THE NUMBER OF POLYTOPES, CONFIGURATIONS AND REAL MATROIDS

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Abstract. We show that the number of combinatorially distinct labelled d-polytopes on n vertices is at most $(n/d)^{d^2n(1+o(1))}$, as $n/d \to \infty$. A similar bound for the number of simplicial polytopes has previously been proved by Goodman and Pollack. This bound improves considerably the previous known bounds. We also obtain sharp upper and lower bounds for the numbers of real oriented and unoriented matroids with n elements of rank d. Our main tool is a theorem of Milnor and Thom from real algebraic geometry.

§1. *Introduction.* Let $c(n, d)$ denote the number of (combinatorial types of) d-polytopes on n labelled vertices and let $c_s(n, d)$ denote the number of simplicial d-polytopes on n labelled vertices. The problem of determining or estimating these two functions (especially for 3-polytopes) was the subject of much effort and frustration of nineteenth-century geometers. Although it follows from Tarski's Theorem on the decidability of first order sentences in the real field that the problem of computing $c(n, d)$ is solvable (*cf.* [Gr. pp 91-92]), it seems extremely difficult actually to determine this number even for relatively small n and d. Both Cayley and Kirkman failed to determine $c(n, 3)$ or $c_s(n, 3)$ despite a lot of effort. Detailed historical surveys of these attempts were given by Brückner [Br] and Steinitz [Ste] (see also [Gr. pp. 288-290]). Brückner [Br] determined $c_s(n, 3)$ for $n \leq 10$. Hermes [He] tried to extend Brückner's work for $n = 11, 12$, but both his enumeration and Brückner's extensive attempts to correct it were incomplete, as shown by Grace [Gra]. Hermes [He] determined $c(n, 3)$ for $n \leq 8$ and Grace [Gra] determined $c_s(11, 3)$. More recently, Grünbaum and Sreedharan [GS] determined $c_s(8, 4)$ and Altshuler and Steinberg [AS1, AS2] determined $c(8, 4)$. Determining $c(n, d)$ and $c_s(n, d)$ for small values of n and d, however, does not, of course, solve the general problem. Write $\beta = n - d$. The cases $d \leq 2$ or $\beta \leq 2$ are quite easy; there is only one polygon with n (unlabelled) vertices and there are $[d^2/4]$ d-polytopes on $d+2$ (unlabelled) vertices, $[d/2]$ of which are simplicial, (see [Gr. pp. 98-101]).

Using a Gale Diagram, Perles (*cf.* [Gr. pp. 112-114]) found an explicit formula for $c_s(d+3, d)$ and determined the asymptotic behaviour of $c(d+3, d)$ as d tends to infinity. An explicit formula for $c(d+3, d)$ was given later by Lloyd [Ll].

The asymptotic behaviour of $c(n, 3)$ and $c_s(n, 3)$ was determined almost precisely by Tutte [Tu] and by Richmond and Wormald [RW], (see also [Gr. pp 289-290]). However, as mentioned in [Gr. p 290], it seems that the determination of $c_s(n, d)$ or $c(n, d)$ for $d \geq 4$ and $n \geq d+4$ is a problem of an entirely different order of magnitude. Until recently, the best general upper bound for $c_s(n, d)$ was $n^{cn^{d/2}}$. This follows easily from the upper bound theorem [Kl, M, St], and the argument applies also to bound the number of triangulated $(d$-1)-spheres. (Recall that the boundary complex of a simplicial d-polytope is a triangulated $(d$-1) sphere, but the converse is false when $d, \beta \geq 4$).

A major development was very recently achieved by Goodman and Pollack [GP2]. A *simple configuration* of n points in R^d is an ordered n-tuple of points in general position in R^d. Two such configurations A and B are isomorphic if there is a bijection $\phi : A \to B$ such that the orientation of each $d+1$ (ordered) points is the same as that of their images. By a clever use of a theorem of Milnor [Mi] from real algebraic geometry Goodman and Pollack showed that the number of simple configurations of n points in R^d is less than $n^{d(d+1)n}$. This is close to the truth at least for fixed d and large n since it is easy to show that this number is at least

$$n^{d^2n(1+O(\log d/\log n))}.$$

Moreover, their result gives immediately that $c_s(n, d) \leq n^{d(d+1)n}$, improving considerably the best previously known bound.

