Trends in Stat. Phys., 1 (1994)

Restricted colourings and flows on graphs and directed percolation

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Abstract A new class of chromatic, or potential, polynomial is considered. These polynomials are obtained by restricting the potential differences on adjacent vertices to either odd or even parity. For colourings involving an odd number of colours, the polynomials can be evaluated to give an indicator for directed cuts on rooted graphs. Corresponding flow polynomials give directed path indicators. It is shown how the polynomials arising from these restricted colourings and flows can be used to generate the correlation function for directed interaction models which generalize the standard Potts model. A further statistical mechanical model is introduced where odd and even potential differences are given different weights. In this case when each vertex has an odd number of states the percolation limit of this model is Redner's oriented diode model.

1 Introduction

This paper is concerned with particular cases of the $Z(\lambda)$ model in statistical mechanics which describes a system of interacting atoms each of which can be in one of λ-states. The interaction energy is taken to be a sum over pairs of atoms and the energy associated with a given pair is assumed to be a function of the two states involved. The function which determines the interaction may be different for each pair but is specified in defining the model. The atoms are assumed to have fixed positions and some pairs may be considered to have zero interaction depending on their separation. In the usual case the atoms form a crystal lattice and the interactions are restricted to nearest neighbours.

A graph theoretic approach will be taken in which the system is defined on a finite directed graph $H = (V, A)$ where the vertices V represent the atoms and the arcs A represent the non-zero interactions. The state of vertex i is specified by a variable c_i, called its colour or potential, which takes on one of the values in $\mathcal{C} = \{0, 1, \ldots, \lambda-1\}$. The Hamiltonian may be written

$$\mathcal{H} = k_B T \sum_{a \in A} h_a(\delta c(a)) \tag{1}$$

where the energy associated with arc $a = (i,j)$ depends only on the potential difference (PD) $\delta c(a) = c_j - c_i$ (mod λ).

The determination of the partition function $Z_\lambda(H)$ becomes the problem of enumeration of potentials subject to specified constraints on the potential differences. For example in the Potts model (Redner 1982) the interaction energy takes on only two values depending on whether or not the vertices are in the same state. The problem then is to enumerate the number of potentials which have non-zero potential difference on each subgraph of H. In graph theory this is the well known problem of finding the number of proper λ-colourings and it is easy to show that this number is given by a polynomial in λ known as the chromatic polynomial (Whitney 1932).

The Potts model may be considered as a spin model in which the spin vector at a given vertex is directed in one of λ symmetrically placed directions in a space of dimension $\lambda - 1$ and the interaction depends on the scalar product of the vectors of interacting spins. In a second spin model, known as the cubic model (Aharony 1977) the n-dimensional spin vectors point to the $\lambda = 2n$ faces of a hypercube and the scalar product then takes on three values so that additional constraints are placed on the enumeration. However it is still relatively easy to show that the required number of potentials is a polynomial in λ.

The correlation function between vertices 1 and 2 is determined in terms of the probability $U_{\beta,\lambda}(H)$ of finding $c_2 - c_1 = \beta$ (mod λ). Let $H_{\bar{a}}^+$ be the graph H with the additional arc $\bar{a} = (1,2)$. The combinatorial problem associated with the correlation function is similar to that for the partition function but the potentials are enumerated on $H_{\bar{a}}^+$ with a fixed PD of β on $\bar{a}$. This is implied by the notation $U_{\beta,\lambda}(H)$ which does not explicitly show the vertices 1 and 2 and in fact the additional arc a could be between any vertex pair.

An alternative approach to the correlation function is in terms of the discrete Fourier transform of $U_{\beta,\lambda}(H)$, known as the transmissivity (Alcaraz & Tsallis 1982)

$$T_{\alpha,\lambda}(H) = \sum_{\beta=0}^{\lambda-1} e^{2\pi i\alpha\beta/\lambda} U_{\beta,\lambda}(H). \tag{2}$$

This is conveniently expanded in terms of the arc transmissivity $t_a(\alpha)$, which is the transmissivity for the graph consisting of the arc a in isolation. The partition function may also be so expanded. The associated combinatorial problem is now one of enumerating restricted flows (Tutte 1954, Essam & Tsallis 1986) on the subgraphs of H and $H_{\bar{a}}^{+}$ which is less well known than graph colouring. For the Potts model and the cubic model the numbers of flows are polynomial in λ.

Flows and PDs are dual concepts in two different senses (Biggs 1976,1977). Firstly on any graph, flows and PDs when considered as vectors on the arc space lie in orthogonal and complementary subspaces (the cycle and cocycle or cut spaces) having ranks $c(H)$ and $r(H)$ respectively such that $c(H) + r(H) = |A|$. Secondly there a bijection between the PDs on any planar graph and the flows on the dual graph. The cocycle and cycle ranks determine the degrees of the above chromatic and flow polynomials.

In general $t_a(0) = 1$ and for the Potts model the values of $t_a(\alpha)$ for $\alpha > 0$ are equal as are the values of $T_{\alpha,\lambda}(H)$. For the cubic model there are two different transmissivities for positive α which depend on whether α is odd or even. We are thus led to consider fixed parity flows. The values of $T_{\alpha,\lambda}(H)$ for $\alpha > 0$ also depend only on the parity of α.

For the cubic model the value of λ is even and consequently the number of fixed parity flows is independent of the directing of the subgraph. This is a manifestation of the fact that the interaction was via a scalar product which is symmetric. In terms of flows, reversing an arc is equivalent to replacing α by $\lambda - \alpha$ which preserves the parity.

Choosing λ to be odd in the case of fixed parity flows and PDs leads to theoretically interesting results. The numbers of PDs and flows now become directing dependent. The interaction therefore has chirality and the model is similar in this respect to the standard Chiral Potts model (Ruse, Ostlund 1981). It is always possible to redirect the graph so that the parity is everywhere even. Thus much of this work will be concerned with even PDs and flows for odd λ. The polynomial property survives for odd λ but in a restricted form. Clearly if the enumerations for odd λ are to be given by a polynomial it must be different from that for λ even since the latter is directing independent. To establish the polynomial property is far from trivial (Arrowsmith & Essam 1994a,b) and we give the results in theorems 1 and 2. The number of flows is polynomial in both λ and α but different polynomials are required for odd and even α.

