Chromatic polynomials and mod λ flows on directed graphs and their applications

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Abstract We survey the problem of enumerating certain types of colourings of graphs which are dependent on orientation. Specifically, we consider colourings in which each of the $\nu = |V|$ vertices of a directed graph is coloured in one of the integer colours in $\mathcal{C} = \{0, 1, \ldots, \lambda - 1\}$ with the constraint that $(c_j - c_i) \bmod \lambda$, when chosen as an element of $\mathcal{C}$, must be non-zero and even for all arcs (i, j), where c_i is the colour of vertex i. Similar enumerations can be made for the case of odd $(c_j - c_i) \bmod \lambda$ and they are distinguished by the parity of λ.

For even λ and a given undirected graph, the number of even colourings is independent of the directing and the problem is easily related to the standard colouring problem.

The main results concern the enumeration and connectedness properties of even colourings for odd λ when the colour difference on a particular arc has the fixed value β. There are corresponding enumerations and properties for mod λ flows. These restricted colourings and flows are applied in various ways. The key results describe the way in which the connectedness of directed graphs is derived from even colourings and flows. Thus they have relevance to the theory of directed percolation. The even flow enumerations have application in the theory of directed polymer networks. A generating function for the numbers of colourings with even and odd colour difference is seen as the partition function of a Potts model in which the colours are the states of spins attached to the vertices of the graph. For odd λ the model exhibits chirality.

Finally, we briefly consider the role of partition functions, and particularly those for the Potts Model, in producing knot invariants.

1 The chromatic polynomial

If the choice of attaching one of λ colours in $\mathcal{C} = \{0, 1, \ldots, \lambda - 1\}$ to each of the vertices of a graph $G = (V, E)$ is unrestricted, then the number of such colourings is counted by the polynomial $P(\lambda, G) = \lambda^\nu$, where $\nu = |V(G)|$ is the number of vertices of G. Furthermore, the number of ways in which the vertices of G can be coloured such that vertices adjacent to a common edge receive different colours, known as the number of *proper* colourings, is also well-known to be counted by a polynomial $P^+(\lambda, G)$ in λ, the *chromatic polynomial*. The graph G can contain loops but then $P^+(\lambda, G) = 0$. Moreover, the number of colourings is not changed if a multiple edge is reduced to a single edge. Every colouring of a graph G is equivalent to a proper colouring on a contracted graph G/G' obtained from G by contracting the edges E' which have identical colours on adjacent vertices and so we can write

$$P(\lambda, G) = \sum_{E' \subseteq E} P^+(\lambda, G/G'). \tag{1}$$

Möbius inversion gives the polynomial form of $P^+(\lambda, G)$ due to Whitney [36]:

$$P^+(\lambda, G) = \sum_{E' \subseteq E} (-1)^{|E \setminus E'|} P(\lambda, G/G') \tag{2}$$

$$= \sum_{E' \subseteq E} (-1)^{|E \setminus E'|} \lambda^{\nu(G/G')}. \tag{3}$$

In the following sections we shall impose further restrictions on the colourings and proper colourings. Equations (1) and (2) will also apply to these restricted colourings.

Hence for each type of colouring considered in this paper, it is possible to restrict the enumeration to those either with or without the proper constraint. Also a proof of a polynomial property without the proper constraint will imply the same property for proper colourings and vice-versa.

1.1 Colour difference restrictions

We now consider $\mathcal{C}$ with mod λ addition as $\mathbf{Z}_\lambda$, the integers mod λ. Let $\mathcal{D}(G)$ be the set of directed graphs obtained by directing the edge set E of G in all possible ways. Let $c : V \to \mathcal{C}$ be a λ-colouring of G and let $H \in \mathcal{D}(G)$ have arc set $A(H)$. Associated with c and H is a mod λ colour difference $\delta c : A(H) \to \mathbf{Z}_\lambda$ where, if $a = (i, j) \in A(H)$, then $\delta c(a) = c_j - c_i \bmod \lambda \in \mathcal{C}$. Clearly signed colour differences around a cycle sum to zero and this is also a possible defining property; that is an edge valuation is a colour difference (i.e. it supports a λ-colouring) iff its signed sum around every cycle is zero mod λ.

In the next section we pose a number of further colouring problems by imposing constraints on the image of the colour difference function δc. First of all we note that the more familiar proper and rooted colourings can be described in this way.

1.1.1 Proper colourings

Note that proper colourings are those for which $c_i \neq c_j$ for all $(i, j) \in A(H)$. Thus a λ-colouring c on $H \in \mathcal{D}(G)$ is *proper* if the colour difference δc satisfies $\delta c(A(H)) \subseteq \mathcal{C}\backslash\{0\}$.

1.1.2 Rooted colourings

A directed graph H becomes a rooted graph $H_{\bar{a}}$ by distinguishing an arc $\bar{a}$. A λ-colouring c on $H_{\bar{a}}$ is said to be *rooted* if the colour difference $\delta c(\bar{a})$ equals β, a fixed value in $\mathcal{C}$. The number of such rooted λ-colourings, $P_{\beta,\lambda}(H_{\bar{a}})$, is independent of the directing of H and is again counted by a polynomial in λ which for *non-zero* β is independent of β and given by

$$P(\beta, \lambda, H_{\bar{a}}) = \lambda^{\nu(H)-1}\eta_{\bar{a}}(H) \qquad (4)$$

where the *cocycle indicator* $\eta_{\bar{a}}(H)$ is 1 if H has a cocycle (or cut) containing $\bar{a}$ (i.e. $\bar{a}$ is not a loop) and is 0 otherwise.

1.2 The number of colour differences.

For any given colour difference on $H \in \mathcal{D}(G)$, there are λ different associated colourings on any given connected component of G. Let the number of mod λ colour differences be denoted by $D_\lambda(H)$ then if $\omega(G)$ is the number of components in G

$$D_\lambda(H) = \lambda^{-\omega(G)} P_\lambda(H). \qquad (5)$$

This formula applies to all the colouring classes in this paper, in particular the number of unrestricted colour differences is counted by the polynomial

$$D(\lambda, H) = \lambda^{\nu(H)-\omega(H)} = \lambda^{r(H)} \qquad (6)$$

where $r(H)$ is known as the cocycle rank of H.

Consideration of the number of colour differences rather than the number of colourings brings out the duality relations with the number of flows to be considered later.

