_{ik'} \\ a_{i'k} & a_{i'k'} \end{pmatrix}$$

that has the form

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Proposition 4.2 *Let M be a matroid on E with the set B of bases and X the set of all extended circuits of M. Let $\hat{A} = \|a_{x,b}\|$, $x \in X$, $b \in B$ be the incidence matrix (5) for the matroid M. The matrix $\hat{A}$ has the T-property.*

Proof. We will show that the matrix $\hat{A}$ cannot have a second order submatrix with all the elements equal to "1". Suppose that the opposite is true, i.e. there are two extended circuits $x, x' \in X$, $x \neq x'$ and two bases $b, b' \in B$, $b \neq b'$ such that $b \subset x$, $b' \subset x$ and $b \subset x'$, $b' \subset x'$. Since $b \subset x$ and $b' \subset x$, then $b \cup b' \subset x$. Note that since $|x| = r + 1$ and $b \neq b'$ then the set $b \cup b'$ has at least $r + 1$ elements. Therefore, $x = b \cup b'$. Similarly, we can obtain that $x' = b \cup b'$. Thus, $x = x'$. We have obtained a contradiction. □

Consider the following geometrical object related to the matrix with the T-property:

Definition 4.3 *Let V be a finite set (a set of vertices) and F a set of subsets $f \subset V$. A pair (V, F) is called a graphoid if the following conditions are satisfied:*

1) $| f \cap f' | \leq 1$ for any $f, f' \in F$ (T-property)

2) $\bigcup_{f \in F} F = V$

3) $| f | > 1$ for any $f \in F$.

Let (V, F) be a graphoid with the set of vertices V. Any subset $f \in F$ can be considered as a k-dimensional simplex (a *k-monade*), where $k = |f| - 1$. The T-property means that any two monades from F cannot have an edge in common. Thus, graphoid is a special case of a simplicial complex.

Proposition 4.4 *There is a one-to-one correspondence between graphoids and* $0-1$ *matrices* $\| a_{c,d} \|$, $c \in C,\ d \in D$ *with the following properties:*

1) T*-property;*

2) $\sum_d a_{c,d} > 0$, *for any* $c \in C$

3) $\sum_c a_{c,d} > 1$, *for any* $d \in D$.

Proof. Consider a $0-1$ matrix $\| a_{c,d} \|$, $c \in C,\ d \in D$. Let us correspond to it the following graphoid: to each row $c \in C$ we correspond a vertex v and to each column $d \in D$ we correspond the set $f_d = \{c : a_{c,d} = 1\}$. It is easy to see that the properties 1)-3) of the matrix $\| a_{c,d} \|$ are equivalent to the properties 1)-3) of a graphoid. □

Graphoid generalizes a graph without isolated vertices and multiple edges. Indeed, if $|f| = 2$ for any $f \in F$, then (V, F) is a graph. (In this case the corresponding T-matrix has no more than two "1" in each column.)

4.2 Matroids on a graphoid

In Section 1 we have constructed matroids $M(\sigma)$ for an incidence matrix. In Subsection 4.1 we have defined a graphoid or equivalently, a $0-1$ matrix with the T-property (see Proposition 4.4). Thus we can study matroids $M(\sigma)$ on a graphoid. Below is the reformulation of the construction from Section 1 of the matroids $M(\sigma)$ in graphoid terms[3].

Let $G = (V, F)$ be a graphoid and $\sigma = (v_{i_1}, v_{i_2}, \ldots)$ be an ordering of V. Denote $F_1 = \{f : v_{i_1} \in f\}$,

$$F_k = \{f : v_{i_k} \in f \text{ and } f \notin F_1 \cup \ldots \cup F_{k-1}\} \tag{6}$$

Consider the following subset of monades $t \subset F$

$$t = \{(F_1 \setminus f_1) \cup (F_2 \setminus f_2) \cup \ldots\}, \text{ where } f_k \in F_k \tag{7}$$

By varying $f_k \in F_k$ we obtain different sets t. Let T_σ be the set of all t constructed for a fixed ordering σ of V.

Similarly, consider a set $t^* \subset F$

$$t^* = \{f_1, \ldots, f_k\}, \text{ where } f_k \in F_k \tag{8}$$

[3]Note that this construction is applied here to any graphoid and not neccessarily to a graphoid that was obtained from the incidence matrix $\| a_{x,b} \|$ of a matroid M.

and denote by T^*_σ the set of all such t^* for a fixed ordering σ.

Let $\hat{A}$ be the incidence matrix of the relation $A \subseteq (V \times F)$. Let $M(\sigma) = (F, P_\sigma)$ and $M^*(\sigma) = (F, Q_\sigma)$ be the matroids constructed (according to Section 1) for the matrix $\hat{A}$ with the ordering σ of V. It is easy to check that the following proposition holds.

Proposition 4.5 *There is a one-to-one correspondence between the set P_σ and the set T_σ. There is a one-to-one correspondence between the set Q_σ and the set T^*_σ.*

4.3 Graph matroids as a special case of matroids M_σ

In this Subsection we continue to use the notations from Subsection 4.2.

Proposition 4.6 *Let $G = (V, F)$ be a graph without multiple edges and isolated vertices. Let $\hat{A}$ be the incidence matrix between vertices $v \in V$ and edges $f \in F$.*

1. Let $\sigma = (v_{i_1}, v_{i_2}, \dots)$ be an ordering of V. Let $t^* = \{f_1, \dots, f_k\}$ be a set (8) for the ordering σ. The set t^* is a tree (not neccessarily connected).