In this paper we apply another (similar) theorem of Milnor and Thom to bound the number of simple and nonsimple configurations of n points in R^d, and hence to bound the number of arbitrary d-polytopes on n-vertices. We also slightly improve the bound of [GP2] and show, in particular, that for fixed $d \geq 2$ the numbers of simple or of nonsimple configurations of n points in R^d both have the form $n^{d^2n(1+o(1))}$ as $n \to \infty$. For polytopes we obtain

$$\left(\frac{n-d}{d}\right)^{nd/4} \leq c_s(n, d) \leq c(n, d) \leq (n/d)^{d^2n(1+O(\log\log(n/d)/\log(n/d)))},$$

and show that the total number of polytopes on n vertices is at most $2^{n^3+O(n^2)}$. Very recently, Kalai [K] showed that the total number of triangulated spheres on n vertices is at least $2^{2^{n/3}}$. Thus, very few of these are boundary complexes of simplicial polytopes.

Our methods also enable us to obtain sharp bounds on the asymptotic number of real and complex matroids with n elements of rank d. For fixed d and $n \to \infty$, these numbers have the form $n^{O(d^2n)}$. The total number of complex matroids on n elements is bounded by $2^{O(n^3)}$, a very small part of the total number of matroids on n points which is at least $2^{\Omega(2^n/n^{3/2})}$, as shown by Knuth [Kn].

Our paper is organized as follows: in Section 2 we apply Milnor's Theorem to obtain a general bound on the number of *sign patterns* of a sequence of polynomials. In Section 3 we deal with the number of real and complex matroids on n points, and in Section 4 we consider the number of configurations of n points in R^d. In the final Section 5 we prove our bounds for the number of d-polytopes.

§2. *The Number of Sign Patterns.* Let $P_j = P_j(x_1, x_2, \ldots, x_l)$, $(j = 1, \ldots, m)$ be real polynomials. For a point $c = (c_1, c_2, \ldots, c_l) \in R^l$ the *sign pattern* of the P_j at c is the m-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \{-1, 0, 1\}^m$, where

$\varepsilon_j = \text{sign } P_j(c_1, c_2, \ldots, c_n)$. The total number of sign patterns as c ranges over all points of R^n, denoted by $s(P_1, \ldots, P_m)$, is clearly at most 3^m. Using a theorem of Milnor [Mi] (see also Thom [Th]) from real algebraic geometry, we bound this number by a function of n and the degrees of the P_j. All our upper bounds in the paper are derived from this bound (and its analogue for complex polynomials).

We first state Milnor's theorem.

THEOREM 2.1 (Milnor [Mi, Theorem 2]). *Let V be a variety in R^l, defined by the polynomial equations*

$$f_i(x_1, x_2, \ldots, x_l) = 0, \qquad (i = 1, \ldots, h).$$

If each polynomial f_i has degree $\le k$, then the sum of the Betti numbers of V is at most $k(2k-1)^{l-1}$. In particular, the number of connected components of V is at most $k(2k-1)^{l-1}$.

Using this theorem we prove

THEOREM 2.2. *Let $P_1 = P_1(x_1, \ldots, x_n), \ldots, P_m = P_m(x_1, \ldots, x_n)$ be real polynomials. Let $d_j = \deg P_j (\ge 1)$ be the degree of P_j, $1 \le j \le m$. Put $J = \{1, 2, \ldots, m\}$ and let*

$$J = J_1 \cup J_2 \cup \ldots \cup J_h$$

be a partition of J into h pairwise disjoint parts. Define

$$k = 4 \max_{1 \le i \le h} \left(\sum_{j \in J_i} d_j \right).$$

Then the number of sign patterns of the P_j satisfies

$$s(P_1, \ldots, P_m) \le k(2k-1)^{n+h-1}.$$

Remark 2.3. By taking the trivial partition of J into one part, we conclude that if $r = \sum_{j=1}^m d_j$ then $s(P_1, \ldots, P_m) \le 4r(8r-1)^n$. By using another theorem of Milnor [Mi, Theorem 3], we can show that in fact

$$s(P_1, \ldots, P_m) \le (2+2r)(1+2r)^{n-1}. \tag{2.1}$$

For our applications, however, Theorem 2.2 will usually give asymptotically better bounds. We omit the detailed proof of (2.1).

