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AN ARRANGEMENT OF REAL HYPERPLANES AND THE PARTITION FUNCTION CONNECTED WITH IT

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Many essential questions in the geometry of Lie groups and the corresponding dual questions in representation theory depend on the structure of a simple function called the partition function, in its continuous [1] and discrete [2] variants. The study of this function leads to geometric problems that are undoubtedly of independent interest. These problems are closely connected with the calculus of Heaviside functions constructed in [7]. The present note is mainly devoted to these geometric problems; an application of them to the investigation of the partition function will be given at the end of the paper.

1. Formulation of the geometric problems: chambers and simplicial cones. Let $L_1, \dots, L_n$ be a finite collection of one-dimensional subspaces and $W_1, \dots, W_m$ a finite collection of subspaces of codimension 1 in an l-dimensional real vector space V. Assume the condition

(C1) Every vector subspace of V spanned by some of the $L_1, \dots, L_n$ can be represented as an intersection of some of the $W_1, \dots, W_m$.

This condition includes, in particular, the requirement that $W_1 \cap \dots \cap W_m = 0$. The situation when $L_1 + \dots + L_n = V$ is the case most important for applications. In this case condition (C1) is equivalent for applications. In this case condition (C1) is equivalent to all the subspaces of codimensional 1 spanned by some of the $L_1, \dots, L_n$ being in the collection $W_1, \dots, W_m$.

For all $i = 1, \dots, n$ we now choose an open half-line $L_i^+ \subset L_i$ with origin at 0 such that

(C2) All the half-lines L_i^+ lie on one side of some hyperplane in V passing through 0.

In an example important for applications the $L_1^+, \dots, L_n^+$ are chosen to be half-lines passing through the positive roots of some root system in V. We are interested in geometric objects of two kinds: simplicial cones C_I generated by the L_i^+, and chambers Γ determined by the subspaces $W_1, \dots, W_m$.

A subset $I \subset [1, n]$ is said to be *independent* if the subspaces L_i with $i \in I$ are in general position, i.e., they generate a vector subspace of dimension $|I|$ in V. Denote by C_I the closed convex cone spanned by the half-lines L_i^+ with $i \in I$. For example, $C_\varnothing$ is the cone consisting of the single point 0. It is clear that all the cones C_I corresponding to independent subsets I are simplicial.

We say that two points $x, y \in V$ lie in a single chamber Γ if for each $j = 1, \dots, m$ the segment $[x, y]$ either does not intersect W_j or lies entirely in W_j. For example, the l-dimensional chambers in V are the connected components of the space $V \backslash (\bigcup_{1 \le j \le m} W_j)$. The chambers are clearly convex polyhedral cones in V. They are all nonclosed, with the exception of the chamber consisting of the single point 0.

It follows from condition (C1) that each cone C_I is a union of chambers. We introduce the incidence matrix M. Its rows are parametrized by the chambers Γ, and its columns are parametrized by the independent subsets $I \subset [1, n]$; the intersection of the row Γ and the column I contains a number (Γ, C_I) equal to 1 if $\Gamma \subset C_I$ and 0 otherwise.

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We solve the following problems:

1. Find a complete system of linear relations among the columns of the matrix M.
2. Construct a basis in the vector space generated by the columns of M.

1′ and 2′. The same problems for the rows of M.

Let M_r, $r = 0, \dots, l$, be the submatrix of M consisting of the rows Γ and columns I with $|I| = \dim \Gamma = r$. We solve these problems also for all submatrices M_r. It is especially important for applications to investigate the submatrix M_l of M corresponding to the chambers and cones of highest dimension.

Denote by φ_I the column of M corresponding to the independent subset I, and by ψ_Γ the row of M corresponding to the chamber Γ. If $|I| = \dim \Gamma = r$, then let φ_I^r and ψ_Γ^r be the corresponding column and row of M_r. Let Φ be the vector space generated by the columns φ_I, and Ψ the vector space generated by the columns ψ_Γ. Similarly, let Φ_r and Ψ_r be the vector spaces generated by the columns and rows, respectively, of M_r.