2 Colourings with even and odd colour differences

For either the proper or rooted cases described above we can impose further conditions on the λ-colourings. Given $H \in \mathcal{D}(G)$, consider those λ-colourings c for which the colour differences $\delta c(a)$ are even values of $\mathcal{C}$, $a \in A(H)$. The number of such " even" λ-colourings is denoted by $P_\lambda^{even}(H)$. If the allowed values of the colour differences are taken to be odd or zero, then we have a further enumerator $P_\lambda^{odd}(H)$. Crucially, for odd λ, the numbers of even and odd colourings are dependent on the directing of the underlying graph.

Moreover, the rooted analogues of these even and odd colourings have corresponding enumerators denoted by $P_{\beta,\lambda}^{even}(H_{\bar{a}})$ and $P_{\beta,\lambda}^{odd}(H_{\bar{a}})$. The notation for colourings can be extended to colour differences of rooted graphs by using $D_{\beta,\lambda}^{even}(H_{\bar{a}})$ and $D_{\beta,\lambda}^{odd}(H_{\bar{a}})$. The properties of such functions are considered in detail in [4]. The origin of the problem addressed here lies in both the study of chiral Potts models and directed percolation [6] and decompositions of the chromatic polynomial [4].

The relationship (5) extends to rooted colourings and also even and odd colourings. Thus, for example, $D_{\beta,\lambda}^{even}(H_{\bar{a}}) = \lambda^{-\omega(H_{\bar{a}})} P_{\beta,\lambda}^{even}(H_{\bar{a}})$.

2.1 Odd λ

The enumerators of λ-colourings described in section 1 were independent of the directing of the underlying graph G. It was shown that these colourings can be counted by polynomials of degree at most $r(G)+1$ and the cocycle indicator can be found by formally setting $\beta = 0$ and $\lambda = 1$ in (4) to obtain

$$P(0, 1, G_{\bar{a}}) = \eta_{\bar{a}}(G). \tag{7}$$

We now describe analogous properties for even λ-colourings which for odd λ are directing dependent. In this case no explicit polynomial formulae have been found for a general graph and the proof of the polynomial property is much more difficult and requires an inductive argument.

We need the following definition. A set of arcs $b \subseteq A(H)$ is a *directed cut* of the graph H if the vertex set V has a non-trivial partition $[S, S']$ such that (i) b is the set of arcs between S and S' and (ii) the arcs of b are all directed from S to S'. Let $\chi_{\bar{a}}(H)$ be a directed cut indicator corresponding to $\eta_{\bar{a}}(G)$.

Theorem 1. *[4] Let $H \in \mathcal{D}(G)$ have a rooted arc $\bar{a}$, then the number of colour differences, $D_{\beta,\lambda}^{even}(H_{\bar{a}})$, for $\beta = 2m$ and $\lambda = 2n+1$ with $m, n \in \mathbf{Z}^+$, may be obtained by evaluating a polynomial $D^{even}(\beta, \lambda, H_{\bar{a}})$, in β and λ, having joint degree at most $r(H)-1$, where $r(H) = \nu(H) - \omega(H)$ is the cocycle rank of H, with the property*

$$D^{even}(0, 1, H_{\bar{a}}) = \chi_{\bar{a}}(H), \tag{8}$$

where $\chi_{\bar{a}}(H)$ is the directed cut indicator for root arc $\bar{a}$. There exists a similar polynomial for β odd. Also, the values of $D^{even}_{0,\lambda}(H_{\bar{a}}) = D^{even}_{\lambda}(H^{\gamma}_{\bar{a}})$, where $H^{\gamma}_{\bar{a}}$ is the graph obtained by contracting the arc $\bar{a}$ of $H_{\bar{a}}$, are given by a polynomial $D^{even}(\lambda, H^{\gamma}_{\bar{a}})$ of degree at most $r(H^{\gamma}_{\bar{a}})$ in λ such that $D^{even}(1, H^{\gamma}_{\bar{a}}) = 1$. More generally, for $H \in \mathcal{D}(G)$, $D^{even}(\lambda, H)$ is a polynomial in λ of degree at most $r(H)$ such that $D^{even}(1, H) = 1$.

The proof of Theorem 1 is given in detail in [4]. The essential idea of the proof is to show by induction that $D^{even}_{2m,2n+1}(H_{\bar{a}})$, where $m, n \in \mathbf{Z}^+$, can be enumerated by polynomials of degree at most $\nu(H) - \omega(H)$ in the variables m and n. An inductive step is set up which relates $D^{even}_{2m,2n+1}(H_{\bar{a}})$ and $D^{even}_{2m-2,2n+1}(H_{\bar{a}})$. Reversal of arcs on cuts are used to reduce the value of β from $2m$ to $2m-2$. Thus the problem of evaluation of polynomials in the rooted case can be related to that of the unrooted case where the specific evaluations which give $D^{even}(0, 1, H_{\bar{a}})$ can be made. The difference equation obtained together with a reversal-deletion-contraction rule (see Section 4.3.2) is sufficient to obtain the general polynomial properties given in the theorem.

Lemma 1. *(Directed cut reversal) [4] Let b be a directed cut containing the root arc $\bar{a}$ and let H^{ρ}_{b} be the graph obtained by reversing all arcs of b other than $\bar{a}$. For odd λ, there exists a bijection between the even mod λ colour differences on $H_{\bar{a}}$ and H^{ρ}_{b} with difference β and $(\beta + 1)$ respectively in the root arc $\bar{a}$.*

The equivalence is obtained by increasing the colour difference on the arcs of b by one and then reversing the non-root arcs of b. The resulting correspondence is given by $\delta c' = \delta c$ on the arc set $A(H)\backslash A(b)$ and for $a \in A(b)$, $a \neq \bar{a}$, $\delta c'(a^{\rho}) = \lambda - 1 - \delta c(a)$ mod λ where $\delta c \in \mathcal{D}^{even}_{\beta,\lambda}(H_{\bar{a}})$, $\delta c' \in \mathcal{D}^{even}_{\beta+1,\lambda}(H^{\rho}_{b})$ and a^{ρ} is the arc a with reversed orientation. Thus

$$D^{even}_{\beta,\lambda}(H_{\bar{a}}) = D^{even}_{\beta+1,\lambda}(H^{\rho}_{b}), \tag{9}$$

where $\beta + 1$ is evaluated mod λ.

