Three Constructions for the Bröcker-Scheiderer-Theorem

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September 2. 1997

1 Introduction

In 1988, Bröcker and Scheiderer solved independently a question raised by the first one. It gives information about the minimal number of inequalities needed to describe a basic open set of a real variety. It turns out that this minimal number depends only on the dimension of the real variety in question: in fact, these two numbers are the same. We state the Theorem of Bröcker and Scheiderer in terms of real spectra (see [9]):

Theorem 1.1 *Let A be any R-algebra of transcendence degree $d > 0$ over the real closed field R, then any basic open set in the real spectrum of A can be written with only d inequalities.*

There are several proofs of this theorem, but they use much machinery, like theory of fans, spaces of orderings or the Tsen-Lang-Theorem. The aim of this paper is to give three constructions which give firstly a constructive proof of Theorem 1.1 in some special cases and secondly some more information about the way the reduction depends on the given description as a basic open set.

We give an algorithmic solution for the first interesting case of Theorem 1.1, namely the description of convex planar polygons with the help of two inequalities. Afterwards, under some assumptions, we generalize this technique to a constructive solution of Theorem 1.1. Finally, a tricky proof shows that Theorem 1.1 can be reduced to the case of finitely generated algebras and that the functions of the reduction can be choosen in the algebra generated by the functions of the starting desription.

The complete proofs can be found in [1].

I want to thank to L. Mahé and E. Becker for their help during the realization of this work.

2 Convex Interpolation

In this chapter, we will show a useful theorem which will give us some information about the two-dimensional case of the Bröcker-Scheiderer-Theorem. Since the Convex Interpolation could be useful for other applications than Real Geometry, we give the proof in the general case, although we only use a special case, namely $n = 2$.

Definition 2.1 *Given m points $y_1, \ldots, y_m \in \mathbb{R}^n$, we will say that they lie in a* **convex position**, *if no point lies in the convex hull of the others.*

Theorem 2.2 *Given m points $y_1, \ldots, y_m \in \mathbb{R}^n$ in convex position, there is a non constant polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ with the following properties:*

- *The sets $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ and $\{x \in \mathbb{R}^n : p(x) > 0\}$ are convex.*
- $p(y_1) = \ldots = p(y_m) = 0$.

Remark 2.3 In [8], a similar theorem is given in the one-dimensional case. Our case is in some sense the multi-dimensional generalization of this theorem.

For the proof, we need the following lemma:

Lemma 2.4 *For a given $\epsilon > 0$, $y_1, \ldots, y_m \in \mathbb{R}^n$ in convex position and $i \in \{1, \ldots, m\}$ there is a polynomial $p_i \in \mathbb{R}[x_1, \ldots, x_n]$ with:*

- $p_i(y_i) = 1$
- $| p_i(y_j) | < \epsilon$ *for* $j \neq i$
- *The function p_i is convex on $\mathbb{R}^n$.*

Proof: Using the convex position of the points $y_1, \ldots, y_m$, we find a linear function $g : \mathbb{R}^n \to \mathbb{R}$ such that $g(y_i) > 0$ and $g(y_j) < 0$ for $j \neq i$. With the help of a linear transformation we can assume that $g(y_i) = 1$ and $-1 < g(y_j) < 1$ for $j \neq i$. For a suitable exponent $2s$, all the values $g(y_j)^{2s}$ for $j \neq i$ are of absolute value $< \epsilon$ and the function $p_i(x) := g(x)^{2s}$ is convex, so the Lemma is proven. □

Let $\epsilon > 0$ be a real number that will be fixed later on. For each y_i we choose a polynomial p_i as in the Lemma. We write

$$p = 1 - \left(\sum_{i=1}^{m} c_i p_i \right) \tag{1}$$

where the c_i are positive real numbers which are to be found. The conditions $p(y_j) = 0$ for $j = 1, \ldots, m$ yield to the system of equations:

$$\sum_{i=1}^{m} c_i p_i(y_j) = 1 \tag{2}$$

With the help of matrices this can be written as

$$\begin{pmatrix} p_1(y_1) & \ldots & p_m(y_1) \\ \vdots & \vdots & \vdots \\ p_1(y_m) & \ldots & p_m(y_m) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \tag{3}$$