It turns out that the correlation function (transmissivity) may be expressed as the expected number of PDs(flows) in a bond percolation process. In the case of the Potts model this was discovered (Kasteleyn and Fortuin 1972) for undirected percolation. More recently (Arrowsmith & Essam 1990) we have shown that for odd λ the fixed parity PD and flow models give rise to directed percolation. In the case of PDs this turns out to be the dual directed percolation model (Dhar et al 1981). One important consequence of the above polynomial dependence is that when the polynomials are used to extrapolate the correlation function (transmissivity) to $\lambda = 1$ and $\beta(\alpha) = 0$, the pair connectedness (blocking probability) functions are found. This places directed percolation theory in a statistical mechanical context.

In a polychromatic bond percolation model the open bonds can be of several types (colours) which occur with different probabilities (Zallen 1977). In general the correlation function for the $Z(\lambda)$ model, when there are k distinct interaction parameters, may be expressed as an expected value in a k-chromatic bond percolation problem. The cubic model is a bichromatic model. We will consider an odd λ, mixed parity model in which different weights are given to odd and even flows (PDs). This bichromatic model turns out to be related to the oriented diode percolation model of (Redner 1982) in which the open bonds may be oriented in either direction with different probabilities.

2 Chromatic and Potential Polynomials

2.1 The unrestricted case

The chromatic polynomial is obtained by enumerating the number of ways, $P_\lambda^+(G)$, that λ colours denoted by $\mathcal{C} = \{0, 1, \ldots, \lambda - 1\}$ can be attached to the vertices of a graph $G = (V, E)$ such that every non-loop arc has distinct colours on its adjacent vertices. Such a colouring is said to be *proper*. To obtain a formula for $P_\lambda^+(G)$, we observe that:

- the number of unrestricted colourings, $P_\lambda(G)$, is given by $\lambda^{\nu(G)}$, where $\nu(G)$ is the number of vertices of the graph G;
- given a colouring of graph G, let G/G' be the graph obtained by contracting those edges E' of the graph G which are not properly coloured, i.e. those that have vertices of the same colour. Then G/G' has an induced proper colouring. Thus, every colouring of G is an induced proper colouring of a suitably contracted graph G/G'. The total number of colourings $P_\lambda(G)$ on G can be counted by summing over the proper colourings, $P_\lambda^+(G/G')$ of the contracted graphs G/G' to obtain

$$P_\lambda(G) = \sum_{\emptyset \subseteq E' \subseteq E} P_\lambda^+(G/G'). \tag{3}$$

Möbius inversion then gives

$$P_\lambda^+(G) = \sum_{\emptyset \subseteq E' \subseteq E} (-1)^{|E \setminus E'|} P_\lambda(G/G') \tag{4}$$

where $G' = (V', E')$. Since $P_\lambda(G)$ is interpolated by the polynomial $P(\lambda, G) = \lambda^{\nu(G)}$,

$$P_\lambda(G/G') = P(\lambda, G/G') = \lambda^{\omega(G')} \tag{5}$$

where $\omega(G')$ is the number of connected components of G'. Hence we have a polynomial interpolation for the number of proper colourings which is of degree at most $\nu(G)$ and is known as the Whitney polynomial (Whitney 1932). Throughout this paper we adopt the convention that if a subscript is changed to a function argument then the domain of the function is the real numbers, thereby providing an interpolation for the values corresponding to the natural domain of the subscript which is always a subset of the positive integers.

An alternative language for the colouring problem is that of 'potentials'. A λ-potential c on G is a map $c : V \to C$. We can now go further and consider potential differences. Let $\mathbf{D}(G)$ be the set of $2^{|E|}$ directed graphs obtained by orienting the arcs of G in all possible ways. For $H \in \mathbf{D}(G)$ it is trivial to see that the signed sum of the potential difference δc is zero (mod λ) around any cycle of G, when the PD on any arc is counted negatively if is directed opposite to some chosen direction for the cycle. Conversely, if $d : E \to C$ is such that it satisfies the property that d sums to zero (mod λ) on every cycle, then there exists a λ-potential c such that its potential difference $\delta c = d$. The potential c is unique up to fixing the value of c at an arbitrary vertex on each connected component of c. In fact $P_\lambda(G) = \lambda^{\omega(G)} D_\lambda(G)$, where $D_\lambda(G)$ is the number of distinct mod-λ potential differences defined on G. This relation restricts to proper potentials and potential differences to give

$$P_\lambda^+(G) = \lambda^{\omega(G)} D_\lambda^+(G) \tag{6}$$

Also, it follows that

$$D_\lambda(G) = \lambda^{r(G)} \tag{7}$$

where $r(G) = \nu(G) - \omega(G)$, the cocycle rank of G.

The dual construction to the mod-λ potential difference is the mod-λ flow. A mod-λ flow on the directed graph H is a function $\phi : A \to C$ such that the $\phi(a)$ have a signed sum to zero (mod λ) over all arcs incident with a given vertex v, for each $v \in V$. The flow on an arc which is directed away from v counts negatively. The number of unrestricted flows is given by

$$F_\lambda(G) = \lambda^{c(G)} \tag{8}$$

where $c(G)$ is the cycle rank of G.

A crucial property of the above polynomials is that they are independent of the directing, H, of the graph G which was chosen in defining the PD. Given any two graphs $H, H' \in \mathbf{D}(G)$ there is a bijection between the PDs on H and H'. The corresponding PDs d and d' satisfy $d = d'$ on arcs which are coherently oriented and $d + d' = 0$ (mod λ) on those arcs of H and H' which are oppositely oriented.