Proof of Theorem 2.2. Let $C \subseteq R^n$ be a finite set of points that represents all the sign patterns of the P_j. (Clearly there is such a C satisfying $|C| \le 3^m$). For $c = (c_1, c_2, \ldots, c_n) \in C$ and $1 \le i \le m$ we denote $P_j(c_1, \ldots, c_n)$ by $P_j(c)$. Define $\varepsilon > 0$ by

$$\varepsilon = \tfrac{1}{2} \min \{|P_j(c)|: c \in C, \quad 1 \le j \le m \quad \text{and} \quad P_j(c) \ne 0\}.$$

Let $\delta > 0$ satisfy $\delta < \varepsilon^{4|J_i|}$, $1 \le i \le h$. Define h polynomials $f_1, f_2, \ldots, f_h$ with variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_h$ by

$$f_i(x_1, \ldots, x_n, y_1, \ldots, y_h) = -y_i^2 - \delta + \prod_{j \in J_i} (P_j(x_1, \ldots, x_n) - \varepsilon)^2 (P_j(x_1, \ldots, x_n) + \varepsilon)^2.$$

The degree of f_i is clearly $4\sum_{j \in J_i} d_j \le k$. Also, if $c = (c_1, \ldots, c_n) \in C$ then

$$\prod_{j \in J_i} (P_j(c) - \varepsilon)^2 (P_j(c) + \varepsilon)^2 > \delta,$$

and hence there exist real values $b_1, \ldots, b_h$ such that

$$f_i(c_1, \ldots, c_n, b_1, \ldots, b_h) = 0, \qquad (1 \le i \le h).$$

For $c \in C$ denote by $\bar{c}$ a real vector $(c_1, \ldots, c_n, b_1, \ldots, b_h)$ that satisfies the last system. Let V be the variety in R^{n+h} defined by

$$f_i(x_1, \ldots, x_n, y_1, \ldots, y_h) = 0 \qquad (1 \le i \le h).$$

By definition every vector $\bar{c}(c \in C)$ is a point of V. We now claim that if $c_1, c_2 \in C$ represent distinct sign patterns of the P_j, then $\bar{c}_1, \bar{c}_2$ are not in the same connected component of V. Indeed, if $c_1, c_2 \in C$ represent distinct sign patterns, there exists some $1 \le j \le m$, such that sign $P_j(c_1) \ne$ sign $P_j(c_2)$. If $j \in J_i$, this implies, by continuity and the choice of ε, that any path in R^{n+h} joining $\bar{c}_1$ to $\bar{c}_2$ contains a point $(x_1, \ldots, x_n, y_1, \ldots, y_h)$ such that $P_j(x_1, \ldots, x_n) = \varepsilon$, or $P_j(x_1, \ldots, x_n) = -\varepsilon$, *i.e.*, a point where

$$f_i(x_1, \ldots, y_h) = -y_i^2 - \delta \le -\delta < 0.$$

This point is thus not in V and our claim follows. Since C represents all the sign patterns of the P_j, we conclude that the number of sign patterns is at most the number of connected components of V which is, by Milnor's Theorem (Theorem 2.1), at most $k(2k-1)^{n+h-1}$. This completes the proof of the theorem.

For our applications we will also be interested in the number of sign patterns of complex polynomials. If $Q_i(z_1, \ldots, z_n)$, $1 \le i \le m$, are complex polynomials and $b = (b_1, \ldots, b_n) \in C^n$, the sign pattern of the Q_j at b is the m-tuple $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^m$, where $\varepsilon_j = \text{sign}\,|Q_j(b)|$. By applying Theorem 2.2 to the real polynomials $P_j = (\text{Re } Q_j)^2 + (\text{Im } Q_j)^2$, $1 \le j \le m$ in the $2n$ real variables Re z_i and Im z_i, $1 \le j \le n$, we can bound the number of sign patterns of the Q_j in terms of their degrees. Moreover, since in this case $P_j \ge 0$, we can slightly improve the estimate by defining here $f_i = -y_i^2 - \delta + \prod (P_j - \varepsilon)^2$. This gives the following theorem, whose detailed proof is omitted.

