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Oriented Matroids and Hyperplane Transversals

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We prove that a finite family $\mathscr{A}$ of compact connected sets in $\mathbf{R}^d$ has a hyperplane transversal if and only if for some k, $0 < k < d$, there exists an acyclic oriented matroid of rank $k+1$ on $\mathscr{A}$ such that every $k+2$ sets in $\mathscr{A}$ have an oriented k-transversal which meets the sets consistently with that oriented matroid. © 1996 Academic Press, Inc.

Let $\mathscr{A}$ be a finite family of compact convex sets in $\mathbf{R}^d$. A *k-transversal* for $\mathscr{A}$ is an affine subspace of dimension k which intersects every member of $\mathscr{A}$. A *hyperplane transversal* for $\mathscr{A}$ is a hyperplane which intersects every member of $\mathscr{A}$. Under what conditions does the family $\mathscr{A}$ have a hyperplane transversal?

Hadwiger in 1957 gave the first such conditions for line transversals in the plane [3]. He noted that a directed line transversal intersects pairwise disjoint sets in a specific order. He used this ordering to give conditions for the existence of line transversals.

THEOREM 1 (Hadwiger's Transversal Theorem [3]). *A finite family $\mathscr{A}$ of pairwise disjoint compact convex sets in the plane has a line transversal if and only if there is a linear ordering of $\mathscr{A}$ such that every three convex sets have a directed line transversal meeting them consistently with that ordering.*

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In 1988 Goodman and Pollack generalized Hadwiger's theorem and proof to hyperplane transversals [2]. Although they did not do so, we restate their theorem in the language of oriented matroids.

THEOREM 2 (Goodman and Pollack [2]). *A $(d-2)$-separated family $\mathscr{A}$ of compact convex sets in $\mathbf{R}^d$ has a hyperplane transversed if and only if there is an acyclic oriented matroid of rank d on $\mathscr{A}$ such that every $d+1$ of the sets have a hyperplane transversal meeting them consistently with that oriented matroid.*

A family of convex sets is k-separated if no subset of size $k+2$ has a k-transversal. A rank r acyclic oriented matroid on $\mathscr{A}$ is a set of orientations on r-tuples of $\mathscr{A}$ which form an acyclic oriented matroid with elements $\mathscr{A}$; i.e., an acyclic oriented matroid on $\mathscr{A}$ is defined by a mapping $\chi: \mathscr{A}^r \rightarrow \{-1, 0, 1\}$, called a *chirotope*, which satisfies certain axioms. (See [1] for axioms defining a chirotope and much other information on oriented matroids.) An oriented k-transversal meets a family of connected sets consistently with a given acyclic oriented matroid if one can choose a point from the intersection of each set and the k-transversal such that the orientation of every $(k+1)$-tuple of points matches the orientation of the corresponding $(k+1)$-tuples of $\mathscr{A}$.

Goodman and Pollack originally formulated Theorem 2 using *order types*, or realizable oriented matroids. An acyclic oriented matroid of rank $k+1$ is *realizable* if it can be represented as the set of orientations of a set of points in $\mathbf{R}^k$. As was subsequently noted, the proof given by Goodman and Pollack does not depend upon the oriented matroid being realizable.

Wenger showed that the pairwise disjointness condition could be dropped from Hadwiger's theorem on line transversal in the plane, giving a new topological proof of Hadwiger's theorem [7]. Subsequently, Pollack and Wenger proved Theorem 2 without the separated condition [6].

THEOREM 3 (Pollack and Wenger [6]). *A family $\mathscr{A}$ of compact connected sets in $\mathbf{R}^d$ has a hyperplane transversal if and only if for some k, $0 \leqslant k < d$, there is a realizable acyclic oriented matroid of rank $k+1$ such that every $k+2$ of the sets have a k-transversal meeting them consistently with that oriented matroid.*

This theorem is stated for compact connected sets instead of just compact convex sets. Note that a hyperplane h intersects a connected set a if and only h intersects the convex hull of a.

The proof in [6] depended upon the oriented matroid being realizable. Gil Kalai asked whether Theorem 3 still held when that assumption was dropped. In this paper we answer that question in the affirmative, giving the following theorem which generalizes all the theorems listed above.