2.2 Parity Constraints

We now introduce new collections of chromatic polynomials, some of which are distinguished by their orientation dependence. For λ even, define $C^{even} = \{0, 2, 4, \ldots, \lambda - 2\}$ and $C^{odd} = \{0, 1, 3, \ldots, \lambda - 1\}$. For λ odd, define $C^{even} = \{0, 2, 4, \ldots, \lambda - 1\}$ and $C^{odd} = \{0, 1, 3, \ldots, \lambda - 2\}$. An even (odd) mod-$\lambda$ PD d on H is a mod-λ PD such that $\delta c(a) \in C^{even}$ (respectively $\in C^{odd}$) for all $a \in A$. The nature of the interpolating polynomials for the constrained λ-potentials depends strongly on the parity of λ. For λ even, odd and even PDs both preserve their parity under reversal of arcs and so odd PDs cannot be changed to even PDs (and vice-versa) by orientation reversal and hence the number of PDs is orientation independent. For $\lambda = 2n$, $n \in \mathbf{Z}^+$, the mod-λ even PDs and the mod-n PDs on G are in 1-1 correspondence and we obtain

$$D_\lambda^{even}(G) = D_{\lambda/2}(G) = (\lambda/2)^{r(G)}. \tag{9}$$

The corresponding numbers of proper PD's are given by a formula similar to (4). For the case of odd mod-λ PDs, G must be bipartite to support a proper odd mod-λ PD; all vertices adjacent to a vertex with odd potential must have even potential and vice versa, hence

$$D_\lambda^{odd+}(G) = (\lambda/2)^{r(G)} b(G) \tag{10}$$

where $b(G) = 1$ if G is bipartite and zero otherwise. The corresponding results for flows are obtained by replacing $r(G)$ by $c(G)$ and $b(G)$ by $\epsilon(G)$ where $\epsilon(G) = 1$ if G has all vertices of even degree and zero otherwise (i.e G is an Euler graph).

For odd λ, if H^ρ is the directed graph obtained by reversing the orientation of all the arcs of H, then $D_\lambda^{even}(H) = D_\lambda^{odd}(H^\rho)$. Thus it is possible to reduce evaluations to those for even PDs. The difficulty in evaluating the number of even or odd mod-λ PDs is that in this case there is no simple formula corresponding to the result $D_\lambda(G) = \lambda^{r(G)}$ for unrestricted mod-λ PDs. Moreover, simple examples show that the number of PDs is orientation dependent. However polynomial interpolation is still possible and we give the results for odd λ after first introducing rooted PDs.

3 Rooted Potential Differences and Flows.

Suppose now that the graph $H \in \mathbf{D}(G)$ has a distinguished, or root, arc $\bar{a}$. The incident vertices of $\bar{a}$ are said to be roots of the graph and we denote the rooted graph by $H_{\bar{a}}$. In this section we consider the problem of evaluating the number of mod-λ PDs, as well as the restricted problem of odd and even PDs, subject to there being a fixed "root PD" of β on the arc $\bar{a}$. Rooted flows are also considered.

3.1 Unrestricted Rooted PDs and Flows

The number of mod-λ PDs with root PD β but which are otherwise unrestricted will be denoted by $D_{\beta,\lambda}(G_{\bar{a}})$. We use G rather than H to emphasize that, just as in the unrooted case, the number of unrestricted PDs is independent of the directing of H. Likewise the numbers of flows with given flow α in the root arc are denoted by $F_{\alpha,\lambda}(G_{\bar{a}})$.

If $\beta > 0$, then $D_{\beta,\lambda}(G_{\bar{a}})$ is zero unless $G_{\bar{a}}$ has a cut containing $\bar{a}$. Notice that $G_{\bar{a}}$ will have such a cut unless $\bar{a}$ is a loop. Let $\chi_{\bar{a}}$ be the cut indicator for G, i.e. $\chi_{\bar{a}}(G) = 1$ if $\bar{a}$ is not a loop and zero otherwise. Then we have

$$D_{\beta,\lambda}(G_{\bar{a}}) = D(\beta, \lambda, G_{\bar{a}}) = \lambda^{r(G)-1}\chi_{\bar{a}}(G) \qquad (11)$$

and using (6)

$$P_{\beta,\lambda}(G_{\bar{a}}) = P(\beta, \lambda, G_{\bar{a}}) = \lambda^{\nu(G)-1}\chi_{\bar{a}}(G) \qquad (12)$$

and hence

$$P(\beta, 1, G_{\bar{a}}) = D(\beta, 1, G_{\bar{a}}) = \chi_{\bar{a}}(G). \qquad (13)$$

For $\alpha > 0$, the number of mod-λ flows, $F_{\alpha,\lambda}(G_{\bar{a}})$, is zero unless there exists a cycle containing $\bar{a}$ (i.e. $\bar{a}$ is not an isthmus). Then, fixing the flow to be α in some cycle through $\bar{a}$, we have

$$F_{\alpha,\lambda}(G_{\bar{a}}) = F(\alpha, \lambda, G_{\bar{a}}) = \lambda^{c(G)-1}\gamma_{\bar{a}}(G), \qquad (14)$$

where $\gamma_{\bar{a}}(G) = 1$ if there is a cycle containing $\bar{a}$ and 0 otherwise. Hence

$$F(\alpha, 1, G_{\bar{a}}) = \gamma_{\bar{a}}(G). \qquad (15)$$

Notice that the presence of a cycle is equivalent to the existence of a path between the root vertices not including $\bar{a}$. We use this fact later when considering the pair connectedness in percolation theory.

3.2 Odd and Even Rooted PDs and Flows for even λ

We now restrict the PD(flow) to be of fixed parity and allow $\beta(\alpha)$ to be odd or even independently of the parity of the PD(flow) on the rest of the arcs.

For even λ and $\beta > 0$, the chosen parity of the PD, and that of the root PD, are independent of the orientation of H and so there are four different parity pairs to consider given by the choice of parity of the PD and that of the root PD. Denoting the odd or even restriction by $*$ it can be seen that

$$D^{*}_{\beta,\lambda}(G_{\bar{a}}) = (\lambda/2)^{r(G)-1}\chi_{\bar{a}}(G)\psi(G_{\bar{a}}) \qquad (16)$$

where $\psi(G_{\bar{a}})$ is given by the following table.

root parity \ PD parity	proper odd	even
odd	$b(G)$	$1-\gamma_{\bar{a}}(G)$.
even	$b(G^{\gamma}_{\bar{a}})$	1

where $b(G)$ is defined as in equation (10) and $G^{\gamma}_{\bar{a}}$ is the graph obtained by contracting the arc $\bar{a}$. Notice that the result for odd PDs is simplest when the proper condition is imposed, the total number of odd PD's is given by (3).