THEOREM 2.4. *Let $Q_1(z_1, \ldots, z_n), \ldots, Q_m(z_1, \ldots, z_n)$ be complex polynomials. Put $d_j = \deg Q_j (\ge 1)$, $J = \{1, 2, \ldots, m\}$ and let $J = J_1 \cup J_2 \cup \ldots \cup J_h$ be a partition of J into h pairwise disjoint parts. Define*

$$k = 4 \max_{1 \le i \le h} \left(\sum_{j \in J_i} d_j \right).$$

Then the number of sign patterns of the Q_j is at most

$$k(2k-1)^{2n+h-1}.$$

§3. *The Number of Rational, Real and Complex Matroids.* There are several known asymptotic estimates for the number of nonisomorphic matroids of several kinds on n points. See [We, pp. 305–308] for bounds on the number of all matroids on n points, the number of transversal matroids on n points, and the number of matroids on n points which are representable over a finite field with q elements. Knuth [Kn] showed that the number of labelled (simple) matroids on n points is at least

$$2^{\binom{n}{\lfloor n/2 \rfloor}/(2n)}.$$

Here we obtain sharp bounds on the numbers $r(n, d, Q)$, $r(n, d, R)$ and $r(n, d, C)$, which are the numbers of matroids on n points with rank d, which are representable over the rationals, the reals and the complex numbers, respectively. Clearly

$$r(n, d, Q) \le r(n, d, R) \le r(n, d, C).$$

Here we show that

$$n^{(d-1)^2 n - O(d^2 n(\log d - \log\log n)/\log n)} \le r(n, d, Q) \le r(n, d, R) \le n^{(d-1)dn + O(nd \log\log n/\log n)}, \tag{3.1}$$

that

$$r(n, d, C) \le n^{2(d-1)dn + O(nd \log\log n/\log n)}, \tag{3.2}$$

and that for every $d \le n$

$$r(n, d, R) \le r(n, d, C) \le 2^{O(n^3)}. \tag{3.3}$$

We first prove the upper bounds. We begin by considering real matroids. It is easy to check that every real matroid of rank d is representable in R^d, *i.e.*, for each point of the matroid we have a vector in R^d and a set of points is independent, if, and only if, the corresponding vectors are linearly independent. Let $(x_{11}, \ldots, x_{1d}), \ldots, (x_{n1}, \ldots, x_{nd}) \in R^d$ be the vectors representing our matroid, and consider the set of all $\binom{n}{d}$ d by d determinants $\det(x_{ij})$, $i \in \{i_1, \ldots, i_d\}$, $j = 1, \ldots, d$, where $1 \le i_1 < i_2 < \ldots < i_d \le n$. Such a determinant is non-zero, if, and only if, the corresponding set is a base of the matroid. Hence the sign pattern of these $\binom{n}{d}$ polynomials of degree d in the dn variables x_{ij} determines the matroid represented by the given values of the x_{ij}. (In fact, the sign pattern determines more, and we get here an upper bound on the number of oriented real matroids—see Section 4). Thus the total number of matroids is at most the number of sign patterns of the $\binom{n}{d}$ degree d polynomials in the dn variables x_{ij}. Divide the polynomials into $[n/\log n]$ groups, each of total degree

$$\sim \binom{n}{d} d/[n/\log n] \le (1 + o(1)) \frac{n^{d-1} \log n}{(d-1)!},$$

and apply Theorem 2.2 with

$$h = [n/\log n], \qquad k = (4 + o(1)) \frac{n^{d-1} \log n}{(d-1)!}$$

(and dn variables) to get the upper bound in inequality (3.1). Inequality (3.2) follows similarly from Theorem 2.4. Inequality (3.3) follows from Remark 2.3 (or Theorem 2.2 with $h = 1$) by noticing that the sum of degrees of our polynomials is at most $\binom{n}{d} d \le 2^n n$.