For odd λ, the various cases of the 'even' and 'odd' constraints and the parities of β are related by the following easy lemma, which shows that the counting polynomials for rooted colourings with odd colour difference can also be expressed in terms of those for colourings with even colour difference.

Lemma 2. *[4] Let $H_{\bar{a}}$ be the rooted directed graph obtained from $H \in \mathcal{D}(G)$ with root arc $\bar{a}$. Let $H^{\rho}_{\bar{a}}$ be the rooted directed graph obtained from $H_{\bar{a}}$ by reversing the orientation of every arc of H except for the root arc $\bar{a}$. The graph H^{ρ} is obtained by reversing the orientation of every arc of H. For λ odd,*

$$(i) \qquad P^{odd}_{\lambda}(H) = P^{even}_{\lambda}(H^{\rho}), \tag{10}$$

and

$$(ii) \qquad P^{odd}_{\beta,\lambda}(H_{\bar{a}}) = P^{even}_{\beta,\lambda}(H^{\rho}_{\bar{a}}). \tag{11}$$

Moreover, if $(H_{\bar{a}})^{\rho}$ denotes $H_{\bar{a}}$ with all arcs reversed, then

$$(iii) \qquad P^{odd}_{\beta,\lambda}(H_{\bar{a}}) = P^{even}_{\lambda-\beta,\lambda}((H_{\bar{a}})^{\rho}), \tag{12}$$

where $\lambda - \beta$ is an element of $\mathcal{C}$ by reducing mod λ if necessary.

2.2 Even λ

The problem of enumerating even and odd colourings is straightforward when λ is restricted to be even [4]. Given $H, H' \in \mathcal{D}(G)$, a λ-colouring c on H with even colour difference δc is also even on H' since δc is the same on H and H' for all coherently oriented arcs and $\delta c(a) = \lambda - \delta c(a') \bmod \lambda$, which preserves parity, when the arcs of the same edge $a \in A(H)$ and $a' \in A(H')$ are oppositely oriented. The same argument can be applied for λ colourings with odd colour differences which are also preserved by arc reversal. The number of colourings is therefore directing independent.

3 Mod λ flows

The concept which is dual to mod λ colour differences is that of mod λ flows [35]. A mod λ flow on the directed graph $H = (V, A) \in \mathcal{D}(G)$ is a map $\phi : A \to \mathbf{Z}_\lambda$ such that

$$\sum_{a \in A_v^+} \phi(a) = \sum_{a \in A_v^-} \phi(a) \tag{13}$$

in $\mathbf{Z}_\lambda$, for every vertex v, where the sets A_v^+, A_v^- are respectively the arcs oriented in and out of the vertex v. The number of such flows is given by a polynomial $F(\lambda, G)$ and depends only on the underlying graph G and not the directing H. Rooted flows on the directed graph $H_{\bar{a}}$ have the extra requirement that $\phi(\bar{a}) = \alpha$, a fixed value.

Furthermore, the structure dual to a directed cut is a *circuit* and is defined to be a directed cycle in which the arcs are coherently oriented. This includes the case of an oriented loop. The directed cut indicator can be related to the indicator for circuits $\pi_{\bar{a}}(H)$ which has value 1 if there is a circuit in H containing $\bar{a}$ and 0 otherwise. The existence of a directed cut containing $\bar{a}$ is equivalent to the non-existence of a circuit containing $\bar{a}$ and so $\chi_{\bar{a}} = 1 - \pi_{\bar{a}}$.

We have the following theorem for flows which is dual to Theorem 1.

Theorem 2. *[4] Let $H \in \mathcal{D}(G)$ have a rooted arc $\bar{a}$, then the number of flows $F^{even}_{\alpha,\lambda}(H_{\bar{a}})$, for $\alpha = 2m$, $\lambda = 2n+1$ with $m, n \in \mathbf{Z}^+$, may be obtained by evaluating a polynomial, $F^{even}(\alpha, \lambda, H_{\bar{a}})$, in α and λ having joint degree at most $c(H) - 1$, where $c(H)$ is the cycle rank of H, with the property*

$$F^{even}(0, 1, H_{\bar{a}}) = \pi_{\bar{a}}(H). \tag{14}$$

There also exists a similar polynomial for α odd. Also, the values of $F^{even}_{0,\lambda}(H_{\bar{a}}) = F^{even}_{\lambda}(H^{\delta}_{\bar{a}})$ are given by a polynomial $F^{even}(\lambda, H^{\delta}_{\bar{a}})$ of degree at most $c(H^{\delta}_{\bar{a}})$ in λ such that $F^{even}(1, H^{\delta}_{\bar{a}}) = 1$. More generally, for $H \in \mathcal{D}(G)$, $F^{even}(\lambda, H)$ is a polynomial in λ of degree at most $c(H)$ such that $F^{even}(1, H) = 1$.

Remark It should be noted that the particular evaluations in Theorems 1 and 2 of the interpolating functions D^{even} and F^{even} with $\beta = 0, \lambda = 1$ and $\alpha = 0, \lambda = 1$ respectively, which give the connectivity properties of a graph, are not the same as the combinatorial evaluations $D^{even}_{0,1}(H)$ and $F^{even}_{0,1}(H)$ which are both identically 1.

4 Applications

4.1 Connectedness of graphs and directed percolation

Percolation theory was introduced by Broadbent and Hammersley [14]. Suppose that each arc of a directed graph H has probability p of being "open" and $1-p$ of being "closed". The central problem in directed percolation theory is to discuss the properties of the random set of vertices D which can be reached from a given vertex i by at least one directed path of open arcs. The main interest is in the case of infinite graphs in which case D has a positive probability, the percolation probability, of being infinite at or above a certain value of p, the critical probability. If we ignore the directings of the arcs of the path then we have the corresponding undirected percolation problem.

Another property which shows critical behaviour is the expected size $S_i(p,H)$ of D, known as the mean cluster size. $S_i(p,H)$ increases with p and becomes infinite at a critical value p_c of p depending on the graph. The value of p_c may be estimated by expanding $S_i(p,H)$ as a power series in p and using Padé approximant methods. To determine such an expansion we use the relation

$$S_i(p,H) = \sum_{j\in V} C_{ij}(p,H) \tag{15}$$

where the *pair connectedness* $C_{ij}(p,H)$ is the probability of an open path from i to j. This may be expanded as a polynomial in p as follows.