THEOREM 4. *A family $\mathscr{A}$ of compact connected sets in $\mathbf{R}^d$ has a hyperplane transversal if and if for some k, $0 \leqslant k < d$, there is an acyclic oriented matroid of rank $k+1$ such that every $k+2$ of the sets have a k-transversal meeting them consistently with that oriented matroid.*

The proof of Theorem 3 in [6] uses a realization of the oriented matroid to apply the Borsuk–Ulam theorem on antipodal mappings from a sphere $\mathbf{S}^k$ to $\mathbf{R}^k$. The Borsuk–Ulam theorem states that if there exists a continuous antipodal mapping from $\mathbf{S}^k$ to $\mathbf{S}^d$, then $k \leqslant d$. To prove our result we will first prove a combinatorial lemma which will play the same role in our proof that the Borsuk–Ulam theorem played in [6].

Recall that if E is a set of points in affine Euclidean space, a *Radon partition* of E is a partition (A, B) of E such that $\text{conv}(A) \cap \text{conv}(B) \neq \varnothing$. This notion can be expressed in terms of oriented matroids, a fact we will use in the proof of Theorem 4.

One way of describing an oriented matroid of rank r on $\mathscr{A}$ is as a set $\mathscr{A}$ together with a collection of *signed circuits* of $\mathscr{A}$, i.e., a collection of subsets of $\mathscr{A}$ of size at most $r+1$ with a sign "+" or "−" assigned to each element of each subset. (See [1].) For example, if $\mathscr{A}$ is a finite set of points in affine rank r space, consider the set $\mathscr{R}$ of all minimal Random partitions (A, B) with $A, B \subset \mathscr{A}$. Then $\{A^+B^- : (A, B) \in \mathscr{R}\}$ is the set of signed circuits of an oriented matroid. Such an oriented matroid will be *acyclic*, i.e., it will have no signed circuits of the form A^+.

The Folkman–Lawrence topological respresentation theorem gives another representation for oriented matroids. An oriented matroid M of rank r can be defined by an *arrangement of oriented pseudospheres* $\{D_a : a \in \mathscr{A}\}$ on $\mathbf{S}^{r-1}$, each dividing $\mathbf{S}^{r-1}$ into open hemispheres D_a^+ and D_a^-. An *arrangement of oriented pseudospheres* is a set of topological spheres (homeomorphic to $\mathbf{S}^{r-2}$) with intersection properties similar to those of great circles on $\mathbf{S}^r$. (See [1].) The signed circuits of the oriented matroid are given by minimal sets of hemispheres whose intersection is empty, with signs determined by whether the positive, D_a^+, or negative, D_a^-, hemisphere is used. The oriented matroid is acyclic if $\bigcap_{a \in \mathscr{A}} D_a^+ \neq \varnothing$. For any $a \in \mathscr{A}$, the *deletion* $M \backslash a$ of a from M is the oriented matroid represented by the pseudospheres in $\mathscr{A} \backslash \{a\}$. The *contraction* of M on a, M/a, is the oriented matroid of rank (rank(M)-1) represented by the set of pseudospheres $\{D_b \cap D_a : b \in \mathscr{A} \backslash \{a\}\}$.

Let M be an oriented matroid on $\mathscr{A}$ represented by an arrangement of pseudospheres $\{D_a : a \in \mathscr{A}\}$ on $\mathbf{S}^{r-1}$, each dividing $\mathbf{S}^{r-1}$ into open hemispheres D_a^+ and D_a^-. We define the *nerve* N_M of M to be the nerve of this collection of hemispheres. That is, if $\Gamma = \{D_a^+ : a \in \mathscr{A}\} \cup \{D_a^- : a \in \mathscr{A}\}$, then $N_M = \{\phi \subseteq \Gamma : \bigcap_{\gamma \in \phi} \gamma \neq \varnothing\}$. N_M is a simplicial complex. Different representations of M by arrangements of pseudospheres lead to isomorphic copies of N_M. M is acyclic if and only if $T=\{D_a^+ : a\in\mathscr{A}\}$ and $-T=\{D_a^- : a\in\mathscr{A}\}$ are simplices of N_M.