These are special cases of a mixed parity problem in which the PD is even (including zero) on the subgraph G_{even} and odd on the remaining edges. The general result is

$$D^{mixed+}_{\beta,\lambda}(G_{\bar{a}}, G_{even}) = (\lambda/2)^{r(G)-1}\chi_{\bar{a}}(G)b(G_{\bar{a}}/G_{even}). \qquad (17)$$

As in §2, the symbol "/" denotes contraction and the superscript + indicates the non-zero condition outside G_{even}. The number of PDs on relaxing this condition is

$$D^{mixed}_{\beta,\lambda}(G_{\bar{a}}, G_{even}) = \sum_{A'\subseteq A\backslash(A_{even}\cup\bar{a})} D^{mixed+}_{\beta,\lambda}(G_{\bar{a}}/G', G_{even}). \qquad (18)$$

Notice that when G_{even} includes the whole of G, except possibly the root arc, $D^{mixed+}_{\beta,\lambda}(G_{\bar{a}}, G_{even}) = D^{even}_{\beta,\lambda}(G_{\bar{a}})$.

The corresponding results for flows are

$$F^{mixed+}_{\alpha,\lambda}(G_{\bar{a}}, G_{even}) = (\lambda/2)^{c(G)-1}\gamma_{\bar{a}}(G)\epsilon(G_{\bar{a}}\backslash G_{even}) \qquad (19)$$

and

$$F^{mixed}_{\alpha,\lambda}(G_{\bar{a}}, G_{even}) = \sum_{A'\subseteq A\backslash(A_{even}\cup\bar{a})} F^{mixed+}_{\alpha,\lambda}(G_{\bar{a}}\backslash G', G_{even}). \qquad (20)$$

Notice that all of the above enumerators have polynomial dependence on λ for fixed $\beta(\alpha)$ and depend only on the parity of the root PD(flow).

3.3 Odd and Even Rooted PDs and Flows for odd λ

As for the unrooted case, the situation for odd λ is simpler in the sense that there are fewer cases to consider but this is counterbalanced by the fact there are no general formulae and the numbers are directing dependent. Reversal of the orientation of the root arc will change the parity of β and the reversal of the remaining arcs will change PDs from odd to even or vice versa. Thus we do not need a table of enumerating polynomials for odd λ. All cases can be reduced to considering even PDs with an even root PD by a suitable change in the orientation of the arcs of the graph.

The various cases of the 'even' and 'odd' constraints and the parities of β are related by the following lemma. Lemma 1 shows that, for the case of odd λ, the interpolating polynomials for rooted potentials with odd potential difference can also be interpreted in terms of potentials with even potential difference.

Lemma 1. *Let $H_{\bar{a}}$ be the rooted directed graph obtained from $H \in \mathbf{D}(G)$. Let $H_{\bar{a}}^{\rho}$ be the rooted directed graph obtained from $H_{\bar{a}}$ by reversing the orientation of every arc of H except for the root arc $\bar{a}$. The graph H^{ρ} is obtained from H by reversing the orientation of every arc of H. For λ odd,*

$$(i) \qquad P_{\lambda}^{odd}(H) = P_{\lambda}^{even}(H^{\rho}), \tag{21}$$

and

$$(ii) \qquad P_{\beta,\lambda}^{odd}(H_{\bar{a}}) = P_{\beta,\lambda}^{even}(H_{\bar{a}}^{\rho}). \tag{22}$$

Moreover, if $(H_{\bar{a}})^{\rho}$ denotes $H_{\bar{a}}$ with all arcs reversed, then

$$(iii) \qquad P_{\beta,\lambda}^{odd}(H_{\bar{a}}) = P_{\lambda-\beta,\lambda}^{even}((H_{\bar{a}})^{\rho}), \tag{23}$$

where $\lambda - \beta$ is an element of C by reducing mod-λ if necessary.

A *circuit* is a directed cycle in which the arcs are coherently oriented. Thus a coherently oriented loop is a circuit. A *directed cut* of the graph $H_{\bar{a}}$ is a minimal set of arcs $b \subseteq A(H)$ containing $\bar{a}$ which form a cut of H with the property that there is no circuit containing $\bar{a}$ for any choice of orientation on the complementary edge set $E(H)\backslash E(b)$. The directed cut indicator, $\chi_{\bar{a}}(H)$, for the graph H is defined by $\chi_{\bar{a}}(H) = 1$ if $H_{\bar{a}}$ has a directed cut containing $\bar{a}$ and zero otherwise. Notice that the same function name, $\chi_{\bar{a}}$, has been used for both undirected and directed cut indicators. We adopt the convention that if the argument of $\chi_{\bar{a}}$ is a directed graph then a directed cut is implied. The function $\chi_{\bar{a}}$ is simply related to the *dual* indicator for circuits $\gamma_{\bar{a}}(H)$ which has value 1 if there is a circuit in H containing $\bar{a}$ and 0 otherwise. The existence of a directed cut containing $\bar{a}$ is equivalent to the non-existence of a circuit containing $\bar{a}$ and so $\chi_{\bar{a}}(H) = 1 - \gamma_{\bar{a}}(H)$. Again we have used $\gamma_{\bar{a}}$ for both cycle and circuit indicators and these are distinguished by whether the graph to which the indicator is applied is undirected or directed respectively.

The following theorem is proved in (Arrowsmith & Essam 1994a).

Theorem 1. *Let $H \in \mathbf{D}(G)$ have a rooted arc $\bar{a}$, then the values of $D_{\beta,\lambda}^{even}(H_{\bar{a}})$ for $\beta = 2m$ and $\lambda = 2n+1$, $m, n \in \mathbf{Z}^+$, are interpolated by a polynomial $D^{even}(\beta, \lambda, H_{\bar{a}})$, in β and λ, having joint degree at most $r(H) - 1$ with the property*

$$D^{even}(0, 1, H_{\bar{a}}) = \chi_{\bar{a}}(H), \tag{24}$$

where $\chi_{\bar{a}}(H)$ is the directed cut indicator for root arc $\bar{a}$. Also, $D_{0,\lambda}^{even}(H_{\bar{a}}) = D_{\lambda}^{even}(H_{\bar{a}}^{\gamma})$ is a polynomial $D^{even}(\lambda, H_{\bar{a}}^{\gamma})$ of degree at most $r(H_{\bar{a}}^{\gamma})$ in λ such that $D^{even}(1, H_{\bar{a}}^{\gamma}) = 1$. There also exists a similar polynomial for β odd.