2. For any spanning tree $\tilde{t}$ of the graph G there exists an ordering σ of V such that $\tilde{t} \in T^*_\sigma$.

Proof. 1. Consider $t^* \in T_\sigma$. Suppose that the opposite statement is true, i.e. $t^* = \{f_1, \dots, f_k\}$ contains a cycle $s \subseteq t^*$. This means that any vertex of the cycle s belongs to at least two edges of the cycle. Note that for any edge $f_k \in t^*$ we have the corresponding (distinguished) vertex v_{i_k}. Among all the edges of the cycle s consider an edge f_j such that the corresponding vertex v_{i_j} has the least number. Then according to formula (6) any F_k, where $k > i_j$, does not contain any edge that has the vertex v_{i_j}. Therefore, $s \subseteq t^*$ does not contain any edge, other than f_j, that has the vertex v_{i_j}. Therefore, the vertex v_{i_j} belongs to only one of the edges of s. The obtained contradiction completes the proof.

2. Suppose $\tilde{t}$ is a spanning tree of the graph G. Note that a spanning tree of a graph G (without isolated vertices) contains all its vertices.

Let us construct the following ordering of V. Consider an "end branch" of the tree $\tilde{t}$, and denote it by f_1. Let v_{i_1} be the vertex of f_1 such that it belongs to only one edge (i.e. v_{i_1} is an end vertex), we have $v_{i_1} \in f_1$. Consider now the subtree $\tilde{t} \setminus f_1$ and let f_2 be its "end branch". We choose an end vertex of f_2 and denote it v_{i_2}. We have $v_{i_2} \in f_2$. By continuing the process we obtain the ordering σ of all $v \in V$ (since $\tilde{t}$ is a spanning tree.) We have also obtained the ordering $(f_1, f_2, \ldots)$ of edges of $\tilde{t}$ and we have $v_{i_k} \in f_k$.

For this ordering σ of V we define F_k according to formula (6). We need to prove that $\tilde{t} \in T^*_\sigma$, i.e. $f_k \in F_k$, where $f_k \in \tilde{t}$. Clearly, $f_1 \in F_1$. Consider f_2. We have $v_{i_2} \in f_2$. Let us show that $f_2 \notin F_1$. Suppose the opposite is true, i.e. $f_2 \in F_1$. This means that there are two edges $f_1,\ f_2 \in \tilde{t}$ such that $v_{i_1} \in f_1,\ v_{i_1} \in f_2$. But this contradicts with the condition that v_{i_1} is the "end vertex" of the tree $\tilde{t}$. Thus, $f_2 \notin F_1$, and we have $f_2 \in F_2$. Similarly, since $\tilde{t}$ is a tree we can prove that $f_k \notin (F_1 \cup \ldots \cup F_{k-1})$ and $f_k \in F_k$. □

Let $G = (V, E)$ be a graph without multiple edges and isolated vertices. Let $M = (E, B)$ be the usual matroid of this graph, i.e. B is the set of all spanning trees on G. From Proposition 4.6 follows Theorem 4.7.

Let σ be an ordering of V. Let $r_\sigma = |t^*|$, where $t^* \in T^*_\sigma$ is defined by formulas (6), (8). Denote $\bar{r} = \max_\sigma r_\sigma$. Denote also

$$T_{\bar{r}} = \{t^* : t^* \in \bigcup_\sigma T^*_\sigma \text{ and } |t^*| = \bar{r}\}.$$

Theorem 4.7 *A pair $(E, T_{\bar{r}})$ is a matroid and it coincides with the matroid $M = (E, B)$ of the graph $G = (V, E)$.*

5 Geometrical bases of a relation.

In Sections 5, 6 we consider again an arbitrary relation $A \subseteq (C \times D)$. Let $\hat{A}$ be its incidence matrix. Let V_D be the linear space generated by the columns $d \in D$ over a field of characteristic 0. Recall that a subset $f \subset D$, $|f| > 1$ is a circuit if all the columns $d \in f$ of the matrix $\hat{A}$ are linearly dependent in V_D but any proper subset of f is a linearly independent set. Denote by F_0 the set of all circuits f (i.e. all the circuits of the column matroid[4] on $\hat{A}$).

In this Section we will define a geometrical basis of the relation A with respect to F, where $F \subset F_0$. Sometimes in applications only a certain subset F of circuits is known (from "geometric reasons") nevertheless the bases has

[4]The column matroid on $\hat{A}$ is a matroid defined by the columns of $\hat{A}$ considered as vectors in V_D.

to be constructed. It seems to us that the notion of a geometrical basis might be helpful in constructing bases using a subset of circuits and also as a way of expressing combinatorially an element (a column $d \in D$) in terms of a basis (see examples in Section 7).

Remark. Note that if $F = F_0$ then any basis of the column matroid on $\hat{A}$ is geometrical with respect to F.

Definition 5.1 *Let $b \subset D$. We say that an element $d \notin b$ is expressed in one step "in terms of the set b using F" if there exists $f \in F$ such that $d \in f$, and $f \setminus d \subseteq b$.*

We say that an element $d \notin b$ can be expressed in k steps in terms of the set b using F if there exists a sequence $d_1, \ldots, d_k$; $d_i \in D$ such that $d_k = d$ and d_1 is expressed in one step in terms of the set b (using F), d_2 is expressed in one step in terms of the set $b \cup d_1$, etc. Finally, d_k is expressed in one step in terms of the set $b \cup d_1 \ldots \cup d_{k-1}$.