It is clear from the definitions that Φ can be naturally identified with the vector space of functions on V that is generated by the characteristic functions of the cones C_I. The elements of Ψ can be naturally thought of as linear functionals on Φ, i.e., as "distributions" on V connected with the space Φ of test functions.

2. Linear relations among the columns φ_I and φ_I^r. A subset $J \subset [1, n]$ is said to be *weakly dependent* if $\dim \sum_{i \in J} L_i = |J| - 1$. With each weakly dependent subset J we associate a linear relation among the columns φ_I that correspond to the independent subsets $I \subset J$.

Choose a vector v_i in L_i^+ for every $i \in J$. Since J is weakly dependent, there is a linear relation $\sum_{i \in J} a_i v_i = 0$ among the vectors v_i that is unique to within a factor. Let $J_+ = \{i \in J : a_i > 0\}$, $J_- = \{i \in J : a_i < 0\}$ and $J_0 = \{i \in J : a_i = 0\}$. It is clear that the partition $J = J_+ \cup J_- \cup J_0$ does not depend on the choice of the vectors v_i and is uniquely determined by J to within an interchange of J_+ and J_-. Note that the subsets J_+ and J_- are nonempty in view of (C2). Obviously, the subset $J \backslash i$ is independent for all $i \in J_+ \cup J_-$.

THEOREM 1. (a) *For each weakly dependent subset* $J = J_+ \cup J_- \cup J_0$

$$\sum_{\varnothing \neq \Omega \subset J_+} (-1)^{|\Omega|-1} \varphi_{J \backslash \Omega} = \sum_{\varnothing \neq \Omega \subset J_-} (-1)^{|\Omega|-1} \varphi_{J \backslash \Omega}. \tag{1}$$

(b) *All the linear relations among the elements $\varphi_I \in \Phi$ are linear combinations of the relations* (1).

THEOREM 2. (a) *For each weakly dependent subset $J = J_+ \cup J_- \cup J_0$ with $|J| = r + 1$*

$$\sum_{i \in J_+} \varphi_{J \backslash i}^r = \sum_{i \in J_-} \varphi_{J \backslash i}^r. \tag{2}$$

(b) *All the linear relations among the elements $\varphi_I^r \in \Phi_r$ are linear combinations of the relations* (2).

3. Linear relations among the rows ψ_Γ and ψ_Γ^r. First of all, it is clear that

$$\psi_\Gamma = 0 \text{ if the chamber } \Gamma \text{ is not contained in any of the cones } C_I. \tag{3}$$

To describe the remaining relations we give some definitions. A vector subspace $U \subset V$ is called a divider if it is an intersection of some of the subspaces $W_1, \dots, W_m$. The dividers spanned by some of the subspaces $L_1, \dots, L_n$ are said to be *essential*, and the rest are said to be *nonessential*. For example, the one-dimensional essential dividers are the subspaces $L_1, \dots, L_n$.

Let Γ be a chamber, and $\overline{\Gamma}$ the closure of Γ in V; we say that Γ *adjoins a divider* U if the intersection $\overline{\Gamma} \cap U$ contains a chamber that is open in U.

THEOREM 3. (a) *Let L be a one-dimensional nonessential divider, W an arbitrary subspace of co-dimension* 1 *in V that contains L and does not contain any other dividers, and V_W^+ one of the two open half-spaces bounded by W. Then*

$$\sum_{\Gamma}(-1)^{\dim\Gamma-1}\psi_\Gamma=0, \tag{4}$$

where Γ runs through the chambers adjoining L and contained in V_W^+.

(b) *All the linear relations among the elements $\psi_\Gamma\in\Psi$ are linear combinations of* (3) *and* (4).