Let $f\colon \mathbf{S}^d\to N_M$ be a function from points of $\mathbf{S}^d$ to simplices of N_M. f is *lower semi-continuous* if every point $p\in\mathbf{S}^d$ has a neighborhood U_p such that $f(p)\subseteq f(p')$ for all $p'\in U_p$. That is, f is lower semi-continuous if it is continuous with respect to the usual topology on $\mathbf{S}^d$ and the topology on N_M generated by open stars. For each face $\phi\in N_M$, let $-\phi=\{D_a^- : D_a^+\in\phi\}\cup\{D_a^+ : D_a^-\in\phi\}$ be the "antipodal" face to ϕ. f is *antipodal* if $f(p)=\phi$ implies $f(-p)=-\phi$.

We can now state our analog to the Borsuk–Ulam theorem. $\bar{T}$ is the closure of T, the set of all subsets of T.

LEMMA 1. *Let M be an acyclic oriented matroid with nerve N_M. If $f\colon \mathbf{S}^d\to N_M$ is a lower semi-continuous antipodal map from points on $\mathbf{S}^d$ to simplices of N_M, then $d\leqslant \operatorname{rank}(M)-1$. If addition $f^{-1}(\bar{T})=\varnothing$, then $d\leqslant\operatorname{rank}(M)-2$.*

The proof of the first half of Lemma 1 involves constructing an antipodal simplicial map from N_M to $\mathbf{S}^{\operatorname{rank}(M)-1}$, and composing it with the map f to get an antipodal map from $\mathbf{S}^d$ to $\mathbf{S}^{\operatorname{rank}(M)-1}$. It then would follow, by the Borsuk–Ulam theorem, that $d\leqslant\operatorname{rank}(M)-1$. For the second half of Lemma 1, we would like to similarly construct a map from "$N_M\backslash\{\bar{T},-\bar{T}\}$" to $\mathbf{S}^{\operatorname{rank}(M)-2}$. However, $N_M\backslash\{\bar{T},-\bar{T}\}$ is not a simplicial complex, since every vertex of N_M is in $\bar{T}$ or $-\bar{T}$. Instead, we form a cell complex from pairs of the positive and negative vertices of N_M.

Let $\mathscr{X}$ be a simplicial complex whose vertices are partitioned into two sets V and W. (For instance, the vertices of N_M are partitioned into $\{D_a^+ : a\in\mathscr{A}\}$ and $\{D_a^- : a\in\mathscr{A}\}$.) For every face $\phi\in\mathscr{X}$, where $\phi\nsubseteq V$ and $\phi\nsubseteq W$, let $\phi^\#=\{(v,w) : v\in\phi\cap V,\ w\in\phi\cap W\}$. Form a cell complex $\mathscr{X}^\#$ whose vertices are $V^\#=\{\{v,w\} : v\in V,\ w\in W\}$ and whose cells are $\{\phi^\# : \phi\in\mathscr{X},\ \phi\nsubseteq V$ and $\phi\nsubseteq W\}$. This is a piecewise linear cell complex, although not a simplicial one. If $|\phi|$ is a geometric realization of ϕ, then $\phi^\#$ is realized as the intersection of $|\phi|$ with a hyperplane separating $\phi\cap V$ from $\phi\cap W$. Note that the dimension of $\phi^\#$ is one less than the dimension of ϕ. Thus $N_M^\#$ is the complex with vertices $\{\{D_a^+, D_b^-\} : a, b\in\mathscr{A}\}$ and cells $\{\phi^\# : \phi\in N_M,\ \phi\nsubseteq T$, and $\phi\nsubseteq -T\}$.

As above, let $-\phi^\#=\{D_a^- : D_a^+\in\phi^\#\}\cup\{D_a^+ : D_a^-\in\phi^\#\}$ be the "antipodal" face to $\phi^\#$. A map $f\colon\mathbf{S}^d\to N_M^\#$ is *antipodal* if $f(p)=\phi^\#$ implies $f(-p)=-\phi^\#$.

$N_{M\backslash a}$ and $N_{M/a}$ are the nerves of the oriented matroids $M\backslash a$ and M/a, respectively. $N_{M\backslash a}$ is a subcomplex of N_M consisting of $\phi\in N_M$, where $D_a^+\notin\phi$ and $D_a^-\notin\phi$. $N_{M/a}$ is a subcomplex of $N_{M\backslash a}$ consisting of $\phi\in N_{M\backslash a}$, where $\phi \cup \{D_a^+\} \in N_M$ and $\phi \cup \{D_a^-\} \in N_M$. $N_{M/a}^{\#}$ is a subcomplex of $N_{M\backslash a}^{\#}$, which is a subcomplex of $N_M^{\#}$.