Definition 5.2 *A geometrical basis b of a relation A with respect to F is a subset $b \subset D$ that satisfies the following conditions:*

1) b is a basis in V_D;

2) For any $d \in D$, $d \notin b$ there exists k such that d can be expressed in k steps in terms of b using F.

Partition $D = K_0 \cup \ldots K_n$. The notion of a geometrical basis leads to the following partition of the set D. Let b be a geometrical basis with respect to F. We define:

$K_0 = b$

$K_1 = \{d : d$ is expressed in one step but not in zero steps, in terms of K_0 using $F\}$

...

$K_i = \{d : d$ is expressed in not less than i steps in terms of $K_0 \cup K_1 \ldots \cup K_{i-1}$ using $F\}$

Since B is a geometrical basis, there exists n such that

$$D = K_0 \cup \ldots \cup K_n, \;\; K_i \cup K_j = \emptyset, \;\; i \neq j. \tag{9}$$

6 Connection between geometrical bases and combinatorial prebases of a relation.

Let $A \subseteq (C \times D)$ be a relation with the incidence matrix $\hat{A}$ and $rank(\hat{A}) = r$. Let V_D be the linear space generated by the elements $d \in D$ over some field of characteristic 0. Let F_0 be the set of all circuits f on $\hat{A}$ (see Section 5). Let $F \subset F_0$. Consider the following relation $U \subseteq (F \times D)$, where $f \in F$, $d \in D$, and

$$(f, d) \in U, \text{ iff } d \in f \tag{10}$$

Denote also by $\hat{U}$ the incidence matrix of this relation.

Theorem 6.1 and Theorem 6.3 establish connections between geometrical bases of the relation $A \subseteq (C \times D)$ with respect to $F \subset F_0$ and combinatorial prebases (see Section 1) of the relation U for an ordering σ of F.

Let σ be an ordering of F. Denote by $M(\sigma)$ the matroid constructed (according to Section 1) for the relation $U = (F \times D)$ and the ordering σ. Let P_σ be the set of bases of $M(\sigma)$ (i.e. the set of combinatorial prebases).

Theorem 6.1 *1. Let b be a geometrical basis of the relation A with respect to F such that for any $d \in b$ there exists $f \in F$ such that $d \in f$.* [5] *Then there exists an ordering σ of F such that b is a combinatorial prebasis of the relation U for the ordering σ, i.e. $b \in P_\sigma$.*

2. For the ordering σ defined in 1. any combinatorial prebasis $p \in P_\sigma$ of the relation U is a geometrical basis of the relation A with respect to F.

Proof. 1. Let b be a geometrical basis of the relation A with respect to F. In order to construct the ordering σ of F let us first rearrange the elements $d \in D$ in the following way. Consider the partition (9), i.e.

$D = K_0 \cup \ldots \cup K_n$, where $K_0 = b$.

First, we write all the elements $d \in D$ that correspond to $d \in K_0$, with an arbitrary order within K_0, then all the elements $d \in K_1$ (with an arbitrary order within K_1), etc.

[5] We assume here that there are no zero columns in the matrix $\hat{U}$. A zero column d_0 in the matrix $\hat{U}$ can appear in two cases: 1) d_0 is a zero column in the matrix $\hat{A}$ and naturally such element d_0 has to be disregarded; or 2) d_0 is a basis element such that every $d \in D, d \neq d_0$ can be expressed in terms of $b \setminus d_0$. In this case the element d_0 has to be added to the combinatorial prebasis b.

By the definition of K_i, for any $d \in K_i$ there is an element $f \in F$ (not necessarily unique) such that 1) $d \in f$; 2) $f \setminus d \subseteq K_0 \cup \ldots \cup K_{i-1}$. Let us assign one such f to each $d \in K_1 \cup \ldots \cup K_n$. We have obtained an ordering of some $f \in F$ and this ordering corresponds to the ordering of $K_1, K_2, \ldots, K_n$. Let all the rest of $f \in F$ follow in an arbitrary order. The obtained ordering of F we define as σ.

For the ordering σ of F we will construct a combinatorial prebasis $p \in P_\sigma$ and will show that $b =: p$. Let $D = D_1 \cup \ldots \cup D_k$ be the partition of D defined by formula (2). We have $D_0 = \emptyset$ and $D_i \neq \emptyset$ for any $i = 1, \ldots, k$. In order to construct p we need to choose $d_i \in D_i$ for $i = 1, \ldots, k$. Note that from the construction of the ordering σ it follows that:

1) each element $d \in K_1 \cup \ldots \cup K_n$ belongs to one and only one set D_i;

2) if $d, d' \in K_1 \cup \ldots \cup K_n$ and $d \in D_i$ then $d' \in D_j$, where $i \neq j$;

3) $k = |K_1| + \ldots + |K_n|$.

This implies that any set D_i, $i = 1, \ldots, k$ contains one and only one element $d \in K_1 \cup \ldots \cup K_n$. Then we can denote by $\tilde{d}_1, \ldots, \tilde{d}_k$, all the elements from $K_1 \cup \ldots \cup K_n$, where $\tilde{d}_i \in D_i$. We define $p = \cup_{i=1}^{k}(D_i \setminus \tilde{d}_i)$. Clearly, $p \in P_\sigma$.