The only remaining part is the lower bound in equality (3.1). It suffices to show that

$$r(n, d, Q) \ge \binom{[n/\log n]}{(d-1)^2}^{n - [n/\log n]}.$$

Let M be a labelled set of $[n/\log n]$ points in Q^{d-1}, *i.e.*, in the real Euclidean space of dimension $d-1$, with all coordinates having rational values. Assume, further, that the set M is in a generic position. For every subset $N \subseteq M$, $|N| = (d-1)^2$, we define a point $p(N) \in Q^{d-1}$ as follows. Let $N = N_1 \cup N_2 \cup \ldots \cup N_{d-1}$ be a partition of N into $d-1$ equal parts, the first consisting of the first $(d-1)$ points of N, the second consisting of the next $(d-1)$ points of N and so on. Let H_i be the hyperplane of R^{d-1} containing the points in N_i and let $p(N)$ be the intersection point of these hyperplanes. By the generic position of our points each H_i is uniquely defined and $p(N)$ is a point. Also, different choices of the N_i yield different points $p(N)$. Moreover, as is easily checked, $p(N) \in Q^{d-1}$ (since it is a solution of a linear system of equations with rational coefficients). We now add to M another labelled set of $n - [n/\log n]$ points, each being one of the $p(N)$. There are

$$\binom{[n/\log n]}{(d-1)^2}^{n-[n/\log n]} = h(n, d)$$

possibilities for this construction, each supplying a set of n labelled points in Q^{d-1}. If $x_i = (x_{i1}, \ldots, x_{i(d-1)})$ are the coordinates of the i-th point, put $y_i = (x_{i1}, \ldots, x_{i(d-1)}, 1)$. The y_i form a representation of a rational matroid on n points with rank d, in which $\{i_1, \ldots, i_d\}$ is independent if $x_{i_1}, \ldots, x_{i_d}$ span R^{d-1}, *i.e.*, are not contained in a hyperplane in R^{d-1}. It is easy to check that all our $h(n, d)$ labelled sets of points supply distinct matroids, and the lower bound of inequality (3.1) follows.

§4. *The Number of Configurations.* If $(P_o, P_1, \ldots, P_d)$ is a sequence of $d+1$ points in R^d, with $P_i = (x_{i1}, \ldots, x_{id})$ for each i, we say they have *positive orientation*, written $P_o \ldots P_d > 0$, if $\det(x_{ij})_{0 \le i,j \le d} > 0$ where $x_{io} = 1$ for each i. The conditions $P_o \ldots P_d < 0$ and $P_o \ldots P_d = 0$ are defined similarly. The *order type* of a configuration C of n labelled points $P_1, P_2, \ldots, P_n$ in R^d is a function w from the set of all $(d+1)$-subsets of $\{1, 2, \ldots, n\}$ to $\{0, \pm 1\}$, where for $S = \{i_0, i_1, \ldots, i_d\}$ with $1 \le i_0 < i_1 < \ldots < i_d \le n$, $w(S) = +1$ if $P_{i_0} \ldots P_{i_d} > 0$, $w(S) = -1$ if $P_{i_0} \ldots P_{i_d} < 0$, and $w(S) = 0$ if $P_{i_0} \ldots P_{i_d} = 0$. The configuration is *simple* if $w(S) \ne 0$ for every such S. Notice that $w(S)$ is just sign $\det(x_{i_k j})$, $0 \le k, j \le d$, where $P_{i_k} = (x_{i_k 1}, \ldots, x_{i_k d})$ and $x_{i_k o} = 1$ for $0 \le k \le d$. The order type of a configuration C of points is sometimes known as the oriented matroid structure determined by C. (See [GP1] for more details.) Let $t(n, d)$ denote the number of distinct order types of configurations of n labelled points in R^d, and let $t_s(n, d)$ denote the number of order types of simple configurations of n labelled points in R^d. In [GP1] Goodman and Pollack showed that $t_s(n, d) \le n^{n^d}$. Very recently [GP2], they found a clever way of using a theorem of Milnor (mentioned in Remark 2.3 above) to prove that