Remark The graph $H_{\bar{a}}^{\gamma}$ can be any directed graph by choosing G and the directing H appropriately. It follows that the function $D^{even}(1, H) = 1$ for $H \in \mathbf{D}(G)$.

There is a dual theorem for flows.

Theorem 2. *Let $H \in \mathbf{D}(G)$, then the values of $F_{\alpha,\lambda}^{even}(H_{\bar{a}})$ for $\alpha = 2m$, $\lambda = 2n+1$, $m, n \in \mathbf{Z}^+$, are interpolated by a polynomial, $F^{even}(\alpha, \lambda, H_{\bar{a}})$, in α and λ having joint degree at most $c(H) - 1$ with the property*

$$F^{even}(0, 1, H_{\bar{a}}) = \gamma_{\bar{a}}(H). \tag{25}$$

Also, $F_{0,\lambda}^{even}(H_{\bar{a}}) = F_{\lambda}^{even}(H_{\bar{a}}^{\delta})$, where $H_{\bar{a}}^{\delta}$ is the graph obtained by deleting the arc $\bar{a}$ of $H_{\bar{a}}$, is a polynomial $F^{even}(\lambda, H_{\bar{a}}^{\delta})$ in λ, such that $F^{even}(1, H_{\bar{a}}^{\delta}) = 1$, and is of degree at most $c(H_{\bar{a}}^{\delta})$. There also exists a similar polynomial for α odd.

The proofs of the above theorems are given elsewhere (Arrowsmith & Essam 1994a,b). They are tedious to prove as the evaluations which give the indicators are at $\beta = 0$ and $\lambda = 1$, yet the polynomial $D^{even}(\beta, \lambda, H_{\bar{a}})$ interpolates for $\beta > 0$ and $\lambda > 1$. This aspect of the problem is most obviously apparent in the previous undirected even λ case where the cycle and cocycle indicators have to be explicitly incorporated into the formula.

The essential idea for the proof of Theorem 1 is to show that the enumerations of $D_{2m,2n+1}^{even}(H_{\bar{a}})$, where $m, n \in \mathbf{Z}^+$, are polynomials of degree at most $\nu(G) - 1$ in the variables m and n. An inductive step is set up which relates $D_{2m,2n+1}^{even}(H_{\bar{a}})$ and $D_{2m-2,2n+1}^{even}(H_{\bar{a}})$. Reversal of arcs on cuts are used to reduce the value of β from $2m$ to $2m - 2$. Thus the problem of evaluation of polynomials in the rooted case can be related to that of the unrooted case where the specific evaluations which give $D^{even}(0, 1, H_{\bar{a}})$ can be made. The difference equation obtained is sufficient to obtain the general polynomial properties given in the theorem.

4 Application to Statistical Physics

4.1 The $Z(\lambda)$ model

Using (1), the partition function of the $Z(\lambda)$ model is

$$Z_\lambda(H) = \sum_{c_1=0}^{\lambda-1} \cdots \sum_{c_\nu=0}^{\lambda-1} \prod_{a\in A} w_a(\delta c(a)) \quad (26)$$

where

$$w_a(\alpha) = e^{-h_a(\alpha)} \quad (27)$$

is the Boltzmann factor associated with the arc a.

In the following section we consider the λ-state Potts model, the correlation function of which is expressed in terms of unrestricted PDs and yields the pair-connectedness for ordinary percolation theory when $\lambda = 1$. We also consider versions of the $Z(\lambda)$ model in which the potential differences are restricted to be of fixed parity. For odd λ these yield a type of Chiral Potts model related to directed percolation theory. The case of even λ is covered later along with the corresponding flow model.

4.2 Odd and Even PD models for odd λ and the Potts model.

A special case of the $Z(\lambda)$ model, which was called the "odd-PD model" in (Arrowsmith & Essam 1990) is defined by restricting the sum in (26) to be over potentials such that $\delta c(a) \in C^{odd}$ for all $a \in A$. Similarly an "even-PD model" may be defined by restricting the PD to be everywhere even. Other restrictions may be imposed and we denote the set of allowed potentials for a general constraint by $\mathcal{P}^*_\lambda(H)$ and the corresponding partition function by $Z^*_\lambda(H)$.

4.2.1 *The Partition Function*

Without loss of generality the partition function may be normalised so that $w_a(0) = 1$ and assuming this condition we obtain the following generalisation of (3)

$$Z^*_\lambda(H) = \sum_{\emptyset\subseteq A'\subseteq A} Z^{*+}_\lambda(H/H'), \quad (28)$$

where

$$Z^{*+}_\lambda(H) = \sum_{c\in\mathcal{P}^{*+}_\lambda(H)} \prod_{a\in A} w_a(\delta c(a)). \quad (29)$$

Suppose now that $w_a(\alpha)$ has the same value for all non-zero values of α which satisfy the $*$ condition, then we may write

$$w_a(\alpha) = \begin{cases} 1 & \text{for } \alpha = 0, \\ w_a & \text{for non-zero } \alpha \text{ allowed by } *, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

This together with (28) and (29) yields

$$Z^*_\lambda(H) = \sum_{\emptyset\subseteq A'\subseteq A} P^{*+}_\lambda(H/H') \prod_{a\in A\setminus A'} w_a \quad (31)$$

which allows the partition function to be seen as a generating function for the numbers of proper restricted PDs. In the case that * represents odd or even or the condition is removed, the work of previous sections shows that the partition function has polynomial dependence on λ. When the parity of the PD is fixed different polynomials are required for odd and even λ.

