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OLOGY OF REAL ALGEBRAIC MANIFOLDS

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1. The issue of the topology of manifolds specified by algebraic equations one of the fundamental and classical problems of mathematics; at the same e, it is still a long way from being solved and it is one of the problems se solution determines to considerable extent the level of progress in mathe- ics. Some problems of this type are contained in Hilbert's 16th problem.

Plane second-order curves were investigated as early as the ancient Greeks, e third- and fourth-order curves were studied by Descartes and Newton. The logy of curves of higher order turned out to be much more difficult; the logy of nonsingular sixth-order curves was completed as recently as 1969, e all the possible positions of ovals of an eighth-order curve are unknown today (see surveys [1,2]).

2. Harnack proved in 1876 [3] that the maximum number of connected compo- s of a real algebraic curve of degree n on a projective plane does not exceed 1, where $g = \frac{(n-1)(n-2)}{2}$ is the genus of the curve. Curves with g + 1 ovals t (examples were given as early as Harnack); I. G. Petrovskii proposed that be called M-curves.

Hurwitz [4] and Klein [2] offered a new proof of Harnack's inequality. They oyed a Riemannian surface formed by complex points of an irreducible curve. complex conjugate defines the involution of a Riemannian surface of this type. eal points of the curve form the set of stationary points of this involu- If one of the ovals is removed, the remaining ones are homologically inde- nt on the Riemannian surface (the film that effects homology together with mage under involution forms a connected component of the Riemannian surface, te the assumed irreducibility). This implies that the number of ovals does xceed g + 1. Thus, Harnack's inequality was not only proved but also extend- arbitrary (not necessarily plane) algebraic curves.

3. In a paper presented at the International Mathematical Congress in 1900, rt included problems of the topology of real algebraic manifolds in his

16th problem. In particular, he specially singled out problems of the arrangement of ovals of sixth-order curves in RP^2 and of fourth-order surfaces in RP^3. The accompanying table gives the number N of "logically possible" arrangements of g + 1 ovals of M-curves of small degree n in RP^2.

n	$g+1$	N	v
2	1	1	1
4	4	9	1
6	11	4766	3
8	22	268 282 855	$10 < ? < 144$

The last column of the table indicates how many of the "logically possible" arrangements are realized. Regarding sixth-order curves, Hilbert wrote that he was convinced that it was not possible for all 11 ovals to lie outside one another, but the proof did not appear. Hilbert [6], and subsequently, Briusotti and Wieman and others, proposed a variety of methods for constructing M-curves; in a series of papers, Roon attempted to prove Hilbert's hypothesis regarding sixth-order curves (see survey [1]).

Read!! you could do it too!!!

4. In 1906, a paper by Ragsdale appeared [7], which contained extensive experimental material and a number of hypotheses. Each oval of a real nonsingular plane curve of even degree divides the projective plane into two parts, one of which is diffeomorphic to a circle and is called the interior of the oval (the other is diffeomorphic to a Moebius sheet). An oval is called positive (or even) if it lies inside an even number of other ones, and negative (or odd) if it lies inside an odd number of ovals.

Ragsdale's main hypothesis was the following. Assume that an M-curve of degree n = 2k consists of p even and m odd ovals. Then

$$p \leqslant \frac{3}{2} k(k-1) + 1, \quad m \leqslant \frac{3}{2} k(k-1).$$

These bounds imply an inequality which was given by Ragsdale as a hypothesis:

$$|2(p-m)-1| \leqslant 3k^2 - 3k + 1. \tag{1}$$

This inequality was subsequently proved by Petrovskii, and we will call it the Petrovskii inequality. Ragsdale gave examples that demonstrated that the inequality cannot be improved. She also pointed out the relationship between the problem of the arrangement of the ovals and bounds for the Kronecker characteristics and indexes of singular points of vector fields.

5. In 1932, Comessatti proved [8] an inequality for the Euler characteristic χ of a nonsingular real algebraic surface; in modern notation, the inequality is

$$|\chi - 2 - t| \leqslant h^{1,1} - d,$$

re t is the trace of the involution of the complex conjugate on the space of logy classes generated by algebraic cycles; d is the dimension of this space; $h^{1,1}$ is the Hodge complexification number. By applying this inequality to a able surface, it is possible to prove the Petrovskii inequality (as was re- ly pointed out by V. M. Kharlamov).