$$t_s(n, d) \le n^{d(d+1)n}. \tag{4.1}$$

This bound has several interesting applications (see [GP2], [AFR]). Goodman and Pollack also showed that

$$t_s(n, d) \geq n^{d^2 n(1+O(\log d/\log n))}. \tag{4.2}$$

Here we apply Theorem 2.2 to show that the total number of order types $t(n, d)$ is not much bigger than $t_s(n, d)$. In fact we also slightly improve (4.1) and prove the following theorem, which supplies very sharp estimates for the asymptotic behaviour of both functions $t(n, d)$ and $t_s(n, d)$, at least for n much greater than d.

THEOREM 4.1.

$$(n/d)^{d^2 n + O(d^2 n \frac{\log d}{\log n})} \leq t_s(n, d) \leq t(n, d)$$
$$\leq (n/d)^{d^2 n(1+O(\frac{1}{\log(n/d)} + \frac{\log\log(n/d)}{d \log(n/d)}))},$$

Proof. The lower bound is just inequality (4.2). (Notice that $d^{d^2 n} = n^{d^2 n \log d/\log n}$). To get the upper bound, notice that $t(n\, d)$ is just the number of sign patterns of $\binom{n}{d+1}$ polynomials of degree d in the dn real variables $(x_{i1}, \ldots, x_{id})$, which are the coordinates of the i-th point. The polynomials are just all the determinants $\det(x_{i_k j})$, $0 \leq k, j \leq d$, where $x_{i_k o} = 1$ for all k and $1 \leq i_o < i_1 \ldots < i_d \leq n$. Split these polynomials into $h = [n/\log(n/d)]$ classes, each of total degree

$$\sim \binom{n}{d+1} d \frac{\log(n/d)}{n} \leq \frac{n^d}{d!} \log(n/d).$$

Apply Theorem 2.2 with

$$h = [n/\log(n/d)], \qquad k = 4\frac{n^d \log(n/d)}{d!}$$

(and dn variables) to conclude that

$$t(n, d) \leq (2k)^{dn + \frac{n}{\log(n/d)}} \leq \left(8\left(\frac{en}{d}\right)^d \log(n/d)\right)^{dn + \frac{n}{\log(n/d)}}$$
$$= (n/d)^{d^2 n(1+O(\frac{1}{\log(n/d)} + \frac{\log\log(n/d)}{d \log(n/d)}))}.$$

Theorem 4.1 implies that if d and n vary and $\log d/\log n \to 0$, then both $t_s(n, d)$ and $t(n, d)$ have the form

$$(n/d)^{d^2 n(1+o(1))} = n^{d^2 n(1+o(1))}. \tag{4.3}$$

In particular we obtain the following.

COROLLARY 4.2. *For fixed $d > 2$, as $n \to \infty$*

$$t_s(n, d) = (n/d)^{d^2 n(1+o(1))} = n^{d^2 n(1+o(1))}$$

and

$$t(n, d) = (n/d)^{d^2 n(1+o(1))} = n^{d^2 n(1+o(1))}.$$

With a somewhat more careful computation one can extend the range in which $t_s(n, d)$ and $t(n, d)$ have the form (4.3). The most important cases, however, seem to be these covered by Corollary 4.2.

Remark 4.3. By a similar application of Theorem 2.2 one can easily show that for every n and d,

$$t_s(n, d) \leq t(n, d) \leq 2^{n^3 + O(n^2)}.$$

We omit the details.

Remark 4.4. A *linear space P* on a set $N = \{1, 2, \ldots, n\}$ of n points is a family L of subsets (called lines) $l_1, \ldots, l_t$ of N, such that every two points belong to a unique line. If P can be realized by embedding the points in the plane R^2 where L is the set of all maximal collinear subsets of N, P is called a *representable linear space.* By Corollary 4.2 (with $d = 2$) the number of distinct representable linear spaces on n labelled points is at most $n^{(4+o(1))n}$. On the other hand, it is easy to see that the number of distinct linear spaces on n labelled points is much bigger—it is at least $2^{(n^2/6)+O(n)}$. Indeed, take a fixed Steiner triple system on $\geq n-3$ of our points and let

$$B_1, B_2, \ldots, B_m \qquad (m = n^2/6 + O(n))$$

be the set of its blocks. For every subset F of the set of these blocks, we define a linear space on N whose lines are all the blocks in F together with all pairs of points $\{i, j\}$ that do not lie in any common block of F. This supplies $2^{(n^2/6)+O(n)}$ distinct linear spaces on N.