The open sets in Γ give an acyclic cover of the sphere $\mathbf{S}^{\operatorname{rank}(M)-1}$. Thus, in light of Čech theory, the following result is not surprising.

LEMMA 2. *If M is an acyclic oriented matroid of rank r, then there is an antipodal simplicial map from a refinement of N_M to $\mathbf{S}^{r-1}$ and an antipodal simplicial map from a refinement of $N_M^{\#}$ to $\mathbf{S}^{r-2}$.*

Proof of Lemma 2. We first give an antipodal PL deformation retract from N_M to $\mathbf{S}^{r-1}$. That is, we give a sequence of antipodal pairs of elementary collapses from N_M to a triangulation of $\mathbf{S}^{r-1}$. Since an elementary collapse is a PL map, this sequence of collapses is a PL map. Thus the collapse from N_M to $\mathbf{S}^{r-1}$ can be realized as an antipodal simplicial map from a refinement of N_M to a triangulation of $\mathbf{S}^{r-1}$. (This part of the proof does not rely on M being acyclic.)

The construction is by induction on r. If $r=1$, then N_M consists of two disjoint simplices, which retracts to $\mathbf{S}^0$. Now assume we have such a sequence of collapses for any oriented matroid of rank less than r. Choose some $e \in \mathscr{A}$. We will collapse N_M to $\mathbf{S}^0 * N_{M/e}$. Our induction hypothesis then collapses this to $\mathbf{S}^{r-1}$.

If ϕ is a maximal simplex of N_M, then either D_e^+ or D_e^- is a vertex of ϕ. On the other hand, if ψ is a simplex of $N_{M\backslash e}$, then either $\psi \cup \{D_e^+\}$ or $\psi \cup \{D_e^-\}$ or both are simplices of N_M. Collapse all maximal simplices $\phi = \psi \cup \{D_e^+\} \in N_M$, where $\psi \cup \{D_e^-\} \notin N_M$, $\psi \in N_{M\backslash e}$, through the face ψ. Similarly collapse all maximal simplices $\phi = \psi \cup \{D_e^-\} \in N_M$, where $\psi \cup \{D_e^+\} \notin N_M$, $\psi \in N_{M\backslash e}$, through the face ψ. Note if $\phi = \psi \cup \{D_e^+\} \in N_M$ and $\psi \cup \{D_e^-\} \notin N_M$, then $-\phi = -\psi \cup \{D_e^-\} \in N_M$ and $-\phi \cup \{D_e^+\} \notin N_M$, so these collapses come in antipodal pairs. Repeat these collapses until no such simplices remain. These collapses reduce $N_{M\backslash e}$ to $N_{M/e}$ and N_M to the join $\{\{D_e^+\}, \{D_e^-\}\} * N_{M/e}$. By induction $N_{M/e}$ collapses to $\mathbf{S}^{r-2}$, and so the join above collapses to $\mathbf{S}^{r-1}$.

To prove the second statement of the lemma, we give a sequence of antipodal pairs of collapses from $N_M^{\#}$ to $\{\mathbf{S}^0 * N_{M/e}\}^{\#}$. This last term is isomorphic to a PL refinement of $N_{M/e}$. The argument above gives a further sequence of collapses to $\mathbf{S}^{r-2}$.

If $\phi^{\#}$ is a maximal cell of $N_M^{\#}$, $\phi \in N_M$, then either D_e^+ or D_e^- is a vertex of ϕ. On the other hand, if $\psi^{\#}$ is a cell of $N_{M\backslash e}^{\#}$, $\psi \in N_{M\backslash e}$, then either $\{\psi \cup \{D_e^+\}\}^{\#}$ or $\{\psi \cup \{D_e^-\}\}^{\#}$ or both are cells of $N_M^{\#}$.

As above, collapse all maximal cells $\{\psi \cup \{D_e^+\}\}^{\#} \in N_M^{\#}$, where $\{\psi \cup \{D_e^-\}\}^{\#} \notin N_M^{\#}$, $\psi \in N_{M\backslash e}$, through the face $\psi^{\#}$. (Since M is acyclic, $\psi^{\#}$ is a face of $N_M^{\#}$.) Similarly, collapse all maximal cells $\{\psi \cup \{D_e^-\}\}^{\#} \in N_M^{\#}$, where $\{\psi \cup \{D_e^+\}\}^{\#} \notin N_M^{\#}$, $\psi \in N_{M\backslash e}$, through the face $\psi^{\#}$. Again these collapses come in antipodal pairs. Let J_e be the join $\{\{D_e^+\}, \{D_e^-\}\} * N_{M/e}$. These collapses reduce $N^{\#}_{M\backslash e}$ to $N^{\#}_{M/e}$ and $N^{\#}_{M}$ to $J^{\#}_{e}$.