The partition function may also be written as a percolation average by making use of the following identity.

$$w_a(\alpha) = w_a\epsilon^*(\alpha) + (1 - w_a)\Delta(\alpha), \quad (32)$$

where

$$\Delta(\alpha) = \begin{cases} 1 & \alpha = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (33)$$

and

$$\epsilon^*(\alpha) = \begin{cases} 1 & \alpha \text{ allowed by } *, \\ 0 & \text{otherwise.} \end{cases} \quad (34)$$

Substitution in (26) and expansion of the product yields

$$Z^*_\lambda(H) = \sum_{c_1=0}^{\lambda-1} \cdots \sum_{c_\nu=0}^{\lambda-1} \sum_{A'\subseteq A} \prod_{a\in A'} [(1 - w_a)\Delta(\delta c(a))] \times \prod_{a\in A\setminus A'} [w_a\epsilon^*(\delta c(a))] \quad (35)$$

and carrying out the summation over the c_i gives

$$Z^*_\lambda(H) = \sum_{A'\subseteq A} P^*_\lambda(H/H') \prod_{a\in A'} (1 - w_a) \prod_{a\in A\setminus A'} w_a \quad (36)$$

which is the expectation value of $P^*_\lambda(H/H')$ in a bond percolation model on H in which the arc a is present with probability $p_a = 1 - w_a$.

4.2.2 *The Correlation Function*

The correlation function $U^*_{\beta,\lambda}(H)$ is given by (26) with the restriction $\delta c(\bar{a}) = \beta$ placed on the summation which is normalized by dividing by $Z^*_\lambda(H)$.

The above analysis for the partition function gives

$$U^*_{\beta,\lambda}(H) = \frac{1}{Z^*_\lambda(H)} \sum_{A'\subseteq A} P^*_{\beta,\lambda}(H^+_{\bar{a}}/H') \prod_{a\in A'} (1-w_a) \prod_{a\in A\setminus A'} w_a \quad (37)$$

where the sum is the percolation average of $P^*_{\beta,\lambda}(H^+_{\bar{a}}/H')$. Notice that

$$U^*_{0,\lambda}(H) = \frac{Z^*_\lambda(\tilde{H})}{Z^*_\lambda(H)} \quad (38)$$

so we will normally consider $\beta > 0$. Here $\tilde{H}$ is the graph H with vertices 1 and 2 identified.

4.2.3 *The Potts Model and Ordinary Percolation*

If $\epsilon^*(\alpha) \equiv 1$ we obtain the standard Potts model in which all PDs are allowed and all non-zero values of the PD get the same weight. The numbers of PDs in (36) and (37) are therefore unrestricted and the correlation function for $\beta > 0$ is a directing independent rational function $U(\beta, \lambda, G)$ of β and λ. Substituting from (5) and (12) gives, $U(\beta, \lambda, G) =$

$$\frac{\sum_{A' \subseteq A} \lambda^{\omega(G')-1}(1-\gamma(G')) \prod_{a\in A'}(1-w_a) \prod_{a \in A\setminus A'} w_a}{\sum_{A'\subseteq A} \lambda^{\omega(G')} \prod_{a\in A'}(1-w_a) \prod_{a\in A\setminus A'} w_a}. \tag{39}$$

where we have used

$$\nu(G_{\bar a}^+/G') = \omega(G'), \tag{40}$$

$$\chi_{\bar a}(G_{\bar a}^+/G') = 1 - \gamma((G')_{\bar a}^+), \tag{41}$$

and γ is the indicator for a connection between vertices 1 and 2 in G. This is the result of Kasteleyn and Fortuin, 1972. Note that setting $\lambda = 1$ in (39) gives

$$U(\beta, 1, G) = 1 - C_{1,2}(1-w, G) \tag{42}$$

where $C_{1,2}(p, G)$ is the probability that 1 and 2 are connected by a path in an undirected bond percolation process on G in which p_a is the probability that the arc a is open.

4.2.4 *A Chiral Potts Model and Dual Directed Percolation for λ odd.*

We have seen that for λ odd, the even and odd chromatic polynomials are directing dependent and hence, using (36) and (37), so are the corresponding partition function and correlation function.

It is sufficient to consider even PDs since any PD may be seen as an even PD on some other directing of the same graph. Using theorem 1 together with (36) and (37) shows that the values of the partition function and correlation function, for odd $\lambda \geq 3$ and even $\beta \geq 2$, are interpolated by a polynomial, $Z^{even}(\lambda, H)$, in λ and a rational function, $U^{even}(\beta, \lambda, H)$, in λ and β, respectively. Using these interpolating formulae, combined with (6), (36) and (37), and setting $\beta = 0$ and $\lambda = 1$ yields

$$U^{even}(0, 1, H) = \sum_{A'\subseteq A} \chi_{\bar a}(H_{\bar a}^+/H') \prod_{a\in A'}(1-w_a) \prod_{a\in A\setminus A'} w_a. \tag{43}$$

In the dual directed percolation model (Dhar et al 1981) fluid is only allowed to percolate parallel to the arc a with probabilty p_a but in both directions with probability $1 - p_a$. In terms of this model, the above percolation average may therefore be interpreted as a blocking probability,

$$U^{even}(0, 1, H) = \bar C_{1,2}(1-w, H) \tag{44}$$

where $\bar C_{1,2}(p, H)$ is the probability that the passage of fluid from 1 to 2 in H is blocked in the Dhar *et al* model. Equation (44) is an extension of Kasteleyn and Fortuin's result (42) for ordinary percolation to directed percolation.

4.3 Odd and Even flow models

It may be shown (Alcaraz & Köberle 1980,1981) that the transmissivity, defined in the introduction, may be written in the form

$$T_{\alpha,\lambda}(H) = \frac{N_{\alpha,\lambda}(H)}{N_{0,\lambda}(H)} \tag{45}$$

where

$$N_{\alpha,\lambda}(H) = \sum_{\alpha-flows} \prod_{a\in A} t_a(\phi(a)) \tag{46}$$

Here, for $\alpha = 1, \ldots, \lambda - 1$, an α-flow is a mod-λ flow on $H_{\bar a}^+$ with $\phi(\bar a) = \alpha$. In the context of flows $H_{\bar a}^+$ will always be the graph H with the additional arc $\bar a = (2, 1)$. The arc transmissivity $t_a(\phi)$ is the transmissivity $T_{\phi,\lambda}(a)$ of the graph consisting of a single arc a and is determined by $w_a(\alpha)$ and vice-versa (Arrowsmith & Essam 1990). It follows that $t_a(0) = 1$. The denominator $N_{0,\lambda}(H)$ is proportional to the partition function of the graph H since when $\alpha = 0$ the α-flow becomes a normal flow on the graph H with $\bar a$ deleted.