There are obvious generalizations of this remark to higher dimensions.

§5. *The Number of Convex Polytopes.* Let $c(n, d)$ denote the number (up to combinatorial isomorphism) of d-polytopes on n labelled vertices and let $c_s(n, d)$ be the number of simplicial d-polytopes on n labelled vertices. The problem of determining or estimating $c(n, d)$ and $c_s(n, d)$ has a long history, part of which, including some previous results, is outlined in Section 1. Very recently, Goodman and Pollack [GP2] used their bound for $t_s(n, d)$ (see inequality (4.1) above) to show that

$$c_s(n, d) \leq t_s(n, d) \leq n^{d(d+1)n}.$$

This follows immediately from the fact that the two vertex sets of two inequivalent simplicial polytopes with vertices in general position in R^d form distinct simple configurations. Indeed, one can easily check (see, *e.g.*, [GP1]) that the order type of a configuration that spans R^d determines which sets of its points lie on supporting hyperplanes of its convex hull. This also holds for non-simple configurations. Hence, the order type of a configuration on a set $N = \{1, 2, \ldots, n\}$ of n points in R^d which is the set of vertices of a convex polytope P determines its facets and thus its complete combinatorial type. This implies that $c_s(n, d) \leq c(n, d) \leq t(n, d)$, and by Theorem 4.1 and the remarks following it we obtain:

THEOREM 5.1.

$$c_s(n, d) \leq c(n,d) \leq (n/d)^{d^2 n(1+O(\frac{1}{\log(n/d)} + \frac{\log\log(n/d)}{d \log(n/d)}))}.$$

In particular, if $n/d \to \infty$ then

$$c_s(n, d) \leq c(n, d) \leq (n/d)^{d^2 n(1+O(1))}. \tag{5.1}$$

Furthermore, for every d and n

$$c(n,d) \le 2^{n^3+O(n^2)},$$

and hence the total number of polytopes on n points is at most $2^{n^3+O(n^2)}$.

As mentioned in [GP2], one can show that the estimate given for $c_s(n,d)$ and $c(n,d)$ by (5.1) is not so far from the truth. Indeed, one can show that for $n \ge 2d$

$$c_s(n,d) \ge \left(\frac{n-d}{d}\right)^{nd/4}. \tag{5.2}$$

To see this, take a cyclic polytope P on the first $n/2$ points (see [Gr]). Then P has $\ge ((n-d)/d)^{d/2}$ facets. Put the last $n/2$ labelled points, in all possibilities, each one "close" to a facet of P. This implies (5.2). In [Sh] Shemer proved that even the number of (unlabelled) distinct *neighbourly* polytopes with n points in R^d is $\ge n^{c_d n}$, where $\lim_{d\to\infty} c_d = 1/2$. By (5.1) this shows that for fixed $d(\ge 4)$ this number is of roughly the same order of magnitude as the total number of d-polytopes on n vertices; quite a surprising fact (especially in view of Motzkin's old conjecture [Mo] that there is only one neighbourly d-polytope on n points).

We conclude our paper by noting that, as observed by G. Kalai, both Theorem 4.1 and Theorem 5.1 can be somewhat improved for the case $n-d = o(n)$. In fact, by being more careful we can prove that, *for fixed* $\beta > 0$

$$c_s(d+\beta,d) \le c(d+\beta,d) \le t(d+\beta,d) \le n^{\beta(\beta-1)d(1+o(1))}.$$

We omit the details.

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Note added in proof. A result similar to Theorem 2.2 but for the number of sign patterns that consist of ±1 terms only was proved by Warren in *Trans. Amer. Math. Soc.*, 133 (1968), 167–178.

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