6. In 1933 and 1938, papers [9] and [10] of Petrovskii appeared, of which second contained a proof of inequality (1) and its generalization to curves dd degree. Petrovskii's proof was based on the Euler-Jacobi formula:

$$\Sigma g(x)/\det(\partial f/\partial x) = 0$$

sum is taken over all n^2 roots of the system of equations $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0$, e f_1 and f_2 are polynomials of degree n; g is a polynomial whose degree is than that of the denominator). Petrovskii proved, in particular, Hilbert's othesis regarding the impossibility of all 11 ovals of a sixth-degree curve g outside one another. This paper by Petrovskii initiated a lengthy series tudies in real algebraic geometry, and the entire subsequent development of theory is related either directly or indirectly to the ideas, methods, and lts of this pioneering study.

7. In 1949, Petrovskii and Oleinik proved [11] an inequality analogous to (1) for smooth real algebraic hypersurfaces in a space of any number m of mea- surements. For hypersurfaces of even dimension in RP^m, e.g., for algebraic sur- aces in three-dimensional real projective space, they obtained an upper bound or their Euler characteristic as a function of the degree n of the hypersurface:

$$|\chi - 1| \leqslant \Pi_{m+1}(n),$$

where $\Pi_d(n)$ is the number of entire points strictly inside cube $(0, n)^d$, which lie on the hyperplane perpendicular to the principal diagonal of the cube and passing through the center of the cube.

Combinatorics !!

For noneven-dimensional hypersurfaces of even degree, e.g., curves in RP^2, Petrovskii and Oleinik bounded the difference between the Euler characteristics of the parts into which this hypersurface divides a projective space. This bound has the form

$$|\chi_+ - \chi_-| \leqslant \Pi_{m+1}(n).$$

For m = 2, it becomes the Petrovskii inequality.

8. In 1951, Oleinik considered [12] spatial real algebraic curves cut out on a surface of degree n_1 by an algebraic surface of degree n_2. The bounds she obtained (see §15 below) are not improvable, at least for curves on surfaces of

degree 2. Oleinik also established separate bounds for the sums of even-dimensional and noneven-dimensional Betti numbers of algebraic hypersurfaces [13].

9. In 1964-1965, papers by Milnor [14] and Thom [15] appeared, containing bounds for the Betti numbers of real algebraic manifolds. These bounds are more weaker than those of Petrovskii and Oleinik. At the end of [15], Thom gave an important inequality—the sum of all Betti numbers of a smooth real algebraic manifold does not exceed the sum of all Betti numbers of the complexification of this manifold (all Betti numbers being modulo 2). This inequality is a direct corollary of Smith's theory applied to the involution of the complex conjugate. Thom ascribes it to Borel.

The Smith inequality precisely yields the Harnack inequality for algebraic curves. Manifolds for which the Smith inequality becomes an equality are called M-manifolds (at the suggestion of V. A. Rokhlin, who employed Petrovskii's term "M-curves" as a starting point).

10. By 1969, D. A. Gudkov had completed his investigation of the topology of plane nonsingular real curves of degree 6 (see [16]). For the 11 ovals of an M-curve of degree 6, only 3 arrangements turned out to be possible: exactly one oval contains other ones inside it, the number of the latter being only 1, 5, or 9. Gudkov's proofs are extremely complicated; they are an extension of the methods of Roon and of the Italian school of algebraic geometry.

By using Gudkov's results, G. A. Utkin obtained (see [16]) considerable information regarding the topology of surfaces of degree 4 in RP^3.

In considering curves of various degrees which he was able to construct, Gudkov arrived at the following "periodicity hypothesis": the difference between the number of even and odd ovals of an M-curve of even degree is congruent modulo 8 to the square of half the degree.

11. In 1971, V. I. Arnol'd (whom Gudkov had informed of his hypothesis) proved a weakened version of it (with 8 replaced by 4) [17]. Arnol'd's proof linked the study of real algebraic curves to the topology of four-dimensional manifolds and the arithmetic of quadratic forms. The use of these relationships led to progress in real algebraic geometry, facilitated by the employment of powerful techniques of modern differential topology and algebraic geometry.

The four-dimensional manifold considered in [17] was a double covering of a complex projective plane, branched along the curve being investigated.