By inclusion-exclusion we have the following expansion [1], [2],

$$C_{ij}(p,H) = \sum_{S\subseteq \mathcal{S}_{ij}(H)} pr(S)(-1)^{|S|+1} \tag{16}$$

where $\mathcal{S}_{ij}(H)$ is the collection of directed paths from i to j and $pr(S)$ is the probability that at least the paths in the subset S are open. By noting that $pr(S) = p^{|A_S|}$, where A_S is the union of the arc sets of the paths in S, and collecting together the terms of the above sum for which $A_S = A' \subset A$, we obtain a subgraph expansion which takes the form

$$C_{ij}(p,H) = \sum_{A'\subseteq A} \vec{d}_{ij}(H')p^{|A'|} \tag{17}$$

where $H' = (V, A')$ and the corresponding coefficient $\vec{d}_{ij}(H')$ is solely a function of the subgraph H', known in the literature as the "d-weight", [6], of H'. By taking $p = 1$, and using Möbius inversion we have

$$\vec{d}_{ij}(H) = \sum_{A'\subseteq A} (-1)^{|A\setminus A'|}\pi_{ij}(H'), \tag{18}$$

where $\pi_{ij}(H') = C_{ij}(1,H')$ is 1 if there a directed path in H' from i to j and zero otherwise.

For percolation on an undirected graph $G = (V,E)$ the "d-weight" in the expansion corresponding to (17) is denoted by $d_{ij}(G')$ and is given by

$$d_{ij}(G) = \sum_{E'\subseteq E} (-1)^{|E\setminus E'|}\gamma_{ij}(G'). \tag{19}$$

where the undirected connectedness indicator $\gamma_{ij}(G')$ is 1 if there a path in G' from i to j and zero otherwise. Thus the connectedness of the subgraphs plays a major part in the calculation of the d-weights required to obtain the pair-connectedness.

Let $H^+_{\bar{a}}$ be the graph obtained from H by adding the "external" arc $\bar{a} = [j, i]$ then from theorem 2

$$\pi_{ij}(H) = \pi_{\bar{a}}(H^+_{\bar{a}}) = F^{even}(0, 1, H^+_{\bar{a}}). \tag{20}$$

If $G^+_{\bar{a}}$ is the undirected graph obtained by ignoring the directing of the arcs of $H^+_{\bar{a}}$ then the number of undirected mod λ flows $F(\alpha, \lambda, G^+_{\bar{a}})$, with a non-zero flow of α on $\bar{a}$ is independent of α and is given by $\lambda^{c(G)}\gamma_{ij}(G)$, where $c(G)$ is the cycle rank of G. It follows that

$$\gamma_{ij}(G) = F(0, 1, G^+_{\bar{a}}) \tag{21}$$

which is directly analogous to (20).

Equation (15) may be generalised to give the expected number $S_i^{(m)}(p, H)$ of vertices which are m-connected to u

$$S_i^{(m)}(p, H) = \sum_j C_{ij}^{(m)}(p, H), \tag{22}$$

where $C_{ij}^{(m)}(p, H)$ is the probability that vertex j is m-connected from i. For the asymptotics of $S_i^{(m)}(p, H)$ a generalisation of the d-weight is needed together with simplifying rules, see [1]. The critical probability $p_c^{(m)}$ for this function on an infinite graph gives the onset of the existence of infinite clusters of vertices which are m-connected to i.

4.2 Vicious walkers and polymer networks on a lattice

Fisher, in his Boltzmann medal award lecture, [18], considered the problem of m lock-step walkers who start from distinct points on the real line having even integer co-ordinates and at each time step simultaneously, but independently, move a unit distance to the left or right with equal probability. *Vicious walkers* shoot one another if they arrive at the same point. The problem Fisher addressed was the determination of the *survival probability* $P_S(t)$ that m vicious walkers all stay alive for at least t steps. He also considered the *reunion probability* $P_R(t)$ that they survive and end up at distance two apart.

The space-time trajectories of the walkers are paths on a directed square lattice the sites of which are the points of the plane having integer co-ordinates the sum of which is even. These paths may be pictured as the embeddings of directed polymer chains of length t no pair of which intersect [17].

To make the problem specific let us suppose that the i^{th} walker has initial position $x_i = 2(i-1)$, then

$$P_S(t) = S_t(m)/2^{mt} \tag{23}$$

where $S_t(m)$ is the number of possible t-step paths of the m walkers which are non-intersecting and end anywhere. In polymer terminology this is the number of *star* *configurations*. To obtain the reunion probability further suppose that after t steps, $x_i = 2(i-1)+2q-t$, where q is the number of positive steps made by each walker, then

$$P_R(t,q) = w_t(m,q)/2^{mt}, \tag{24}$$

where $w_t(m,q)$ is the number of non-intersecting path configurations subject to the above initial and final conditions. Again in polymer terminology this is the number of *watermelon configurations*.

In [7], we formulated the reunion problem in terms of even flows on the directed square lattice. This is made possible by the following bijection between the paths of m-vicious walkers and flows.

Consider the configuration of the m walkers in the x,t-coordinate plane. The assumption of non-intersection means that if we translate the path of the i-th walker by $2(i-1)$ in the negative x-direction we then have m paths all beginning at the point $(0,0)$ and terminating at (k,t) where $k = 2q - t$. Moreover, the translated paths taken pairwise do not cross over although they might share common arcs or vertices. If a unit flow is attached to each walk, the above translation of paths then provides an essentially rooted and directed integer flow from root $(0,0)$ to root (k,t) with a flow of m through the "root" vertices. Note that this is a **Z**-flow in the sense that the Kirchhoff constraint at each vertex is zero over the *integers*. This is equivalent to considering an even mod λ flow, for odd $\lambda > 2m$, with root flow $2m$ obtained by attaching a flow of 2 to each walker. The odd mod λ Kirchhoff constraint for an even flow at a grid vertex with two edges oriented in and out forces a **Z**-flow.

Conversely, a $2m$ rooted even flow of the type described above can be uniquely unfolded to produce m parallel walks with a flow of 2 for each walk.

It follows from Theorem 2 that these walk configurations are enumerated by a polynomial in m, see also [7,17]. This is not apparent in the Fisher approach where m appears as the dimension of a matrix.