Let us show that $p = b$. Indeed, if $d \in p$, then $d \in D \setminus (K_1 \cup \ldots \cup K_n) = K_0 = b$. If $d \in b = K_0$ then since $D_0 = \emptyset$ we have $d \in D_1 \cup \ldots \cup D_k$. But $d \notin K_1 \cup \ldots \cup K_n$, therefore, $d \in \cup_{i=1}^{k}(D_i \setminus \tilde{d}_i) = p$. Thus, $b \in P_\sigma$.

2. Let us prove that for the above ordering σ of F any combinatorial prebasis $p \in P_\sigma$ of the relation $U \subseteq (F \times D)$ is a geometrical basis of the relation $A \subseteq (C \times D)$ with respect to F. Let $p \in P_\sigma$, $p \neq b$. Since $b, p \in P_\sigma$ and b is a basis in V_D, we have $|p| = r$. Therefore, we only need to prove that for any $d \in D$ there exists l such that d is expressed in l steps in terms of p using F.

Let $\sigma = (f_1, f_2, \ldots)$ be the ordering defined above. Consider the partition (2). We have $D_0 = \emptyset$ and $D_i \neq \emptyset$ for any $i = 1, \ldots, k$ (since b is a geometrical basis). Let $d \notin p$. Then $d \in D_i$ for some i.

First, consider the case when $d \in D_1$. According to (3) for any $d' \neq d$, $d' \in D_1$ we have $d' \in p$. This means that d is expressed in one step in terms of p using f_1 , i.e. $d \in K_1$.

Let $d \in D_2$. Then using f_2 the element d can be expressed in terms of $p \cup K_1$, i.e. since $d \notin p$ we have $d \in K_1 \cup K_2$.

Finally, if $d \in D_k$ we obtain that $d \in K_1 \cup \ldots \cup K_k$.

Thus, we have proved that for any $d \notin p$ there exists i such that d is expressed in i steps in terms of p using F, i.e. $p \in P_\sigma$, $p \neq b$ is a geometrical basis of the relation A with respect to F. □

Remark. If we know one geometrical basis of a relation A with respect to F then Theorem 6.1 gives the way of constructing more geometrical bases of this relation (with respect to F).

We will now describe (Theorem 6.3) some conditions when a combinatorial prebasis of the relation $U \subseteq (F \times D)$ with an ordering σ of F is a geometrical basis of the relation $A \subseteq (C \times D)$ with respect to F.

Let $A \subseteq (C \times D)$ be a relation with the incidence matrix $\hat{A}$ and $r = rank(\hat{A})$. Let F_0 be the set of all circuits f on $\hat{A}$ and $F \subset F_0$. For the relation U defined by formula (10) and an ordering σ of F let us consider the partition (2). Let k_σ be the number of nonempty sets D_i in this partition and r_σ the rank of the matroid $M(\sigma)$ (see Theorem 1.1).

Proposition 6.2 *Let σ be an ordering of F such that in the partition (2) $D_0 = \emptyset$. Then $r \leq r_\sigma$.*

Proof. Let $\sigma = (f_1, f_2, \ldots)$ be an ordering of F such that $D_0 = \emptyset$. Let $p \in P_\sigma$ (see formula (3).) We need to prove that $r \leq |p|$. Since $D_0 = \emptyset$, we have $D = p \cup (d_1 \cup \ldots \cup d_{k_\sigma})$.

Let us prove that any element $d_1, \ldots, d_{k_\sigma}$ can be expressed in terms of p using F. Consider d_1. We have $d_1 \in f_1$, and for any $d \in f_1$, $d \neq d_1$ we have $d \in p$, i.e. d_1 is expressed in one step in terms of p using f_1. Consider d_2. We have $d_2 \in f_2$. Any other element $d \neq d_2$ from f_2 either belongs to p or is equal to d_1. But d_1 is expressed in terms of p using F, therefore, d_2 can be expressed in terms of p using F. Similarly, we obtain that any element d_i, $i = 1, \ldots, k_\sigma$ is expressed in terms of p using F.

Thus, if the ordering σ is such that $D_0 = \emptyset$ then any element $d \in D$ is expressed in terms of p using F. Therefore $r \leq |p| = r_\sigma$. □

From Proposition 6.2 it is clear that a combinatorial prebasis $p \in P_\sigma$ can be a geometrical basis of A with respect to F only if the ordering σ of F is such that $r = |D| - k_\sigma$ (see Theorem 1.1).

Theorem 6.3 *Let $A \subseteq (C \times D)$ be a relation with the incidence matrix $\hat{A}$ and $rank(\hat{A}) = r$. Let $U \subseteq (F \times D)$ be the relation defined by formula (10).*

If there exists an ordering σ of F such that in the partition (2) for the relation U:

1) $D_0 = \emptyset$,

2) $k_\sigma = |D| - r$,

then any combinatorial prebasis $p \in P_\sigma$ defined by (3) for the relation U is a geometrical basis of the relation A with respect to F.

Proof. Let $\sigma = (f_1, f_2, \ldots)$ be an ordering satisfying the above conditions. We have $D_0 = \emptyset$ and $D_1 \neq \emptyset, \ldots, D_{k_\sigma} \neq \emptyset$. Let $p \in P_\sigma$, then $D = p \cup (d_1 \cup \ldots \cup d_{k_\sigma})$ (according to (3)). Then any element $d \in D$ can be expressed in terms of p using F (see proof of Proposition 6.2). Since $|p| = |D| - k_\sigma = r$ we conclude that p is a geometrical basis of the relation A with respect to F. □

7 Examples of geometrical bases.

7.1 Geometrical bases of simplices.

Let $E = (e_1, e_2, \ldots, e_N)$, $N > n$ be a finite set of points in an n-dimensional affine space V. Let $P = conv(E)$ be the convex hull of E. Let $\sigma = \sigma(e_{i_1}, \ldots, e_{i_{n+1}})$ be an n-dimensional simplex with the vertices $e_{i_1}, \ldots, e_{i_{n+1}} \in E$. Denote by Σ the set of all such simplices σ. All the simplices σ (as a rule overlapping) cover the polytope P. The simplices σ divide the polytope P into a finite number of chambers γ (see Definition 7.1). Denote by Γ the set of all chambers γ in P.