12. Many real entities have complex analogs, sometimes quite unexpected ones. For example, the complex analog of group Z_2 is group Z; the complex analog

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a permutation group is an Artinian braid group; the complex analog of spaces 0) are spaces $K(\pi, 1)$; and the complex analog of the Morse theory is the Picard-schetz theory. (It would be very interesting to find a complex analog of ebraic K-theory using the Cerf-Hutcher-Wagoner-Volodin theory as a "real" rting point.) What is done upon complexification with a manifold with an e? To answer this question, we specify a manifold with an edge by the in-ality $f(x) \geqslant 0$. For complexification, we algebrize this inequality, i.e., we e it in the form $f(x)=z^2$. Thus, we arrive at a double covering which is ched along the complexification of the edge of the initial real manifold.

The parts into which an algebraic curve of even degree divides RP^2 give rise wo double coverings—manifolds of real dimension 4. The intersection index wo-dimensional integer homologies of this manifold specifies a unimodular near form, while the complex conjugate defines a form-preserving involution.

Thus, the arithmetic of quadratic forms appear in the topology of real e algebraic curves.

In addition to proofs of Gudkov's hypothesis modulo 4, Arnol'd's paper [17] ained a number of new constraints on the arrangement of ovals of plane alge- curves.

13. Gudkov's hypothesis was fully proved by Rokhlin [18] in 1972. Rokhlin nded Gudkov's congruence to regular complete intersections [19], then to ngular algebraic manifolds [20]. For M-manifold A of even dimension, the in congruence has the form

$$\chi(A) \equiv \sigma(CA) \bmod 16$$

the Euler characteristic, σ is the signature, and CA is the complexifica- of A). In the noneven-dimensional case, the Rokhlin congruence involves a ture of a double covering of a surface with branching along A. Rokhlin ed several proofs for Gudkov's congruence; the later proofs are simpler and general, but the earlier ones contain some latent additional information.

14. The congruences of Gudkov and Rokhlin were soon generalized by Kharla- 1,22], and also by Gudkov and Krakhnov [23] for M - 1- and M - 2-manifolds - i-manifolds the left side of the Smith inequality is 2i smaller than ght). Thus, for a plane curve of degree n = 2k consisting of $\frac{1}{2}(n-1)(n-2)$ (a M - 1-curve), the difference between the number of even and odd ovals gruent modulo 8 with $k^2 \pm 1$.

5. Kharlamov obtained [24,25] new generalizations of the Petrovskii and k inequalities. In these generalizations, the Euler characteristics of

even-dimensional real algebraic manifolds are bounded in terms of the Hodge numbers of suitable complex manifolds. For the Euler characteristic of a real smooth manifold of dimension 2k, Kharlamov obtained the inequality

$$|\chi - 1| \leqslant h^{k,k} - 1, \qquad (2)$$

where $h^{k,k}$ is the average Hodge number of the complexification of this manifold. Zvonilov proved in [26] that for a nonsingular hypersurface of degree n in CP^m the right side of inequality (2) coincides with the right side of the Petrovskii-Oleinik inequality. Thus, Kharlamov's inequality is a direct generalization of the Petrovskii-Oleinik inequality.

Kharlamov regards a noneven-dimensional manifold as the edge of a 2k-dimensional manifold with an edge, and proves the following inequality for the Euler characteristic of the latter:

$$|2\chi - 1| \leqslant h^{k,k} - 1,$$

where $h^{k,k}$ is the Hodge number of the double covering of the complexification of the manifold with edge which is branched along the complexification of the edge. For noneven-dimensional hypersurfaces in CP^m, he obtained precisely the Petrovskii-Oleinik inequality, while for curves on surfaces he obtained the Oleinik inequality.

16. Other generalizations of the Petrovskii-Oleinik inequality were obtained by Arnol'd in [27], on the basis of recent progress in the local theory of singularities of differentiable mappings. Arnol'd considered a finite-multiple critical point of a real analytic function of 2k variables, and bounded the index of the gradient of this function at the critical point in terms of the Hodge numbers of the mixed Hodge structure defined by Steenbrink [28] in homologies of the nonsingular local manifold of the level of the function. This bound has the form

$$|\mathrm{ind}| \leqslant h_1^{k,k} \qquad (3)$$

(the subscript 1 indicating the Hodge number on the space adjoint to the eigenvalue, equal to unity, of the monodromy operator).