A direct argument which shows that the number of the above **Z**-flows may be counted by a polynomial in m is as follows. We let P be the set of all possible walker paths from source $(x,t) = (0,0)$ to sink (k,t). They can be given the structure of a partially ordered set where two given paths ϕ, ϕ' satisfy $\phi < \phi'$ if the path ϕ' is to the right of ϕ in the x,t-plane. The number of flows is then obtained by distributing the flow m among the paths in any totally ordered set of P. If we denote the family of non-empty totally ordered subsets of P by Θ, the number of vicious walker configurations [7] is

$$w_t(m,q) = \sum_{\theta\in\Theta} \frac{(m-|\theta|+1)_{|\theta|-1}}{(|\theta|-1)!} \tag{25}$$

where we have used the Pochhammer symbol $(a)_k = a(a+1)(a+2)\dots(a+k-1)$ for $k > 0$ and $(a)_0 = 1$. The RHS of (25) is clearly polynomial in m.

For $0 \le q \le \frac{1}{2}t$ we have the explicit product form of the polynomial,

$$w_t(m,q) = \prod_{j=1}^{q} \frac{(m+j)_{t-2j+1}}{(j)_{t-2j+1}} \tag{26}$$

This formula was conjectured in [7] and proved in [17]. An equivalent formula, valid for $0 \leq q \leq t$ but not explicitly of polynomial form in m, is

$$w_t(m,q) = \prod_{i=0}^{m-1} \frac{(t-q+i+1)_q}{(i+1)_q} = \prod_{i=0}^{m-1} \left[\frac{i!}{(t+i+1)_i} \binom{t+2i}{q+i} \right]. \tag{27}$$

A Gaussian approximation to the binomial coefficient in (27) gives the asymptotic form as $t \to \infty$

$$w_t(m,q) \cong \left(\prod_{i=1}^{m-1} i! \right) [(2\pi)^{-1/2} 2^{m+t} t^{-m/2} \exp(-x_1^2/2t)]^m. \tag{28}$$

In [17] it was shown that equation (26) can be generalised to the case when the i^{th} walker makes q_i positive steps and $q_i \geq q_{i-1}$. The number of t-step configurations is given by

$$w_t(m, q_1, q_2, \ldots, q_m) = \prod_{1 \leq i < j \leq m} (q_j - q_i + j - i) \prod_{j=1}^{m} \frac{(t+m-j)!}{(q_j + j - 1)!(t - q_j + m - j)!}. \tag{29}$$

The number of m vicious walker configurations $S_t(m)$ which finish anywhere after t steps may be obtained by summing (29) over the values of q_i subject to $0 \leq q_1 \leq q_2 \leq \ldots \leq q_m \leq t$. In [7] it was conjectured that

$$S_t(m) = \prod_{j=1}^{\lfloor (t+1)/2 \rfloor} \frac{(m+2j-1)_{2t-4j+3}}{(2j-1)_{2t-4j+3}} \tag{30}$$

where $\lfloor \quad \rfloor$ is the floor function. The asymptotic form of $S_t(m)$ as $t \to \infty$ is

$$S_t(m) \cong \left(\frac{2^{e(m-e/2)/2}}{\pi^{e/4}} \prod_{k=1}^{e/2} (m-2k)! \right) 2^{mt} t^{-m(m-1)/4} \tag{31}$$

where $e = 2\lfloor m/2 \rfloor$. This result was also obtained by Fisher [18] using a continuum approximation.

More recently it has been brought to our attention [26] that (30) is a special case of the Bender-Knuth conjecture [11] on plane partitions which was first proved by Gordon [19]. The first published proof was by Andrews [10]. For a more general discussion between plane partitions and parallel walks, see [31], [32] and [20]. To see the connection with plane partitions for the above case of vicious walkers, number the steps of each walk from 1 to t and let $n_{i,j}$ be the number of the j^{th} positive step of the $(m-i+1)^{th}$ walk, counting from the end of the walk, n_{ij} is zero for $i > m$ or if $j > t$. Clearly $n_{ij} > n_{i,j+1}$ and the mutual avoidance condition is expressed by $n_{ij} \geq n_{i+1,j}$. Such a matrix subject to $\sum_{ij} n_{ij} = n$ is known by Bender and Knuth as an m-rowed strict plane partition of n with no part exceeding t. They conjectured ([11] eq. 8) that the number of such partitions is the coefficient of z^n in the expansion of the generating function

$$\prod_{i=1}^{t}\prod_{j=i}^{t}\frac{1-z^{m+i+j-1}}{1-z^{i+j-1}}. \tag{32}$$

The number of star polymer configurations is the number of different matrices satisfying all of the above conditions except $\sum_{ij} n_{ij} = n$ and is therefore obtained by taking the limit $z \to 1$ in (32) with the result

$$S_t(m) = \prod_{i=1}^{t}\prod_{j=i}^{t}\frac{m+i+j-1}{i+j-1} = \prod_{i=1}^{t}\frac{(m+2i-1)_{t-i+1}}{(2i-1)_{t-i+1}} \tag{33}$$

which may be rearranged to give (30).

The result corresponding to equation (27) for plane partitions is found in [28] where $w_t(m,q)$ counts tableaux, or plane partitions, of rectangular shape with bounded entries. The more general enumeration in equation (29) counts tableaux of a given arbitrary shape. This, in turn, can be interpreted in terms of dimension formula for the irreducible representation of $SL(n)$ given in [27].

4.3 Potts models

It is possible to extend the simple counting of proper colourings on a graph to more refined weighted enumerations. Enumerations in this broader class are often referred to as interaction models, see [13] for a general discussion. Here we consider a subclass of these models, originally introduced by Potts in [30], called *Potts models*.

Let $A(G)$ be the arc set of some arbitrarily chosen directing of the graph G. The standard Potts model partition function is defined by

$$Z_\lambda(G,u,v) = \sum_{c}\prod_{a\in A(G)} w(\delta c(a)), \tag{34}$$

where the sum is over all vertex colourings and the arc weight function w is given by

$$w(\alpha) = \begin{cases} u & \text{if} \quad \alpha \neq 0 \bmod \lambda, \\ v & \text{if} \quad \alpha = 0 \bmod \lambda. \end{cases} \tag{35}$$

Here $u, v \in \mathbf{C}$, or more generally these weights can take values in a given ring. The partition function so defined is independent of the chosen directing and we call this the undirected Potts model.

The origin of these models is in physics where macroscopic properties of many particle structures, e.g. magnetization of materials, need to be explained in terms of local interactions between neighbouring atoms. In these physical applications the summation $Z_\lambda(G)$ is called the *partition function* and determines the thermodynamic properties of the system being modelled.