Definition 7.1 *Let $\sigma \in \Sigma$ and $\tilde{\sigma}$ be the boundary of σ. Let $\tilde{\Sigma} = \bigcup_{\sigma\in\Sigma} \tilde{\sigma}$ and $\bar{P} = P \setminus \tilde{\Sigma}$. Let $\bar{\gamma}$ be a connected component of $\bar{P}$ and γ be closure of $\bar{\gamma}$. We call γ a chamber and $\bar{\gamma}$ an open chamber.*

Let $A \subseteq (\Sigma \times \Gamma)$ be the inclusion relation of chambers γ in simplices σ, i.e.

$$(\sigma, \gamma) \in A \text{ iff } \gamma \subset \sigma.$$

Let $\hat{A}$ be the incidence matrix of A. Consider the linear space V_Σ generated by the rows of $\hat{A}$ over some field of characteristic 0. In Section 7.1 we will prove (Theorem 7.4) that the bases in V_Σ constructed in [2] are geometrical with respect to F defined in [1]. Let us show first (Theorem 7.3) that the linear relations $f \in F$ are circuits of the row matroid on $\hat{A}$.

Consider a subset $S \subseteq E$ consisting of $n+2$ points and such that S contains at least $n+1$ points in general position. Denote

$$f = \{\sigma : \sigma(e_{i_1}, \ldots, e_{i_{n+1}}) \in \Sigma \text{ and } e_{i_k} \in S\} \tag{11}$$

Thus, with each such S we associate a subset $f \subset \Sigma$ (clearly, $f \neq \emptyset$). Let F be the set of all such f corresponding to all possible $S \subseteq E$.

Definition 7.2 *We say that a point p of affine space is visible from the point e, $e \neq p$, with respect to the simplex σ if the open segment (e,p) does not intersect σ, i.e. $(e,p) \cap \sigma = \emptyset$.*

We say also that a subset S of points is visible from the point e if every point of this subset is visible from e.

Theorem 7.3 [6] *Let $\sigma \in \Sigma$ be a simplex and $e \in E$ a point that is not a vertex of σ.*

1. There is the following linear relation in V_Σ among simplices:

$$\sigma = \sum_{q_i \in Q^+} \sigma(q_i, e) - \sum_{q_i \in Q^-} \sigma(q_i, e), \tag{12}$$

where

Q^+ is the set of all facets (i.e. $(n-1)$-dimensional faces) q_i of the simplex σ that are not visible from e (with respect to σ);

Q^- is the set of all facets q_i of the simplex σ that are visible from e (with respect to σ);

$\sigma(q_i, e)$ is the n-dimensional simplex spanned by the facet q_i of the simplex σ and the point e.

2. Any proper subset of simplices σ; $\sigma(q_i, e)$, where $q_i \in Q^+$; $\sigma(q_i, e)$, where $q_i \in Q^-$ is a linearly independent set.

Proof. 1. Let us reformulate what a linear relation among simplices is. Let $\sigma_1, \ldots, \sigma_k \in \Sigma$. Denote by $\phi_\sigma(x)$ the characteristic function of σ, i.e. $\phi_\sigma(x) = 1$, if $x \in \sigma$ and $\phi_\sigma(x) = 0$ if $x \notin \sigma$. Note that each of these

[6] The statement 1. of this theorem is stated in [1] in another form. Here we use the important geometric notion of visibility.

characteristic functions $\phi_\sigma(x)$ is constant within every open chamber $\gamma \in \Gamma$. Therefore, it is easy to check that the relation $\sum \lambda_i \sigma_i = 0$ in V_Σ is equivalent to the relation $\sum \lambda_i \phi_{\sigma_i}(x) = 0$ for any $x \in \cup\gamma$.

Then, instead of the relation (12) we need to prove the following:

Let $\sigma \in \Sigma$ and $e \in E$ be a point that is not a vertex of σ, then

$$\phi_\sigma(x) = \sum_{q_i \in Q^+} \phi_{\sigma(q_i,e)}(x) - \sum_{q_i \in Q^-} \phi_{\sigma(q_i,e)}(x), \text{ for any } x \in \bigcup_{\gamma \in \Gamma} \gamma \qquad (13)$$

Two cases are possible: 1) $e \in \sigma$; 2) $e \notin \sigma$. Consider the case 1). In this case the system of simplices $\sigma(q_i, e)$, where q_i is a facet of the simplex σ gives a subdivision of σ into these simplices. Therefore, for any point $x \in conv(\sigma, e)$ we have: $x \in \sigma$ and there exists only one j such that $x \in \sigma(q_j, e)$. Since in the case 1) any facet q_i of the simplex σ is nonvisible from e with respect to σ, we have $q_j \in Q^+$. Then in the relation (13) there are only two summands that cancel out and, therefore, (13) is satisfied.