To bound the index of the gradient of a function of an odd number of variables, it suffices to increase the number of variables by 1, adding the square of the new variable to the function.

When the function is a homogeneous polynomial, inequality (3) becomes the Petrovskii-Oleinik inequality.

17. Progress in singularity theory has made it possible to bound from ove not only the index of a singular point of a gradient vector field, but so the index of an isolated singular point of an arbitrary vector field. It s proved by Arnol'd in [27] that the modulus of the index of a singular point a vector field in RP^m whose components are homogeneous polynomials of degree - 1 is bounded from above by the Petrovskii-Oleinik number:

$$|\mathrm{ind}| \leqslant \Pi_{m+1}(n).$$

e proof is based on the Levine-Eisenbud-Khimshiashvili formula [29], which presses the index of a singular point of a vector field in the form of the gnature of a quadratic form on a local algebra corresponding to a finite-ltiple singular point ("algebra of functions on merged singular points").

When the vector field is the gradient of a homogeneous polynomial of degree the inequality in question becomes the Petrovskii-Oleinik inequality. There-e, Arnol'd called his inequality the generalized Petrovskii-Oleinik inequality.

18. The question of whether equality is attained in the Petrovskii-Oleinik quality has been resolved only in the simplest cases: for plane curves (Rags-e); for curves on second-order surfaces (Oleinik); for fourth-order surfaces. ality is attained in all these cases. Khovanskii proved in [30] that equality ttained in the generalized Petrovskii-Oleinik inequality for vector fields; onstructed homogeneous polynomial vector fields of degree n - 1 in R^m with gest possible index (equal to $\Pi_m(n)$) . Moreover, he constructed fields with trary indexes from $-\Pi_m(n)$ to $+\Pi_m(n)$ to $\Pi_m(n)$ congruent modulo 2 with $\Pi_m(n)$ (the xes of all fields of fixed degree are identical modulo 2, since complex roots n pairs).

19. Khovanskii also obtained [30] bounds for the sum ind_+ of indexes of ular points of a polynomial vector field in a region bounded by an algebraic rsurface. Assume that the components of the field are of degree $n_1-1, \dots, n_m-1$, e the hypersurface is of degree n_0. Consider the parallelepiped $\prod_{k=1}^{m}(0, n_k)$. We te by $\Pi(n, n_0)$ the number of integral points z inside this parallelepiped for h $\Sigma n_k - n_0 \leqslant 2\Sigma z_k \leqslant \Sigma n_k + n_0$, and by $O(n,n_0)$ the number of integral points for which $n_0 \leqslant 2\Sigma z_k \leqslant \Sigma n_k$. For some condition of nondegeneracy (which is usually satisfied)

$$|\mathrm{ind}_+| \leqslant O(n, n_0), \quad |\mathrm{ind}_+ - \mathrm{ind}_-| \leqslant \Pi(n, n_0), \quad |\mathrm{ind}_+ + \mathrm{ind}_-| \leqslant \Pi(n, 0).$$

e inequalities together with the obvious congruences modulo 2 comprise the lete system of equations of existence of the field with these data, in par-lar with sum of indexes ind_+ for singular points on one side of the hyper-ace and ind_- on the other side of it. The proof employs an expression for the indexes in the form of signatures of suitable forms, and the Eulier-Jacobi formula. It is a direct extension of the work of Petrovskii [10] and Petrovskii and Oleinik [11], and shows, incidentally, that Petrovskii's paper [10] already contained many elements of the proof of the signature formula for the index, which was obtained in [29].

20. Kharlamov extended generalized Petrovskii inequalities to algebraic manifolds with singularities. One of his generalizations involves a mixed Hodge structure of a nonsingular local manifold of level $\mathcal{A}$ of a homogeneous function constructed on the basis of a given hypersurface A of dimension 2k. In this case Kharlamov obtained the following generalization of the Petrovskii inequality:

$$|\chi(A)-1| \leqslant h_1^{k,k}(\mathcal{A}).$$

A double covering is adduced in the noneven-dimensional case. For plane curves of degree 2k with s double points and t return points, the Kharlamov inequality has the form

$$|\chi_+ - \chi_-| \leqslant 3k^2 - 3k + 1 - s - 2t,$$

where $\chi_\pm$ are the Euler characteristics of the parts into which the curve divides the projective plane.