Partitioning the colourings in (34) according to the edge subset E' on which the colour difference is zero gives

$$Z_\lambda(G,u,v) = \sum_{E'\subseteq E} P_\lambda^{+}(G/G')v^{|E'|}u^{|E\setminus E'|} \tag{36}$$

which evaluates to the chromatic polynomial of G when we make the substitutions $u = 1$ and $v = 0$. This result together with (3) shows that $Z_\lambda(G, u, v)$ is a polynomial in λ of degree at most $r(G) + 1$ as well as being polynomial in the variables u and v.

In this section we consider, for odd λ, chiral Potts models which have partition functions which do depend on the chosen directing of G. These partition functions can be expressed in terms of the directed polynomials discussed in this survey, see [3]. We also present here, for the first time, 'reversal-deletion-contraction' rules for chiral models which generalise those previously obtained for the directed chromatic polynomials in [4]. Finally, the partition functions described here will be used to develop knot invariants in the next section.

4.3.1 Polynomial form of the λ-state chiral Potts model partition function

The arc weights for the chiral Potts model are given by

$$w(\alpha) = \begin{cases} u & \text{if} \quad \alpha \bmod \lambda \in \mathcal{C}^e, \\ \bar{u} & \text{if} \quad \alpha \bmod \lambda \in \mathcal{C}^o, \\ v & \text{if} \quad \alpha = 0 \bmod \lambda, \end{cases} \tag{37}$$

where $\mathcal{C}^e \subset \mathcal{C}$ is the set of non-zero even integers and $\mathcal{C}^o \subset \mathcal{C}$ is the set of odd integers. For $H \in \mathcal{D}(G)$ define the partition function $\vec{Z}$ by

$$\vec{Z}_\lambda(H, u, \bar{u}, v) = \sum_c \prod_{a \in A(H)} w(\delta c(a)) \tag{38}$$

which reduces to the standard undirected Potts model partition function when $\bar{u} = u$ but if $\bar{u} \neq u$ it will depend on the chosen directing H of G.

The formula for the chiral Potts model given in equation (38) has an explicit polynomial form in the variables $u, \bar{u}, v$. To show that the partition function $\vec{Z}$ also has polynomial dependence on λ our approach is to expand $\vec{Z}$ with coefficients which are generated by enumerations of even λ colourings.

For $H, H' \in \mathcal{D}(G)$ define $\epsilon_{H,H'} : E \to \{0, 1\}$ by $\epsilon_{H,H'}(e)$ is zero or one according as the orientations of e in H, H' are respectively the same or opposite.

Consider $\vec{Z}$ on the directed graph $H = (V, A)$ of $\mathcal{D}(G)$ and firstly suppose that $v = 0$ so that the defining sum (34) becomes a sum over proper colourings.

$$\vec{Z}_\lambda(H, u, \bar{u}, 0) = \sum_{c \in \mathcal{C} \setminus \{0\}} \prod_{a \in A} w(\delta c(a)). \tag{39}$$

The proper colourings may be enumerated by considering only colourings with even colour difference on some directing $H' \in \mathcal{D}(G)$ and then summing over all such directings, thus

$$\vec{Z}_\lambda(H, u, \bar{u}, 0) = \sum_{H' \in \mathcal{D}(G)} \sum_{c \in \mathcal{C}^e} \prod_{e \in E(G)} u^{1-\epsilon_{H,H'}(e)} \bar{u}^{\epsilon_{H,H'}(e)} \tag{40}$$

$$= \sum_{H' \in \mathcal{D}(G)} u^{|A|} \left(\frac{\bar{u}}{u}\right)^{\rho(H,H')} P_\lambda^{even+}(H'), \tag{41}$$

where $\rho(H, H')$ is the number of arcs which must be reversed to obtain H' from H and $P_\lambda^{even+}(H')$ is the number of proper even colourings of H'. Together with Theorem 1 this shows that the values of $\vec{Z}_\lambda(H, u, \bar{u}, 0)$ for odd λ may be found by evaluating a polynomial of degree at most $r(G) + 1$ which is also a property of $\vec{Z}_\lambda(H, u, \bar{u}, v)$ since

$$\vec{Z}_\lambda(H, u, \bar{u}, v) = \sum_{A' \subseteq A} v^{|A'|} \vec{Z}_\lambda(H/H', u, \bar{u}, 0). \tag{42}$$

The function $\vec{Z}_\lambda(H, u, \bar{u}, v)$ may also be written as a sum over directings of G. Since the summand in (42) is independent of the directing of the arcs in A' and there are $2^{|A'|}$ such directings, using (41) we find

$$\vec{Z}_\lambda(H, u, \bar{u}, v) = \sum_{H' \in \mathcal{D}(G)} \sum_{A'' \subseteq A'} \left(\frac{v}{2}\right)^{|A''|} u^{|A' \setminus A''|} \left(\frac{\bar{u}}{u}\right)^{\rho(H,H',H'')} P_\lambda^{even+}(H'/H'') \tag{43}$$

where $\rho(H, H', H'')$ is the number of edges of H/H'' which must be reversed to obtain H'/H''.

4.3.2 The reversal-deletion-contraction rule for λ-state chiral Potts models

The RDC rule with respect to an arc $\bar{a} \in A(H)$ takes the form

$$\vec{Z}_\lambda(H, u, \bar{u}, v) + \vec{Z}_\lambda(H_{\bar{a}}^\rho, u, \bar{u}, v) = (u+\bar{u})\vec{Z}_\lambda(H_{\bar{a}}^\delta, u, \bar{u}, v) + (2v-u-\bar{u})\vec{Z}_\lambda(H_{\bar{a}}^\gamma, u, \bar{u}, v) \tag{44}$$

where $H_{\bar{a}}^\rho$, $H_{\bar{a}}^\delta$ and $H_{\bar{a}}^\gamma$ are the directed graphs obtained from H by respectively reversing, deleting, contracting the arc $\bar{a} \in A$. Let the arc $\bar{a} = [i, j]$, then the rule can be shown using the following observations.