If we write C for $\begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix}$ .1 for $\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$ and I for the matrix unity, the condition is

$$(I + A)C = 1 \tag{4}$$

where A is a matrix of which the entries on the diagonal are zero and the others of absolute value smaller than ϵ. We consider in $\mathbb{R}^m$ the maximum norm, that is

$$\left\| \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \right\| = \max_{i=1}^{m} | a_i | \tag{5}$$

We also consider the associated matrix norm. Is $A = (a_{ij})_{i=1,\dots,m;j=1,\dots,m}$ so

$$\| A \| = \max_{j=1}^{m} \sum_{i=1}^{m} | a_{ij} | \leq (m-1)\epsilon \tag{6}$$

We choose $0 < \epsilon < \frac{1}{2(m-1)}$, consequently we have $\| A \| < \frac{1}{2}$. Now, we consider the equation

$$(I + A)C = 1 \tag{7}$$

For $\| A \| < \frac{1}{2} < 1$, $(I + A)$ is invertible, hence there is a unique vector C that fulfils this equation. An elementary calculation shows that

$$\| C - 1 \| \leq \frac{\| A \|}{1 - \| A \|} < 1 \tag{8}$$

From the definition of the norm we conclude that $| c_i - 1 | < 1$, hence that $c_i > 0$ for $i = 1, \dots, m$.

The function p constructed in this way is the function we sought in Theorem 2.2: It vanishes on the y_i by construction and it is a linear combination with negative coefficients of convex functions, hence it is a concave function. Consequently, the sets $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ and $\{x \in \mathbb{R}^n : p(x) > 0\}$ are convex. This finishes the proof. □

Theorem 2.5 *Given linear polynomials $f_1, \dots, f_n \in \mathbb{R}[x, y]$, then there are two polynomials $f, g \in \mathbb{R}[x, y]$ such that*

$$\{(x, y) \in \mathbb{R}^2 : f_1(x, y) > 0, \dots, f_n(x, y) > 0\}$$

$$= \{(x, y) \in \mathbb{R}^2 : f(x, y) > 0, g(x, y) > 0\}$$

Constructive Proof: Let $S = \{(x,y) \in I\!R^2 : f_1(x,y) > 0,\ldots,f_n(x,y) > 0\}$. With the help of a projective change of coordinates, we can easily restrict to the case that S is bounded and non-empty. Assume furthermore that every of the functions $f_1,\ldots,f_n$ is necessaire for the description. S is a convex polygon and according to 2.2 we find a function g such that the set $\{g > 0\}$ is convex and has the vertices of S on the border. Finally, we set $f = \prod f_i$ and it is easy to show that $S = \{f > 0, g > 0\}$. □

3 Reductions with the help of symmetric means

Let s_i denote the ith elementary symmetric polynomial.

Proposition 3.1 *Let R be a real closed field. Then*

$$\{(x_1,\ldots,x_n) \in R^n : x_1 > 0,\ldots,x_n > 0\}$$

$$= \{(x_1,\ldots,x_n) \in R^n : s_1(x_1,\ldots,x_n) > 0,\ldots,s_n(x_1,\ldots,x_n) > 0\} \quad (9)$$

and

$$\{(x_1,\ldots,x_n) \in R^n : x_1 \geq 0,\ldots,x_n \geq 0\}$$

$$= \{(x_1,\ldots,x_n) \in R^n : s_1(x_1,\ldots,x_n) \geq 0,\ldots,s_n(x_1,\ldots,x_n) \geq 0\} \quad (10)$$

Proof:

Consider the polynomial

$$f(t) = \prod_{i=1}^{n}(t - x_i) = \sum_{i=1}^{n}(-1)^i * s_i * t^{n-i}$$

All the roots are real and not zero, so we can count the number of strictly positive roots with the help of Descartes' Rule, it is the number of variations of signs in the sequence $1, -s_1, s_2, \ldots, (-1)^n s_n$. If the s_i's are all positive, then this number is n. The second part of the proposition can be proved in a similar way.