In [31], Viro gave a number of other constraints which involve not only the type of the singular points but also their mutual disposition.

Extensive studies of disintegrating sixth-order curves were made by Polotovskii [32,33]. He discovered, in particular, the link between this issue and the "complex orientations" to be discussed below.

21. In addition to the issue of the topological classification proper of real algebraic curves, which was resolved by Harnack's theorem, and the issue of isotopic classification (Hilbert's question of mutual disposition of ovals), there is the no less important issue of the connectivity components of a manifold of real nonsingular plane curves of fixed degree.

The points of such a component constitute curves of the same isotopic type. Rokhlin pointed out in [34], however, that curves of the same isotopic type do not always belong to the same component; isotopy cannot always be realized in the class of smooth algebraic curves of fixed degree $n \geqslant 5$.

An invariant which makes it possible to distinguish connectivity components was essentially considered by Klein [35], who thoroughly investigated fourth-degree curves. The ovals of an M-curve divide a Riemannian complexification surface into two parts. Each part is an oriented real two-dimensional manifold, whose edge is made up of ovals. Consequently, each oval obtains an orientation efined to within simultaneous variation of the orientation of all ovals). This ientation was called the complex orientation by Rokhlin. Two M-curves from e connectivity component of a manifold of curves of given degree are isotopic th allowance for complex orientations. Therefore, M-curves with nonequivalent mplex orientations that are isotopic without allowance for orientations belong different connectivity components.

A detailed survey of all findings concerning complex orientations of M-rves, and of complex characteristics of real curves in general, may be found Rokhlin's paper [36], which also gives a number of known constraints on the otopic type of a plane curve that do not appear in the present survey trengthened Petrovskii and Arnol'd inequalities; inequalities of Mishachev and nilov, Viro-Zvonilov, Rokhlin [34], and others).

22. The problem of classification of surfaces of fourth degree in RP^3 was initively solved by Kharlamov as a result of a series of studies [37-39], ich yielded both new constraints on the topological and isotopic types of the faces and new methods of constructing surfaces of specified types. Kharlamov the first to employ the powerful resources of modern algebraic geometry in s area. A complex nonsingular surface of fourth degree in CP^3 is a manifold type K - 3. Such manifolds have been studied in detail; the corresponding ule spaces have been investigated; it has been determined how algebraic mani-ds are arranged among all such manifolds. Kharlamov was able to use this ormation to study real surfaces. Thus, he obtained a completely new method constructing examples of fourth-degree surfaces.

Kharlamov's results yield not only a topological but also an isotopic clas-ication of nonsingular surfaces. His proofs made some use of arithmetic ormation obtained by Nikulin in [40].

23. Nikulin was able in [40] to completely arithmecize the problem of rmining the connectivity components of manifolds of nonsingular real alge-c surfaces of type K - 3. He placed every even-dimensional real algebraic fold in correspondence with a triple consisting of the following: 1) an ger-valued symmetrical bilinear unimodular form (form of intersections on e of integer-valued homologies of moderate dimension, reduced modulo the ion); 2) an involution (induced by complex conjugacy); and 3) a "polariza-," specified by an invariant or anti-invariant cycle (suitable degree of rplane section).

Nikulin constructed the complete system of invariants of such triples and, number of particularly interesting cases, he also found all the relation-s among invariants. In particular, he established a one-to-one correspondence between classes of triples with suitable values of the invariants and the connectivity components of a manifold of nonsingular real surfaces of type K - 3. He obtained the components, e.g., for a manifold of double coverings of a plane which are branched along plane curves of sixth degree (thus, for a manifold of most plane curves of sixth degree); for a manifold of nonsingular surfaces of fourth degree in RP^3; and for intersections of cubics and quadratics in RP^4 and intersections of three quadratics in RP^5.

24. Viro [41] developed entirely different methods of constructing examples of real algebraic manifolds with the given topology. In particular, he constructed M-surfaces and M-manifolds, i.e., he established the unimprovable nature of the inequality (a consequence of Smith's theory) for the sum of all Betti numbers of a real manifold modulo 2.