$$\begin{aligned}
\vec{Z}_\lambda(H, u, \bar{u}, v) &= \sum_{c_i = c_j} \prod + \sum_{c_j - c_i \in \mathcal{C}^e} \prod + \sum_{c_j - c_i \in \mathcal{C}^o} \prod \\
&= v \sum_{c_i = c_j} \prod{}' + u \sum_{c_j - c_i \in \mathcal{C}^e} \prod{}' + \bar{u} \sum_{c_j - c_i \in \mathcal{C}^o} \prod{}',
\end{aligned}$$

where $\prod$ abbreviates the usual product of weights over all arcs and where $\prod'$ is the product over the arcs of the graph H' obtained by deleting the arc $\bar{a}$. Similarly,

$$\begin{aligned}
\vec{Z}_\lambda(H_{\bar{a}}^\rho, u, \bar{u}, v) &= \sum_{c_i = c_j} \prod + \sum_{c_j - c_i \in \mathcal{C}^e} \prod + \sum_{c_j - c_i \in \mathcal{C}^o} \prod \\
&= v \sum_{c_i = c_j} \prod{}' + \bar{u} \sum_{c_j - c_i \in \mathcal{C}^e} \prod{}' + u \sum_{c_j - c_i \in \mathcal{C}^o} \prod{}'.
\end{aligned}$$

Thus

$$\vec{Z}_\lambda(H, u, \bar{u}, v) + \vec{Z}_\lambda(H_{\bar{a}}^\rho, u, \bar{u}, v) = 2v \sum_{c_i = c_j} \prod{}' + (u + \bar{u}) \left[\sum_{c_j - c_i \in \mathcal{C}^e} \prod{}' + \sum_{c_j - c_i \in \mathcal{C}^o} \prod{}' \right] \tag{45}$$

Finally,

$$\begin{aligned}
&\sum_{c_j-c_i\in\mathcal{C}^e}\prod{}' + \sum_{c_j-c_i\in\mathcal{C}^o}\prod{}' \\
&= \left[\sum_{c_j-c_i\in\mathcal{C}^e}\prod{}' + \sum_{c_j-c_i\in\mathcal{C}^o}\prod{}' + \sum_{c_j-c_i=0}\prod{}'\right] - \sum_{c_j-c_i=0}\prod{}' \\
&= \vec{Z}_\lambda(H^\delta_{\bar{a}}, u, \bar{u}, v) - \vec{Z}_\lambda(H^\gamma_{\bar{a}}, u, \bar{u}, v).
\end{aligned}$$

Note that the number of proper even- and odd- colourings can be evaluated from $\vec{Z}_\lambda(H, u, \bar{u}, v)$.

Lemma 3. *Let $H = (V, A)$ be a planar directed graph, then the chiral partition function $\vec{Z}_\lambda(H, u, \bar{u}, v)$ satisfies*

$$\vec{Z}_\lambda(H, 1, 0, 0) = P^{even+}_\lambda(H), \quad \vec{Z}_\lambda(H, 1, 0, 1) = P^{even}_\lambda(H), \tag{46}$$

the proper and improper even colour difference chromatic polynomials respectively, and

$$\vec{Z}_\lambda(H, 0, 1, 0) = P^{odd+}_\lambda(H), \quad Z^*_\lambda(H, 0, 1, 1) = P^{odd}_\lambda(H), \tag{47}$$

the proper and improper odd colour difference chromatic polynomials respectively.

Clearly, the RDC rule for the chiral Potts partition function generalises the deletion-contraction rule for the partition function defined in (34), which can be obtained by identifying $G = H = H^\rho$ and $u = \bar{u}$, to give with $Z_\lambda(G, u, v) = \vec{Z}_\lambda(H, u, u, v)$

$$Z_\lambda(G, u, v) = uZ_\lambda(G^\delta_a, u, v) + (v-u)Z_\lambda(G^\gamma_a, u, v). \tag{48}$$

The partition functions given by equations (34,38) can be extended to correspond to the rooted case of the colouring polynomials [3]. Essentially, a particular arc $\bar{a}$ of H is distinguished and the colour difference $\delta c(\bar{a}) = \beta$ is prescribed. The RDC rule for proper even difference potentials used for the inductive proof of Theorem 1 is derived from (44) with the substitutions $u = 1, \bar{u} = v = 0$ and $u = v = 1, \bar{u} = 0$ to give respectively

$$P^{even+}_\lambda(H) + P^{even+}_\lambda(H^\rho_{\bar{a}}) = P^{even+}_\lambda(H^\delta_{\bar{a}}) - P^{even+}_\lambda(H^\gamma_{\bar{a}}) \tag{49}$$

and

$$P^{even}_\lambda(H) + P^{even}_\lambda(H^\rho_{\bar{a}}) = P^{even}_\lambda(H^\delta_{\bar{a}}) + P^{even}_\lambda(H^\gamma_{\bar{a}}) \tag{50}$$

Finally, consider the correlation function

$$U^{even}(\beta, \lambda, H, u) = \frac{\vec{Z}^{12}_\lambda(H, u, 0, 1)}{\vec{Z}_\lambda(H, u, 0, 1)} \tag{51}$$

where $\vec{Z}^{12}_\lambda(.)$ is defined by (38) with the sum restricted to states for which the colour difference between vertices 1 and 2 has the fixed value β. Let $H^+_{\bar{a}}$ be the graph obtained from H by adding the arc $\bar{a} = \{1, 2\}$. Then, [3], [8],using theorem 1,

$$U^{even}(0, 1, H, u) = \sum_{A'\subseteq A} \chi_{\bar{a}}(H^+_{\bar{a}}/H')(1-u)^{|A'|}u^{|A\backslash A'|} \tag{52}$$

and so (cf. equation (15)) we have

$$U^{even}(0,1,H,u) = \bar{C}_{12}(1-u,H) \tag{53}$$

where $\bar{C}_{12}(p,H)$ is the probability of no open path from 2 to 1 in the dual directed percolation model of Dhar *et al* [15] for which there is probability p of an arc being open in both directions and probability $1-p$ of being open only in the direction of the arc.

Very recent progress by Tsuchiya and Katori in the further use of chiral Potts models in walk problems and directed percolation is reported in [34].

4.4 Invariants for knots

One of the standard ways of representing a knot K is to project its 3-dimensional form onto a planar knot diagram D_K which records the over and under- crossings. As we see below a graph can be associated with the knot diagram and a partition function can be attached to the graph. Clearly, there are many knot diagrams and therefore different partition functions, that can be produced depending on the different projections of K. The different knot diagrams of K are related by the 'Reidemeister' rules. Thus an invariant for the knot K can be found if the various weights can be chosen so that the resulting partition function remains identical over all knot diagrams of the knot K. This means that the partition function has to be left unchanged under the effect of each of the Reidemeister rules on graphs. Such invariants can be found by this approach and one of the simplest is the bracket polynomial.