Consider the case 2), i.e. $e \notin \sigma$. Let $x \in conv(\sigma, e)$. Consider the ray from the point e that passes through the point x, i.e. the points $\alpha e + \beta x$, where $\alpha + \beta = 1,\ \beta \geq 0$. This ray intersects the boundary of the simplex σ in two points: $x' = \alpha' e + \beta' x$ and $x'' = \alpha'' e + \beta'' x,\ \ 0 < \beta' < \beta''$. Clearly, the point x' is visible from the point e with respect to σ and the point x'' is not visible.

For the point x $(\alpha = 0, \beta = 1)$ there are two possibilities: a) $x \in (e, x')$, i.e. $0 < \beta < \beta' < \beta''$ or b) $x \in (x', x'')$, i.e. $0 < \beta' < \beta < \beta''$.

Case a). Note that the point x' is an interior point of only one of the facets of σ, i.e. $x' \in q_k$ for some k. Indeed, if x' belongs to the two facets q_k and q_j then the point $x \in (e, x')$ would lie on the boundary of some chamber γ but we need to consider only the points x that lie in an open chamber $\gamma \in \Gamma$.

For the same reason the point x'' is an interior point of only one facet q_j of the simplex σ. We have $q_k \in Q^-$ and $q_j \in Q^+$.

Then it is easy to see that the point x belongs only to the following simplices: $x \in \sigma(q_k, e)$ and $x \in \sigma(q_j, e)$. Therefore, in the equality (13) there are only two summands that cancel out and, therefore, (13) is satisfied.

Case b). Using the same reasoning as in case a) we obtain that the point x belongs only to the followings simplices: $x \in \sigma$ and $x \in \sigma(q_j, e)$, where $q_j \in Q^+$ is the only facet such that the point x'' is its interior point. We obtained again that there are only two summands in the equality (13) and they cancel out.

Proof of the statement 2. We need to prove that any proper subset of simplices σ; $\sigma(q_i, e)$, $q_i \in Q^+$; $\sigma(q_i, e), q_i \in Q^-$ is a linearly independent set.

Throughout the proof we will again use the ray (e, x) from the point e that passes through a point $x \in conv(\sigma, e)$. We will assume that the points x', x'' are the two points on this ray such that x', x'' lie on the boundary of the simplex σ and x' is visible from e (with respect to σ), while x'' is not visible from e.

Consider the following linear combination

$$\lambda\sigma + \sum_{q_i \in Q^+} \lambda_i \sigma(q_i, e) + \sum_{q_i \in Q^-} \lambda_i \sigma(q_i, e) = 0 \tag{14}$$

We will consider three cases.

Case 1. Let us prove that all these simplices without the simplex σ are linearly independent, i.e. if $\lambda = 0$ then in (14) all the other coefficients are 0.

First let us show that if $\lambda = 0$ then all the λ_i corresponding to the nonvisible facets $q_i \in Q^+$ are equal to 0. Indeed, consider a point $x \in conv(\sigma, e)$ such that the corresponding point x'' (on the ray (e, x)) is an interior point of some facet $q_j \in Q^+$. Then it is easy to see that the pont x belongs only to the two simplices: $x \in \sigma$ and $x \in \sigma(q_j, e)$. Since $\lambda = 0$, we conclude that $\lambda_j = 0$, where λ_j corresponds to the term $\sigma(q_j, e)$, $q_j \in Q^+$.

Now let us show that the coefficients λ_i corresponding to the visible facets are equal to 0. Consider a point x such that the corresponding point x' (on the ray (e, x)) is an interior point of a visible facet q_k of the simplex σ. Then x belongs only to the two following simplices: $x \in \sigma(q_k, e)$ and $x \in \sigma(q_j, e)$, where $x'' \in q_j$ and $q_j \in Q^+$. We may assume (see reasoning in the proof of 1, case a) of this theorem) that x'' is an interior point of the facet q_j. Since $q_j \in Q^+$ then, as we have shown above, $\lambda_j = 0$. Therefore, the coefficient λ_k by the term $\sigma(q_k, e)$ is also equal to 0. Thus, all the coefficients in the linear combination (14) are equal to zero and therefore all the simplices without the simplex σ are linearly independent.

Case 2. Let in the equality (14) a simplex $\sigma(q_j, e)$, $q_j \in Q^+$, be missing, i.e. let $\lambda_j = 0$. Let us show that then $\lambda = 0$. Indeed, consider a point x such that the corresponding point x'' is an interior point of the facet q_j. Then x

belongs only to the two simplices: $x \in \sigma(q_j, e)$ and $x \in \sigma$. Since $\lambda_j = 0$, we obtain that $\lambda = 0$. Thus, we have reduced Case 2 to Case 1.

Case 3. Let in the equality (14) a simplex $\sigma(q_k, e)$, $q_k \in Q^-$, be missing, i.e. $\lambda_k = 0$. Consider a point x such that the corresponding point x' is an interior point of the facet q_k. The corresponding point x'' (on the ray (e, x)) lies on some facet $q_j \in Q^+$ (we may again assume that x'' is an interior point of q_j.) It is easy to see that the point x belongs only to the two following simplices: $x \in \sigma(q_k, e)$ and $x \in \sigma(q_j, e)$. Since $\lambda_k = 0$ then the coefficient λ_j by the term $\sigma(q_j, e)$, $q_j \in Q^+$, is equal to 0. Thus, we have reduced Case 3 to Case 2. The theorem is proved. □

Now we will describe the bases of simplices constructed for the case [7] $n = 2$ in [2].