25. Studies of bounds of topological invariants of singularities in terms of Newton polygons led A. G. Kushnirenko to the hypothesis that what is essential for the topological complexity of an object is not the degree of the equation, but rather the number m of monomials that appear in the polynomial with nonzero coefficients. An elementary example of a theorem of this type is Descartes' theorem, which estimates the number of positive roots of a polynomial in terms of the number of sign changes in the sequence of its coefficients. If there are few monomials, then there are also few sign changes. Thus, we have the problem of "oligomials," i.e., the topology of objects specified by polynomials of arbitrarily large degree but with a restricted number of monomials.

It was recently shown by Sevast'yanov that the number of zeros of an m-omial on a plane algebraic curve of degree n is bounded by a constant C(m,n) which is independent of the degree of the m-omial. Viro improved his bound, obtaining

$$C(m, n) = \frac{n^3}{24}(m-1)\,m\,(m+1)\,(m+2) + n\left[\frac{(m+1)\,m\,(m-1)\,(m-2)}{24} + 3(m-1)\right].$$

Khovanskii and Gel'fond proved a number of corollaries of the oligomial hypothesis. Thus, they obtained a bound for the number of zeros of superpositions of oligomials. Their results can be carried over to the case of a field of real Liouville functions, obtained from a field of rational functions by attachment of a finite number of integrals and exponents of integrals. They proved that in the real case, the Liouville functions have a finite number of zeros. For example, cos x is not a Liouville function in the real sense, although it is in the complex sense. These results give hope that the oligomial hypothesis is also valid in the more general situation, e.g., that the number of ovals of curve $f = 0$ can be bounded in terms of the number of monomials of polynomial f.

26. Some unresolved issues.

1) Give an asymptotically exact bound for the number of connected components a space of nonsingular real algebraic hypersurfaces of degree n.

2) Is equality attained in the Petrovskii-Oleinik inequality?

3) Is the Ragsdale hypothesis valid? This hypothesis can be reformulated as llows. Assume that $f(x, y, z)$ is a homogeneous polynomial of even degree; $=f\pm t^2$; $RV_\pm$ is a local manifold of level $F_\pm=\pm\varepsilon$. Ragsdale's hypothesis involves unding the numbers of components

$$b_0(RV_+)\leqslant h_1^{2,2}(F_+),\quad b_0(RV_-)\leqslant h_1^{2,2}(F_-)+1$$

terms of a mixed Hodge structure (for suitable sign of f).

4) Give unimprovable bounds (in terms of the degree or the Hodge number) for e individual Betti numbers of real algebraic hypersurfaces, in particular for e number of components b_0. Perhaps it is easier to bound the numbers b_0, b_0-b_1, $b_1+b_2, \ldots$, as well as combinations of local Morse type numbers M_0, M_0-M_1, $M_0-M_1+M_2, \ldots$ is the number of critical points of index i that merge at zero for any Morse-tion of a homogeneous equation of a hypersurface).

5) What is the largest number of handles of a component of an algebraic sur-e of degree n in RP^3?

6) Bound the number of ovals of a curve whose equation is an oligomial, in ms of the number of its terms.

7) How many nonconvex ovals can a plane algebraic curve of degree n have?

8) Does the isotopic type of a pair (plane M-curve, its complex orientation) ermine the connectivity component in space of nonsingular curves of fixed de-e?

9) Investigate the fundamental complement group π_1 to the set of singular ersurfaces in the complex projective space of all hypersurfaces of fixed de-e in CP^m, and find the corresponding monodromy group (representation of π_1 by omorphisms of the homology group of the hypersurface).

10) Assume that P, Q, and H are polynomials of fixed degrees of x and y, and ume that $I(h)=\oint Pdx+Qdy$ over the oval H = h. What is the largest number of os of I(h) when I(h) is not identically zero?

11) Assume that P and Q in the system $\dot{x}$ = P, $\dot{y}$ = Q are polynomials of second ree, while H is the first integral (not necessarily polynomial). How many it cycles can be generated from the components of the level lines of H for ll perturbations of P and Q that leave them as second-degree polynomials?

12) Assume that P and Q in the system $\dot{x} = P$, $\dot{y} = Q$ are power series that begin with homogeneous polynomials P_n and Q_n of degree n. Is it true that for almost all pairs (P_n, Q_n) the number of limit cycles that are generated from zero under small perturbations of the system is bounded by a constant that depends only on n?

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