We now claim $J^{\#}_{e}$ is isomorphic to a refinement of $N_{M/e}$. For each $\phi \in N_{M/e}$ there are three possibilities:

- If $\phi \subseteq T$, then $\{\phi \cup \{D_e^-\}\}^{\#}$ is a cell of $J^{\#}_{e}$ of the same dimension as ϕ. The vertices of ϕ are $\{\{D_a^+, D_e^-\} : D_a^+ \in \phi\}$ and its faces are $\{\{\psi \cup \{D_e^-\}\}^{\#} : \psi \subseteq \phi\}$. Thus, the face lattice of $\{\phi \cup \{D_e^-\}\}^{\#}$ is isomorphic to the face lattice of ϕ. Neither $\phi^{\#}$ nor $\{\phi \cup \{D_e^+\}\}^{\#}$ are cells of $J^{\#}_{e}$.

- Similarly, if $\phi \subseteq -T$, then $\{\phi \cup \{D_e^-\}\}^{\#}$ is a cell of $J^{\#}_{e}$ with face lattice isomorphic to the face lattice of ϕ. Again neither $\phi^{\#}$ nor $\{\phi \cup \{D_e^-\}\}^{\#}$ are cells of $J^{\#}_{e}$.

- For all other ϕ, the sets $\phi^{\#}$, $\{\phi \cup \{D_e^-\}\}^{\#}$, and $\{\phi \cup \{D_e^-\}\}^{\#}$ are cells in $J^{\#}_{e}$ whose union is $\{\{\{D_e^+\}, \{D_e^-\}\} * \phi\}^{\#}$, a PL ball of the same dimension as ϕ. Cutting a realization $|\phi|$ of ϕ by a hyperplane h separating $\phi \cap T$ from $\phi \cap -T$ gives a refinement $\tilde{\phi}$ of ϕ which is isomorphic to $\{\{\{D_e^+\}, \{D_e^-\}\} * \phi\}^{\#}$. The original vertices of $|\phi|$ corresponding to D_a^+ or D_b^- map to vertices $\{D_a^+, D_e^-\}$ or $\{D_b^-, D_e^+\}$, respectively, of $\{\{\{D_e^+\}, \{D_e^-\}\} * \phi\}^{\#}$. The new vertices of $h \cap |\phi|$ correspond to the intersection of h and the realization of an edge (D_a^+, D_b^-) of ϕ. These vertices map to vertices $\{D_a^+, D_b^-\}$ of $\{\{\{D_e^+\}, \{D_e^-\}\} * \phi\}^{\#}$.

Conversely, every cell of $J^{\#}_{e}$ arises in exactly one of these cases. Thus $J^{\#}_{e}$ is isomorphic to a refinement of $N_{M/e}$. ∎

The *barycentric subdivision* $\text{Bar}(\mathscr{X})$ of a simplicial complex $\mathscr{X}$ is the simplicial complex of all chains in the face lattice of $\mathscr{X}$. $\text{Bar}(\mathscr{X})$ is a refinement of $\mathscr{X}$, with one vertex $\langle \sigma \rangle$ for every simplex σ of $\mathscr{X}$.

Proof of Lemma 1. Let M be an acyclic oriented matroid with nerve N_M. Let $f: \mathbf{S}^d \to N_M$ be a lower semi-continuous antipodal map from points on $\mathbf{S}^d$ to simplices of N_M. For each $p \in \mathbf{S}^d$ choose a suitably small convex ball, U_p, around p so that:

1. $f(p) \subseteq f(p')$ for every $p' \in \text{cl}(U_p)$,
2. $U_{-p} = -U_p$, and
3. any non-empty intersection $\bigcap_{p \in P} U_p$ is a ball.