Let E be a finite set of N points on the affine plane and Σ a set of all triangles with vertices from E. Let us introduce the following notations. Let $e_s, e_t, e_u \in E$. Then $\sigma(e_s, e_t, e_u)$ is the triangle with the vertices e_s, e_t, e_u. We also denote by $(e_s|e_t, e_u)$ the cone (an angle) with the vertex e_s and bounded by the rays (e_s, e_t) and (e_s, e_u).

Let $e_1, \ldots, e_N$ be an ordering [8] of the points E such that for any k the following condition is satisfied:

$$conv(e_1, \ldots, e_k) \cap conv(e_{k+1}, \ldots, e_N) = \emptyset \qquad (15)$$

Consider the point e_1. This point is connected with the points $e_2, \ldots, e_N$. Consider the rays $(e_1, e_2), \ldots, (e_1, e_N)$. The minimal convex cone with the vertex e_1 that contains $P = conv(E)$ is subdivided by these rays into the non-intersecting cones (open cones) of the form $(e_1|e_t, e_u)$. Let us enumerate these rays consecutively (for example, clockwise) by $q_1, \ldots, q_l$. Note that $l \leq N-1$ (there can be several points from E on the same ray). On each ray q_k we choose two points e_{i_k}, e_{j_k} from E (they can coincide). With the point e_1 we associate the following set b_1 of triangles: $\sigma(e_1, e_{i_1}, e_{j_2}), \sigma(e_1, e_{i_2}, e_{j_3}), \sigma(e_1, e_{i_3}, e_{j_4}), \ldots$

Consider the point e_2 of the ordering and the polygon $P_1 = conv(E \setminus e_1)$. Disregarding the point e_1 we associate similarly the set b_2 of triangles with the point e_2.

[7] A question of existence of geometrical bases of simplices in the n-dimensional space is a matter of a separate study.

[8] the proof of existence of such an ordering one can find, for example, in [2]

Repeating the construction for the points $e_3, \ldots$ we will obtain the set $b = b_1 \cup b_2 \cup \ldots$ of triangles. Note that there are no more than $N-2$ sets b_i. As it is proved in [2] the set $b = b_1 \cup b_2 \cup \ldots$ is a basis in V_Σ.

Theorem 7.4 *A basis of simplices b is a geometrical basis with respect to F defined by (11).*

Proof. We need to prove that for any simplex $\sigma \in \Sigma$ there exists k such that σ can be expressed in k steps in terms of b using F.

Let $e_1, \ldots, e_N$ be the ordering of the points $e_i \in E$ which was used to construct the basis b. For a simplex $\sigma(e_{i_1}, e_{i_2}, e_{i_3})$ we will assume that the point e_{i_1} is the minimal point among $e_{i_1}, e_{i_2}, e_{i_3}$ with respect to the ordering $e_1, \ldots, e_N$. Let us denote by $\Sigma_i \subset \Sigma$ the set of simplices $\sigma(e_{i_1}, e_{i_2}, e_{i_3})$ for which $i_1 = N - i$. We have $\Sigma_0 = \emptyset$, $\Sigma_1 = \emptyset$, Σ_2 either contains only one simplex $\sigma(e_N, e_{N-1}, e_{N-2})$ or $\Sigma_2 = \emptyset$. Clearly, we have $\Sigma = \Sigma_2 \cup \ldots \cup \Sigma_{N-1}$, (where $\Sigma_i \cap \Sigma_j = \emptyset,\ i \neq j$).

Let $\sigma(e_{N-i}, e_{i_2}, e_{i_3}) \in \Sigma_i$. Consider two rays (e_{N-i}, e_{i_2}) and (e_{N-i}, e_{i_3}) and the open domain (an angle) $(e_{N-i}|e_{i_2}, e_{i_3})$ formed by the two rays.

Consider $E \cap (e_{N-i}|e_{i_2}, e_{i_3})$. Note that if $e_k \in E \cap (e_{N-i}|e_{i_2}, e_{i_3})$ then $k > N - i$ (due to the condition (15)).

Denote by $\Sigma_i^j \subset \Sigma_i$ the set of all simplices $\sigma(e_{N-i}, e_{i_2}, e_{i_3})$ such that $|E \cap (e_{N-i}|e_{i_2}, e_{i_3})| = j$. We have

$$\Sigma_i = \bigcup_{j=0}^{i-2} \Sigma_i^j, \text{ where } \Sigma_i^j \cap \Sigma_i^k = \emptyset, j \neq k.$$

We will prove the theorem by induction. Suppose that any $\sigma \in \Sigma_{i-1}$ can be expressed in terms of b using F. Let us prove that this statement is true for $\sigma \in \Sigma_i$. We know that $\Sigma_i = \cup \Sigma_i^j$.

First, let $\sigma(e_{N-i}, e_{i_2}, e_{i_3}) \in \Sigma_i^0$, i.e. there are no points from E in the open domain $(e_{N-i}|e_{i_2}, e_{i_3})$. If there are also no points from E on the rays (e_{N-i}, e_{i_2}) and (e_{N-i}, e_{i_3}) then $\sigma(e_{N-i}, e_{i_2}, e_{i_3}) \in b$ by the construction of b.