An "odd-flow model" was defined in (Arrowsmith & Essam 1990) in a similar manner to the odd-PD model by restricting the flow ϕ on any arc to be zero or odd. As in the case of PDs, more general constraints are possible and we again denote the constraint by $*$.

Formulae which parallel those for PDs may be obtained but in general H/H' is replaced by H' since setting a PD to zero is equivalent to contracting an arc whereas zero flow corresponds to arc deletion. Thus if for any flow, H' is the subgraph on which the flow is positive we may write

$$N_{\alpha,\lambda}^*(H) = \sum_{A'\subseteq A} F_{\alpha,\lambda}^{*+}((H')_{\bar a}^+) \prod_{a\in A'} t_a \tag{47}$$

and hence $N_{\alpha,\lambda}^*(H)$ is the generating function for the numbers of restricted proper flows on the subgraphs of H which satisfy the constraint $*$.

An interpretation in terms of percolation theory is again possible by assuming that all non-zero flows satisfying $*$ have the same transmissivity. Thus the flow analogue of (32) is

$$t_a(\phi) = t_a\epsilon^*(\phi) + (1-t_a)\Delta(\phi). \tag{48}$$

Substituting (48) in (46) and carrying out an analysis similar to that for $U_{\beta,\lambda}$ we find that

$$N_{\alpha,\lambda}^*(H) = \sum_{A'\subseteq A} F_{\alpha,\lambda}^*((H')_{\bar a}^+) \prod_{a\in A'} t_a \prod_{a\in A\setminus A'} (1-t_a) \tag{49}$$

where $F^*_{\alpha,\lambda}((H')^+_{\bar{a}})$ is the number of restricted α-flows on $(H')^+_{\bar{a}}$.

Using (14) in the case of unrestricted flows shows that for fixed $\alpha > 0$, $N_{\alpha,\lambda}(H)$ is a polynomial in λ which is independent of α. In the special case $\alpha = 0$, $F_{0,\lambda}(H) = F_\lambda(H)$ and (8) gives a second polynomial. Putting these together we find for $\alpha > 0$ the transmissivity is the rational function (Essam & Tsallis 1986)

$$T(\alpha,\lambda,G) = \frac{\sum_{A'\subseteq A} \lambda^{c(G')}\gamma(G')\prod_{a\in A'} t_a \prod_{a\in A\setminus A'}(1-t_a)}{\sum_{A'\subseteq A} \lambda^{c(G')}\prod_{a\in A'} t_a \prod_{a\in A\setminus A'}(1-t_a)}. \tag{50}$$

Setting $\lambda = 1$ in this formula leads to

$$T(\alpha,1,G) = C_{1,2}(t,G) \tag{51}$$

which is the pair connectedness for undirected percolation with $p_a = t_a$.

In the case of even flows theorem 2 shows that the transmissivity $T_{\alpha,\lambda}(H)$ for odd $\lambda \geq 3$ and even $\alpha \geq 2$ is interpolated by a rational function $T(\alpha,\lambda,H)$. Setting $\alpha = 0$ and $\lambda = 1$ in this function gives a formula similar to (50) with $\lambda = 1$ but now $\gamma(G')$ is now the indicator for a directed path from 1 to 2 in H'. Equation (51) is still valid with $C_{1,2}(t,H)$ as the pair connectedness for directed percolation.

We now consider the case of even λ in a more general form which allows both odd and even flows to occur but with different weight.

4.4 Mixed Parity Flows and Polychromatic percolation

The standard Potts model (Redner 1981) is from a vector spin model. An n-dimensional unit spin vector $\mathbf{s}_i$ is located at each vertex i of G and the vertex states are the allowed directions of this vector. These are the position vectors of the $\lambda = n+1$ corners of a generalised tetrahedron relative to its centre. The interaction energy associated with the arc $a = (i,j)$ is $K_a(1 - \mathbf{s}_i.\mathbf{s}_j)$ where the additive constant is chosen so that the energy is zero when the spins are parallel. Because of the symmetry, the scalar product has the same value whenever the spins are not parallel and hence for the standard Potts model

$$w_a^{Potts}(\alpha) = \begin{cases} 1 & \text{for } \alpha = 0, \\ w_a & \text{otherwise} \end{cases} \tag{52}$$

where $w_a = \exp(\frac{-\lambda K_a}{\lambda - 1})$.

4.4.1 *The n-component cubic model*

Now suppose instead that the vectors point to one of the $\lambda = 2n$ centres of the faces of an n-dimensional hypercube. The scalar product has three values depending on whether the spins are parallel, anti-parallel or orthogonal. In the *cubic model* an additional interaction parameter L_a is introduced in the case that the vectors are orthogonal and the interaction energy associated with the arc a is $K_a(1 - \mathbf{s}_i.\mathbf{s}_j) + L_a(1 - (\mathbf{s}_i.\mathbf{s}_j)^2)$. Thus positive K_a favours the ferromagnetic state in which the spins are parallel and positive L_a gives equal enhancement to the probability of finding the spins aligned parallel or antiparallel. We choose the potential c_i to have values $0, 1, \ldots n-1$ when the spin vector is parallel to the cartesian axes $1, 2, \ldots n$ respectively and suppose that the values $n, n+1, \ldots 2n-1$ correspond to the reversed vectors. The Boltzmann factor may then be written

$$w_a^{cubic}(\alpha) = \begin{cases} 1 & \text{for } \alpha = 0, \\ w_a & \text{for } \alpha = n, \\ w_a u_a & \text{otherwise.} \end{cases} \tag{53}$$

where $w_a = \exp(-2K_a)$ and $u_a = \exp(K_a - L_a)$. Thus when $K_a = L_a$ we have the standard $2n$-state Potts model and if $L_a \to \infty$ we have a two state Potts model which is the Ising model.