An *r-flag* is defined to be an increasing chain $F=(0=U_0\subset U_1\subset\cdots\subset U_r)$ of dividers, where $\dim U_s=s$ for all s. We say that an r-flag F is *oriented* if one of the two connected components U_s^+ in $U_s\backslash U_{s-1}$ has been chosen for all $s=1,\dots,r$; an oriented flag F is denoted by $\vec{F}$. A chamber Γ is said to *adjoin the flag F* if it adjoins all the dividers of F. If a chamber Γ adjoints an oriented r-flag $\vec{F}$, then we let $\varepsilon(\Gamma,\vec{F})=+1$ or -1, depending on the parity of the number of $s\in[1,r]$ for which $\Gamma\cap U_s^+=\varnothing$.

Following [7], for each oriented r-flag $\vec{F}$ we define an element $\psi^r(\vec{F})\in\Psi_r$ by the formula $\psi^r(\vec{F})=\sum_\Gamma\varepsilon(\Gamma,\vec{F})\cdot\psi_\Gamma^r$, where Γ runs through the r-dimensional chambers adjoining $\vec{F}$ (it is easy to see that there are exactly 2^r such chambers). It is clear that the elements $\psi^r(\vec{F})$ connected with different orientations of the same flag can differ only by a sign.

THEOREM 4. (a) *For each oriented r-flag $\vec{F}$ with a nonessential* 1*-dimensional divider,*

$$\psi^r(\vec{F})=0. \tag{5}$$

(b) *All the linear relations among the elements $\psi_\Gamma^r\in\Psi_r$ are linear combinations of the relations* (5) *and the relations $\psi_\Gamma^r=0$ for all r-dimensional chambers Γ not contained in any of the cones C_I.*

REMARK. The relations (4) and (5) can be included in a unified system of relations. We do not give it for lack of space.

4. Bases in the spaces Φ_r and Φ. We choose a mapping τ of the set of nonzero essential dividers into $[1,n]$ that satisfies the condition

(C3) If $\tau(U)=i$, then $L_i\subset U$.

We define the class $\mathscr{I}_\tau$ of independent subsets of $[1,n]$ by the following requirements:

(a) $\varnothing\in\mathscr{I}_\tau$;

(b) if I is a nonempty independent subset of $[1,n]$ and $\tau(\sum_{i\in I}L_i)=i_0$, then $I\in\mathscr{I}_\tau$ if and only if $i_0\in I$ and $I\backslash i_0\in\mathscr{I}_\tau$.

THEOREM 5. *The elements φ_I^r for all $I\in\mathscr{I}_\tau$ with $|I|=r$ form a basis in Φ_r.*

THEOREM 6. *The elements φ_I for all $I\in\mathscr{I}_\tau$ form a basis in Φ.*

REMARK. The definition of the class $\mathscr{I}_\tau$ was in essence given in [3]. The class of "sets without open cycles" considered in [4] and [7] appears as $\mathscr{I}_\tau$ for a special choice of the mapping τ.

5. Bases in the space Ψ_r and Ψ. Let τ be the same mapping as in §4. Denote by $\mathscr{F}_\tau^r$ the set of all oriented r-flags $\vec{F}=(U_0\subset\cdots\subset U_r)$ such that: (a) all the dividers U_s are essential; and (b) if $\tau(U_s)=i_s$, then $L_{i_s}^+\subset U_s^+$, $s=1,\dots,r$.

THEOREM 7. *The elements $\psi^r(\vec{F})$ for all $\vec{F}\in\mathscr{F}_\tau^r$ form a basis in Ψ_r.*

For each oriented r-flag $\vec{F}$ we define an element $\psi(\vec{F})\in\Psi$ by the formula $\psi(\vec{F})=\sum_\Gamma\varepsilon(\Gamma,\vec{F})\psi_\Gamma$, where Γ runs through the r-dimensional chambers adjoining $\vec{F}$. Let $\mathscr{F}_\tau=\bigcup_{0\le r\le l}\mathscr{F}_\tau^r$.

THEOREM 8. *The elements $\psi(\vec{F})$ for all $\vec{F} \in \mathscr{F}_\tau$ form a basis in Ψ.*

The basis in Theorem 7 consists not of the elements ψ_Γ^r themselves, but of linear combinations of them; in this sense it is "more complicated" than the basis in Theorem 5. However, it has the following agreeable property.