If there exists a point $e_k \in E$ such that $e_k \in (e_{N-i}, e_{i_2})$ then for the set S of the points $(e_{N-i}, e_{i_2}, e_{i_3}, e_k)$ we have the set $f \in F$ of simplices $f = \{\sigma(e_{N-i}, e_{i_3}, e_k), \sigma(e_{N-i}, e_{i_2}, e_{i_3}), \sigma(e_{i_2}, e_{i_3}, e_k)\}$. Note that $\sigma(e_{N-i}, e_{i_2}, e_k)$ does not define any 2-dimensional simplex. We have the simplex $\sigma(e_{i_2}, e_{i_3}, e_k) \in \Sigma_{i-1}$ and, as we have supposed, it can be expressed in terms of b using F.

One of the simplices $\sigma(e_{N-i}, e_{i_2}, e_{i_3})$ or $\sigma(e_{N-i}, e_k, e_{i_3})$ belongs to b by the construction of b. Thus, two out of three simplices in f can be expressed in terms of b using F. Therefore, using f we can express the third simplex in terms of b.

Suppose that for $\sigma \in \Sigma_i^{j-1}$ the statement of the theorem is proved. Let us prove it for $\sigma \in \Sigma_i^j$. Consider a simplex $\sigma(e_{N-i}, e_{i_2}, e_{i_3}) \in \Sigma_i^j$, i.e. there exist exactly j points from E in the open domain $(e_{N-i}|e_{i_2}, e_{i_3})$. Let $e_k \in E$ be one of such points. Then the set S of the points $(e_{N_i}, e_{i_2}, e_{i_3}, e_k)$ defines $f \in F$, where $f = \{\sigma(e_{N-i}, e_{i_2}, e_{i_3}), \sigma(e_{N-i}, e_{i_2}, e_k), \sigma(e_{N-i}, e_{i_3}, e_k), \sigma(e_{i_2}, e_{i_3}, e_k)\}$.

Three of these simplices can be expressed in terms of b using F. Indeed, we have:

$\sigma(e_{i_2}, e_{i_3}, e_k) \in \Sigma_{i-1}$ and, as we have supposed, it can be expressed in terms of b using F;

$\sigma(e_{N-i}, e_{i_2}, e_k) \in \Sigma_i^m$, where $m < j$. Indeed, there is the following strict inclusion for the angles: $(e_{N-i}|e_{i_2}, e_k) \subset (e_{N-i}|e_{i_2}, e_{i_3})$.

Similarly, we have $\sigma(e_{N-i}, e_{i_3}, e_k) \in \Sigma_i^{m'}$, where $m' < j$. But as we have supposed any $\sigma \in \Sigma_i^{j-1}$ can be expressed in terms of b using F.

We have three simplices in f that can be expressed in terms of b using F. Therefore, the simplex $\sigma(e_{N-i}, e_{i_2}, e_{i_3}) \in \Sigma_i^j$ can be expressed in terms of b using F.

We also need to check the first nontrivial step of induction. We have $\Sigma_0 = \Sigma_1 = \emptyset$. Let Σ_{i_0} be the first nonempty set, i.e. the points $e_N, \ldots, e_{N-(i_0-1)}$ lie on a straight line but the point e_{N-i_0} does not lie on this line. In this case Σ_{i_0} consists of all the triangles $\sigma(e_{N-i_0}, e_k, e_m)$, where $k, m > N - i_0$.

Note that due to the condition (15) all the points $e_N, \ldots, e_{N-(i_0-1)}$ are ordered on the straight line according to their numbers, i.e. if $e_{i_1}, e_{i_2}, e_{i_3}$ are such points and $i_1 < i_2 < i_3$ then the point e_{i_2} lies between the points e_{i_1} and e_{i_3}.

We need to prove that any $\sigma \in \Sigma_{i_0}$ can be expressed in terms of b using F. We have $\Sigma_{i_0} = \cup \Sigma_{i_0}^j$. By construction the set b includes the following triangles

$$\sigma(e_{N-i_0}, e_N, e_{N-1}), \sigma(e_{N-i_0}, e_{N-1}, e_{N-2}), \ldots, \sigma(e_{N-i_0}, e_{N-i_0+2}, e_{N-i_0+1}),$$

i.e. all the triangles $\sigma \in \Sigma_{i_0}^0$. Repeating the induction on j (as we have already done above for Σ_i^j) we will obtain that any $\sigma \in \Sigma_{i_0}$ can be expressed in terms of b using F. The theorem is proved. □

7.2 Geometrical bases of chambers.

In Subsection 7.1 we have considered the relation $A \subseteq (\Sigma \times \Gamma)$ and have proved that the bases of simplices constructed in [2] are geometrical with respect to a certain subset F of circuits among simplices. For the same relation $A \subseteq (\Sigma \times \Gamma)$ we can also consider bases of chambers. In [1] there is a theorem describing geometrically a certain subset F' of circuits of the column matroid on $\hat{A}$. Roughly speaking, if there are N points $e \in E$ in the n-dimensional affine space then for any vertex w of a chamber such that $w \notin E$ there is a linear relation f' among all the chambers[9] adjacent to the vertex w. Any linear relation among chambers $\gamma \in \Gamma$ is a linear combination of $f' \in F'$ (i.e. the system of linear relations F' is complete).

In [2] there is an algorithm of construction of bases of chambers and it is proved that these bases are geometrical with respect to F'. There is also an example showing that not every basis $b \subset \Gamma$ is geometrical with respect to F' though F' is a complete system of linear relations.

We want to mention that the construction of bases of chambers described in [2] played an important role in the results of the present paper. Note that the present paper gives the algorithm different from the ones in [2] for constructing bases of simplices and bases of chambers.

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