The interest of this model in the present context is that its arc transmissivity for non-zero flow depends only on the parity of the flow. Thus

$$t_a^{cubic}(\phi) = \begin{cases} 1 & \text{for } \phi = 0, \\ t_a^{(1)} & \text{for } \phi \text{ odd}, \\ t_a^{(2)} & \text{for } \phi \text{ even.} \end{cases} \tag{54}$$

where

$$t_a^{(1)} = \frac{1 - w_a}{1 + w_a + 2(n-1)w_a u_a} \tag{55}$$

and

$$t_a^{(2)} = \frac{1 - 2w_a u_a + w_a}{1 + w_a + 2(n-1)w_a u_a}. \tag{56}$$

When $K_a = 0$, the odd transmissivity variable is zero and we have the even flow model discussed in the previous section. This is just an n-state Potts model.

4.4.2 *The general mixed flow model*

Equation (54) may be taken as the definition of a general mixed flow model and may be extended to odd λ as discussed below. Again we have an expansion in terms of proper flows, thus

$$N_{\alpha,\lambda}^{mixed}(H) = \sum_{A'\subseteq A} \sum_{A'_{even}\subseteq A'} F_{\alpha,\lambda}^{mixed+}((H')^+_{\bar{a}}, H'_{even}) \times \prod_{a\in A\setminus A'_{even}} t_a^{(1)} \prod_{a\in A'_{even}} t_a^{(2)} \tag{57}$$

The formulation as a percolation problem is achieved using the identity

$$t_a(\phi) = t_a^{(1)}\epsilon_1(\phi) + t_a^{(2)}\epsilon_2(\phi) + (1 - t_a^{(1)} - t_a^{(2)})\Delta(\phi). \quad (58)$$

which leads to

$$N_{\alpha,\lambda}^{mixed}(H) = \sum_{A'\subseteq A}\sum_{A'_{even}\subseteq A'} F_{\alpha,\lambda}^{mixed}((H')_{\bar{a}}^{+}, H'_{even})\times \prod_{a\in A'\backslash A'_{even}} t_a^{(1)} \prod_{a\in A'_{even}} t_a^{(2)} \prod_{a\in A\backslash A'} (1 - t_a^{(1)} - t_a^{(2)}), \quad (59)$$

where $F_{\alpha,\lambda}^{mixed}((H')_{\bar{a}}^{+}, H'_{even})$ is the number of flows which are even on the subgraph H'_{even} and odd elsewhere. For even λ this is given by (20) which shows that $N_{\alpha,\lambda}^{mixed}(H)$ has polynomial dependence on λ. There are three different polynomials depending on whether α is even, odd or zero. The sum (59) is the expected value of $F_{\alpha,\lambda}^{mixed}(H', H'_{even})$ in a polychromatic bond percolation problem (Zallen 1977) in which the open arcs may be of two different types which occur independently with probabilities $t_a^{(1)}$ and $t_a^{(2)}$. In the more general k-chromatic percolation problem there are k different values of $t_a^{(\alpha)}$ for $\phi > 0$ where $1 < k < \lambda$ which is the percolation formulation of the general $Z(\lambda)$ model.

4.4.3 *Mixed flow models for odd λ*

Suppose now that λ is odd so that the number of flows is directing dependent. In this case

$$F_{\alpha,\lambda}^{mixed}(H, H_{even}) = F_{\alpha,\lambda}^{even}(H|H_{odd}) \quad (60)$$

where $H_{odd} = H\backslash H_{even}$ and $H|H'$ is the graph H with the arcs of the subgraph H' reversed. It is therefore possible to write equation (59) in the form

$$N_{\alpha,\lambda}^{mixed}(H) = \sum_{A'\subseteq A}\sum_{A''\subseteq A'} F_{\alpha,\lambda}^{even}((H')_{\bar{a}}^{+}|H'')\times \prod_{a\in A''} t_a^{(1)} \prod_{a\in A'\backslash A''} t_a^{(2)} \prod_{a\in A\backslash A'} (1 - t_a^{(1)} - t_a^{(2)}) \quad (61)$$

which is the expected value of $F_{\alpha,\lambda}^{even}((H')_{\bar{a}}^{+}|H'')$ in an oriented percolation model (Redner 1982) where the arcs are present with probability $t_a^{(1)} + t_a^{(2)}$ and are reversed relative to H with probability $t_a^{(1)}$. From theorem 2 it follows that $N_{\alpha,\lambda}^{mixed}(H)$, for even $\alpha \geq 2$ and odd $\lambda \geq 3$ is interpolated by a polynomial in α and λ which when evaluated at 0 and 1 gives the pair connectedness $C_{1,2}(t^{(1)}, t^{(2)})$ for this oriented model.

These results have an obvious extension to mixed PD models where a different Boltzmann factor is attached to arcs of odd and even PD. In the odd λ case, setting $\beta = 0$ and $\lambda = 1$ in the interpolated correlation function gives the blocking probability in an extension of the Dhar et al, 1981 dual directed percolation model (Redner 1982). Every edge in this model is open in one direction or the other or in both directions simultaneously with different probabilities.

We note that the mixed flow model can also be considered as a mixed PD model. If we normalize $w_a(\beta)$ for this model so that $w_a(0) = 1$ then for $\beta = 1, \ldots, \lambda - 1$ and $n = (\lambda - 1)/2$

$$w_a(\beta) = \frac{1}{1 + n(t_a^{(1)} + t_a^{(2)})}\times \left[1 - \frac{1}{2}(t_a^{(1)} + t_a^{(2)}) - \frac{i}{2}(t_a^{(1)} - t_a^{(2)})\tan\left(\frac{\pi\beta}{\lambda}\right)\right]. \quad (62)$$

The mixed PD model may also be considered as a flow model with transmissivity given by $t(0) = 1$ and

$$t_a(\phi) = \frac{1}{1 + n(w_a^{(1)} + w_a^{(2)})}\times \left[1 - \frac{1}{2}(w_a^{(1)} + w_a^{(2)}) + \frac{i}{2}(w_a^{(1)} - w_a^{(2)})\tan\left(\frac{\pi\phi}{\lambda}\right)\right] \quad (63)$$

for $\phi = 1, \ldots, \lambda - 1$.

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