Theorem 3.2 *Let V be a bounded real affine variety over $I\!R$ and let $f_1,\ldots,f_{n+1} \in I\!R[V]$ with $n \geq 1$. We set*

$$P = \{x \in V : f_1(x) > 0,\ldots,f_{n+1}(x) > 0\} \quad (11)$$

Assume that the following condition holds:

- *There are no points $x \in V$ such that all the functions $f_1,\ldots,f_{n+1}$ vanish and the number of points $x \in V$ such that exactly n among the functions f_i vanish is finite.*

Then there is an equivalent system of only n functions $g_1, \dots, g_n \in \mathbb{R}[V]$ i.e.

$$\{x \in V : f_1(x) > 0, \dots, f_{n+1}(x) > 0\}$$

$$= \{x \in V : g_1(x) > 0, \dots, g_n(x) > 0\} \tag{12}$$

Every g_j lies in the $\mathbb{R}$-algebra generated by $f_1, \dots, f_{n+1}$.

Proof (Sketch): We define the function $\Phi : V \mapsto \mathbb{R}^{n+1}$ by setting $\Phi(x) = (f_1(x), \dots, f_{n+1}(x))$. We denote by W the image of V under Φ and by Q the image of P. Consequently, we have $Q = W \cap \{t_1 > 0, \dots, t_{n+1} > 0\}$. We claim that we can find functions $h_1, \dots, h_n \in \mathbb{R}[t_1, \dots, t_{n+1}]$ such that

$$\{(t_1, \dots, t_{n+1}) \in W : h_1(t_1, \dots, t_{n+1}) > 0, \dots, h_n(t_1, \dots, t_{n+1}) > 0\}$$

$$= \{(t_1, \dots, t_{n+1}) \in W : t_1 > 0, \dots, t_{n+1} > 0\} \tag{13}$$

Then we have

$$P = \{x \in V : h_1(f_1(x), \dots, f_{n+1}(x)) > 0, \dots, h_n(f_1(x), \dots, f_{n+1}(x)) > 0\} \tag{14}$$

So let us prove the claim.

By the assumptions of the theorem, there are only finitely many intersections $\{x_1, \dots, x_m\}$ of W with the coordinate lines. Firstly, we produce a reduction which works in a neighbourhood of these points and afterwards we use a sort of partition of unity in order to find a global reduction. We set $h_2 := s_3, \dots, h_n := s_{n+1}$ and we want to find the last function h_1. With $W := \{t_1 + \dots + t_{n+1} > 0\}$ and $h := s_2$ it follows from Proposition 3.1 that

$$Q \cap W = \{h > 0, h_2 > 0, \dots, h_n > 0\} \cap W \tag{15}$$

Next we take two functions $p_1, p_2 \in \mathbb{R}[t_1, \dots, t_{n+1}]$ such that

a) $p_1 > 0$ on W.

b) $p_1(x_i) > 1$ for $i = 1, \dots, m$

c) $\{p_1 \geq 1\} \subseteq W$

d) $p_2 \geq 0$ on W.

e) $p_2(x_i) = 0$ for $i = 1, \dots, m$.

f) $\{p_2 \leq 1\} \subseteq \{p_1 > 1\}$

The existence is easy to prove using the Stone–Weierstrass–Theorem.

Now we consider the closed sets

$$M_1 := (W - \{p_2 < 1\}) \quad \cap \quad \overline{Q} \tag{16}$$

$$M_2 := \overline{(W - \{p_2 < 1\}) \quad \cap \quad \{s_3 > 0, \ldots, s_{n+1} > 0\} - Q} \tag{17}$$

M_1 and M_2 are disjoint closed bounded sets and again by the Stone-Weierstrass-Theorem we find a polynomial $h_2 \in \mathbb{R}[t_1, \ldots, t_{n+1}]$ such that $h_2 > 1$ on M_1 and $h_2 < -1$ on M_2.