Since $\mathbf{S}^d$ is compact, we can choose a finite subset $\mathscr{W}$ of these neighborhoods covering $\mathbf{S}^d$ with the additional condition that if $U_p \in \mathscr{W}$ then $U_{-p} \in \mathscr{W}$. The closed sets $\{\text{cl}(U_p) : U_p \in \mathscr{W}\}$ give a decomposition of

$\mathbf{S}^d$ into regular cells. The barycentric subdivision of this cell decomposition is a simplicial complex Σ^d. For each face $\sigma \in \Sigma^d$, there is an antipodal face $-\sigma \in \Sigma^d$.

For every face σ of Σ^d, let $\mathscr{W}(\sigma) \subseteq \mathscr{W}$ be the sets of $\mathscr{W}$ containing σ. Define $\tilde{f}: \mathrm{Bar}(\Sigma^d) \rightarrow \mathrm{Bar}(N_M)$ as

$$\tilde{f}(\langle \sigma \rangle) = \left\langle \bigcup_{U_p \in \mathscr{W}(\sigma)} f(p) \right\rangle.$$

For all $p' \in \sigma$, we have

$$\bigcup_{U_p \in \mathscr{W}(\sigma)} f(p) \subseteq f(p').$$

For any face $\sigma' \subseteq \sigma$, we have $\mathscr{W}(\sigma') \subseteq \mathscr{W}(\sigma)$, and so

$$\bigcup_{U_p \in \mathscr{W}(\sigma')} f(p) \subseteq \bigcup_{U_p \in \mathscr{W}(\sigma)} f(p).$$

Thus $\tilde{f}$ defines a simplicial map from $\mathrm{Bar}(\Sigma^d)$ to $\mathrm{Bar}(N_M)$. Since it takes $\langle -\sigma \rangle$ to $-\langle \sigma \rangle$, it is antipodal as well.

Lemma 2 gives an antipodal simplicial map from an antipodal subdivision of N_M to $\mathbf{S}^{\mathrm{rank}(M)-1}$. This induces an antipodal simplicial map from an antipodal subdivision $\mathrm{Bar}'(N_M)$ of $\mathrm{Bar}(N_M)$ to a triangulation $\Sigma^{\mathrm{rank}(M)-1}$ of $\mathbf{S}^{\mathrm{rank}(M)-1}$. Thus we have antipodal simplicial maps

$$\mathrm{Bar}'(\Sigma^d) \rightarrow \mathrm{Bar}'(N_M) \rightarrow \Sigma^{\mathrm{rank}(M)-1}$$

where $\mathrm{Bar}'(\Sigma_d)$ and $\mathrm{Bar}'(N_M)$ are suitably chosen antipodal subdivisions of $\mathrm{Bar}(\Sigma_d)$ and $\mathrm{Bar}(N_M)$, respectively.

The composition of these maps gives an antipodal map from $\mathrm{Bar}'(\Sigma^d)$ to $\Sigma^{\mathrm{rank}(M)-1}$. This map defines a continuous antipodal map from $\mathbf{S}^d$ to $\mathbf{S}^{\mathrm{rank}(M)-1}$. By the classical Borsuk–Ulam theorem, d is less than or equal to $\mathrm{rank}(M)-1$. This proves the first part of our lemma.

If $f^{-1}(\bar{T}) = \varnothing$, then consider the map $f^{\#}$ taking $p \in \mathbf{S}^d$ to $\{f(p)\}^{\#} \in N_M^{\#}$. Define $\mathscr{W}$, Σ^d, and $\tilde{f}^{\#}: \Sigma^d \rightarrow \mathrm{Bar}(N_M^{\#})$ as above. Lemma 2 gives an antipodal simplicial map from an antipodal subdivision $\mathrm{Bar}'(N_M^{\#})$ of $\mathrm{Bar}(N_M^{\#})$ to its subcomplex $\mathbf{S}^{\mathrm{rank}(M)-2}$. Thus we have maps

$$\mathrm{Bar}'(\Sigma^d) \rightarrow \mathrm{Bar}'(N_M^{\#}) \rightarrow \Sigma^{\mathrm{rank}(M)-2}.$$

The same argument as before then gives $d \leqslant \mathrm{rank}(M)-2$. ∎

The proof of Theorem 4 now follows the same lines as the proof of Theorem 3 in [6].