THEOREM 9. *For each r-dimensional chamber Γ all the coefficients in the decomposition of the element ψ_Γ^r with respect to the basis in Theorem 7 are equal to* 0 *or* 1.

6. Applications to the structure of the partition function. We consider the space $\mathbf{R}^n$ and the "positive orthant" $\mathbf{R}_+^n$ in it. Let $V_0 \subset \mathbf{R}^n$ be an $(n-l)$-dimensional vector subspace such that $V_0 \cap \mathbf{R}_+^n = 0$. Then for every $x \in \mathbf{R}^n$ the parallel plane $V_0 + x$ intersects $\mathbf{R}_+^n$ in the compact (possibly empty) convex polyhedron $\Delta_x = (V_0 + x) \cap \mathbf{R}_+^n$. The partition function is by definition the "volume" of Δ_x; it clearly depends only on the image of x in the quotient space $V = \mathbf{R}^n/V_0$. The "volume" is understood in the following sense. We define in $\mathbf{R}^n$ a differential $(n-l)$-form ω with polynomial coefficients and choose mutually compatible orientations in all the planes $V_0 + x$; the partition function is defined by the formula $P_\omega(v) = \int_{\Delta_x} \omega$, where $v \in V$ is the image of the point $x \in \mathbf{R}^n$ under the natural projection $p\colon \mathbf{R}^n \to V$.

The function $P_\omega(v)$ is piecewise polynomial. To study it we use the technique worked out above in the following situation. Let $e_1, \dots, e_n$ be the standard basis in $\mathbf{R}^n$, and define the one-dimensional subspaces $L_1, \dots, L_n$ of V by $L_i = p(\mathbf{R}e_i)$. As $W_1, \dots, W_m$ we take all the subspaces of codimension 1 in V that have the form $p(\mathbf{R}^I)$ for some coordinate subspace $\mathbf{R}^I$ of $\mathbf{R}^n$. Finally, we choose the half-line $L_i^+ \subset L_i$, $i = 1, \dots, n$, passing through the point $p(e_i)$. Conditions (C1) and (C2) are easy to verify.

Let $V' = V \setminus (\bigcup_j W_j)$ be the union of all the l-dimensional chambers in V. For each independent l-element subset $I \subset [1, n]$ let χ_I stand for the characteristic function of the convex open cone $C_I \cap V'$.

It is proved in [4] that the function P_ω on V' admits the decomposition

$$P_\omega = \sum_I P_\omega^I \cdot \chi_I, \tag{6}$$

where the P_ω^I are certain polynomials on V (see also [5] and [6]). The decomposition (6) is not unique in general, because the functions χ_I can be linearly dependent. A complete system of linear relations among the functions χ_I is given by Theorem 2 (it is clear from the definitions that these are the same relations that exist among the elements $\varphi_I^l \in \Phi_l$). By Theorem 5, the decomposition (6) becomes unique if we leave in it only the terms with $I \in \mathscr{F}_\tau$.

It follows from (6), in particular, that on each l-dimensional chamber Γ the function P_ω is equal to some polynomial P_ω^Γ. Specifying all the polynomials P_ω^Γ describes P_ω "in the dual way" with respect to the decomposition (6). Theorem 4 gives a complete system of universal (i.e., independent of ω) linear relations among the polynomials P_ω^Γ (it follows easily from the definitions that these are the same relations that exist among the elements $\psi_\Gamma^l \in \Psi_l$). As Theorem 7 shows, to compute all the polynomials P_ω^Γ it suffices to compute the "flag" linear combinations $\sum_\Gamma \varepsilon(\Gamma, \vec{F}) P_\omega^\Gamma$ for all the oriented l-

flags $\vec{F} \in \mathscr{F}_\tau^{\,l}$; by Theorem 9, each P_ω^Γ can be represented as a sum of some of these "flag" combinations.

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