We set

$$h_1 := p_1^M * h + p_2^M * h_2 \tag{18}$$

where M is a sufficiently large natural number. Now it is very technical, but not difficult to show that (if we choose M large enough)

$$Q = \{(t_1, \ldots, t_{n+1}) \in W : h_1(t_1, \ldots, t_{n+1}) > 0,$$

$$h_2(t_1, \ldots, t_{n+1}) > 0, \ldots, h_n(t_1, \ldots, t_{n+1}) > 0\} \tag{19}$$

This finishes the proof of Theorem 3.2.

4 Polynomial Reductions

Theorem 4.1 *Let R be a real closed field. Let A be a R-algebra of finite transcendence degree d. Then every basic open set P of $\mathrm{Spec}_r A$ can be written with at most d inequalities. Moreover, given any description of P as a basic open set, we can achieve that the functions of the reduced description lie in the R-algebra generated by the function of this description.*

Proof: It is sufficient to show this theorem for a basic open set described with $d+1$ inequalities. So let $P = \{\alpha \in \mathrm{Spec}_r A : a_1(\alpha) > 0, \ldots, a_{d+1}(\alpha) > 0\}$. Since the transcendence degree of A is d, we find a non constant polynomial $p \in R[t_1, \ldots, t_{d+1}]$ such that $p(a_1, \ldots, a_{d+1}) = 0$. We set

$$B := R[t_1, \ldots, t_{d+1}]/(p) \tag{20}$$

Consider the application

$$\phi : B \longmapsto A \tag{21}$$

defined by $\overline{t_i} \mapsto a_i$. It is clear that ϕ is well defined. It induces a map

$$\phi^* : \mathrm{Spec}_r A \mapsto \mathrm{Spec}_r B \tag{22}$$

Set

$$Q = \{\beta \in \mathrm{Spec}_r B : t_1(\beta) > 0, \ldots, t_{d+1}(\beta) > 0\} \tag{23}$$

Then $P = \phi^{*-1}(Q)$:

$$\alpha \in P = \{a_1 > 0, \ldots, a_{d+1} > 0\} \iff \forall i \; -a_i \notin \alpha \iff \forall i \; -\phi(t_i) \notin \alpha$$

$$\iff \forall i \; -t_i \notin \phi^*(\alpha) \iff \phi^*(\alpha) \in Q = \{t_1 > 0, \ldots, t_{d+1} > 0\} \tag{24}$$

Since p is a non-zero polynomial, the transcendence degree of B is at most d. By the Theorem of Bröcker-Scheiderer we can write Q with only d functions $\overline{g_1}, \ldots, \overline{g_d} \in B$ such that

$$Q = \{\beta \in \mathrm{Spec}_r B : \overline{g_1}(\beta) > 0, \ldots, \overline{g_d}(\beta) > 0\} \tag{25}$$

We choose for each $\overline{g_i}$ a representant $g_i \in R[t_1, \ldots, t_{d+1}]$ We claim that

$$P = \{\alpha \in \mathrm{Spec}_r A : g_1(a_1, \ldots, a_{d+1}) > 0, \ldots, g_d(a_1, \ldots, a_{d+1}) > 0)\} \tag{26}$$

For the proof, let $\alpha \in P$. Then

$$\forall i \; g_i(a_1, \ldots, a_{d+1}) \notin \alpha \iff \forall i \; \phi(\overline{g_i}(t_1, \ldots, t_{d+1})) \notin \alpha$$

$$\iff \forall i \; \overline{g_i}(t_1, \ldots, t_{d+1}) \notin \phi^*(\alpha) \tag{27}$$

Consequently,

$$\alpha \in P \iff \phi^*(\alpha) \in Q \iff \phi^*(\alpha) \in \{\overline{g_1} > 0, \ldots, \overline{g_d} > 0\}$$

$$\iff \alpha \in \{g_1(a_1, \ldots, a_{d+1}) > 0, \ldots, g_d(a_1, \ldots, a_{d+1}) > 0\} \tag{28}$$

This shows the claim and Theorem 4.1. □

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