Proof of Theorem 4. Let $\mathscr{A}=\{a_1, a_2, ..., a_n\}$ be a finite family of compact connected sets in $\mathbf{R}^d$. If $\mathscr{A}$ has a hyperplane transversal τ, then choose a point p_i from each intersection $a_i \cap \tau$, $a_i \in \mathscr{A}$. These points span some oriented affine subspace of dimension k. The set of orientations of the points p_i in that affine subspace generate an acyclic oriented matroid of rank $k+1$ on the points $\{p_i\}$ and a corresponding oriented matroid on $\mathscr{A}=\{a_i\}$. This proves the necessary condition.

Assume there is an acyclic oriented matroid M of rank $k+1$ such that every $k+2$ of the sets have a k-transversal meeting them consistently with that oriented matroid. Represent this oriented matroid by an arrangement of oriented pseudospheres on $\mathbf{S}^k$, each dividing $\mathbf{S}^k$ into open hemispheres D_a^+ and D_a^-. For each direction $v \in \mathbf{S}^{d-1}$, let $H^+(v)$ be the unique hyperplane with normal v such that every $a \in \mathscr{A}$ meets the negative closed half-space bounded by H and some $a \in \mathscr{A}$ is contained in the positive one. Let $H^-(v)$ equal $H^+(-v)$. Let $H(v)$ be the hyperplane perpendicular to v lying halfway between $H^+(v)$ and $H^-(v)$. The family $\mathscr{A}$ has a hyperplane transversal with normal v if and only if $H(v)$ is a hyperplane transversal of $\mathscr{A}$. Note that $H(v)$ and $H(-v)$ represent the same unoriented hyperplane.

Let $A^+(v)$ be the set of elements of $\mathscr{A}$ contained in the positive open half-space bounded by $H(v)$, and let $A^-(v)$ be the set of elements of $\mathscr{A}$ contained in the negative open half-space bounded by $H(v)$. Note that $\mathscr{A}^+(v)=\varnothing$ if and only if $A^-(v)=\varnothing$ and if and only if $H(v)$ is a hyperplane transversal to $\mathscr{A}$.

Define a map f which takes $v \in \mathbf{S}^{d-1}$ to

$$f(v)=\{D_a^+ : a \in A^+(v)\} \cup \{D_b^- : b \in A^-(v)\}.$$

Since $A^+(v) \neq \varnothing$ if and only if $A^-(v) \neq \varnothing$, we have $f^{-1}(\bar{T})=\varnothing$, and $f(v)=\varnothing$ if and only if $H(v)$ is a hyperplane transversal to $\mathscr{A}$. Assume that $f(v) \neq \varnothing$ for all v. We will then show that f is a lower semi-continuous, antipodal map from $\mathbf{S}^{d-1}$ to N_M and derive a contradiction using Lemma 1.

If $f(v)$ is not in N_M, then

$$\left(\bigcap_{a \in A^+(v)} D_a^+\right) \cap \left(\bigcap_{b \in A^-(v)} D_b^-\right)=\varnothing.$$

This implies (cf. 3.7.2 in [1]) that for some minimal $A_1 \subseteq A^+(v)$ and $A_2 \subseteq A^-(v)$, $|A_1|+|A_2| \leqslant k+2$, the intersection of the pseudohemispheres $\{D_a^+ : a \in A_1\} \cup \{D_a^- : a \in A_2\}$ is also empty. Thus $A_1^+ A_2^-$ is a signed circuit of M.

By the hypothesis of Theorem 4, $A_1 \cup A_2$ has a k-transversal τ' which meets it consistently with M. Thus there is a set of points P_1 from $\{\tau' \cap a : a \in A_1\}$ and a set of points P_2 from $\{\tau' \cap a : a \in A_2\}$ such that

$P_1^+ P_2^-$ form a signed circuit of M, i.e., $\text{conv}(P_1) \cap \text{conv}(P_2) \neq \emptyset$. However, $H(v)$ strictly separates P_1 from P_2, a contradiction. We conclude that $f(v) \in N_M$ for all v.

The map f is clearly antipodal. Its semi-continuity follows from the compactness of the sets in $\mathscr{A}$ and $H(v)$. Thus f is a lower-semi-continuous antipodal map from $\mathbf{S}^{d-1}$ to N_M where $f^{-1}(\bar{T}) = \emptyset$. By Lemma 1, $d-1 \leqslant \text{rank}(M) - 2 = k-1 \leqslant d-2$, a contradiction. Thus $f(v) = \emptyset$ for some v and $H(v)$ is a transversal of $\mathscr{A}$. This proves the sufficient condition. ∎

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