4.4.1 Bracket polynomial

The following construction is developed by Kauffman in [22]. There are minor modifications to reflect the notation developed earlier in this paper. Given a planar graph G we can associate a knot universe $M(G)$ by the method of introducing a cross at the mid point of each edge $e \in E$ and then joining adjacent arms of the crosses.

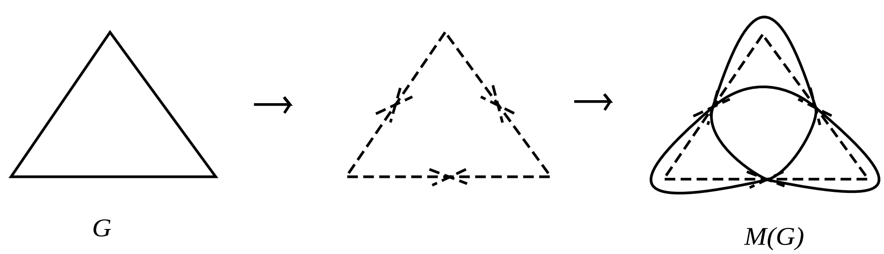

Figure 1. *The construction of the knot universe $M(G)$ from a graph G.*

Now consider $K(G)$, one of the two knots obtained by introducing alternating crossings to the planar universe $M(G)$ to convert it to the knot diagram D_K. Note that one is a reflection of the other. Thus $K(G)$ is a knot associated with the graph G. We define the bracket polynomial $\{K\}$ of a knot K to satisfy:

1. $\{\quad\} = u\{\quad\} + (v-u)\lambda^{-\frac{1}{2}}\{\quad\}$;

2. $\{\bigcirc \cup K\} = \lambda^{\frac{1}{2}} \{K\}$;

3. $\{\bigcirc\} = \lambda^{\frac{1}{2}}$.

These properties should be compared with those of the following partition function on the graph $G = (V, E)$ with λ-colourings denoted by $c : V \to \mathbf{Z}_\lambda$. Consider a new partition function modified from (34)

$$\hat{Z}_\lambda(G, u, v) = \lambda^{-\frac{\nu}{2}} Z_\lambda(G, u, v). \tag{54}$$

The standard Potts partition function has weights given by

$$w_e(c_i, c_j) = u + (v - u)\delta_{c_i, c_j} \tag{55}$$

where $e = [i, j] \in E$ and $\delta_{\cdot,\cdot}$ is the delta-function. This gives only two values, u and v. The deletion-contraction rule for $\hat{Z}$ (cf. 48) is given by

$$\hat{Z}_\lambda(G, u, v) = u\hat{Z}_\lambda(G\backslash e, u, v) + (v - u)\lambda^{-\frac{1}{2}} \hat{Z}_\lambda(G/e, u, v), \tag{56}$$

where $G\backslash e$ and G/e denote respectively the graph G with the edge e deleted and contracted. This is rule 1 for the Kauffman bracket. Rules 2 and 3 for the partition function, correspond to

1. the disjoint union of a single vertex and a graph G is given by

$$\hat{Z}_\lambda(\bullet \cup G, u, v) = \lambda^{\frac{1}{2}} \hat{Z}_\lambda(G, u, v), \tag{57}$$

and,

2. for a single vertex

$$\hat{Z}_\lambda(\bullet, u, v) = \lambda^{\frac{1}{2}}. \tag{58}$$

It can now be seen that the multiplicative factor in the definition of $\hat{Z}_\lambda(G, u, v)$ is used the ensure the value $\lambda^{\frac{1}{2}}$ for the trivial graph of a single vertex.

Proposition 1 *[22] Let $G = (V, E)$ be a planar graph and let $K(G)$ be an associated alternating knot, then*

$$\{K(G)\} = \lambda^{-\frac{\nu}{2}} Z_\lambda(G, u, v). \tag{59}$$

Hence the knot bracket $\{K(G)\}$ can be written in the Potts formalism by using (34) to obtain

$$\{K(G)\} = \lambda^{-\frac{\nu}{2}} \sum_c \prod_{e=[i,j]\in E} (u + (v - u)\delta_{c_i, c_j}). \tag{60}$$

Furthermore, if we simplify the coefficients of the Kauffman bracket and introduce $[K] = [K](A, B, d)$, defined by

1. $[\quad] = A[\quad] + B[\quad]$;

2. $[\bigcirc \cup K] = d[K]$;

3. $[\bigcirc] = d$,

is an invariant, [24], for *reduced* alternating knots for all A, B and d. Thus we have a knot invariant $[K](A, B, d) = \{K(G)\}$ which is given explicitly in terms of the Potts model partition function $Z_\lambda(H, u, v)$ by choosing

$$\lambda = d^2, \qquad u = B, \qquad v = Ad^2 + B. \tag{61}$$

The bracket $[\cdot]$ is enumerated in an integer ring $\mathbf{Z}[A, B, d]$. This follows from the inductive nature of the definition of the bracket where rule 1 gives the bracket of a knot with n-crossings in terms of the bracket for knots with fewer crossings.

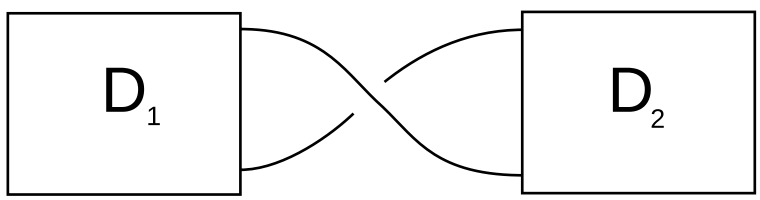

Figure 2. *A knot K is reduced if its diagram D_K does not divide into subsets D_1 and D_2 connected by a two-strand bridge.*

The reason that the bracket $[\cdot]$ is an invariant of reduced alternating knots *for all* A, B, d is that there is an alternative equivalence for alternating knots. The Reidemeister rules RII and RIII can be replaced by imposing invariance under 'flyping moves', [24]. The property was originally conjectured by Tait in [33] and was recently proven in [29].

Tait's Third Conjecture. *Any two reduced knot diagrams D_K and D'_K of an alternating knot K are equivalent by performing a finite number of flypes.*

A *flype* is a rotation of a (2,2)-tangle in a knot as indicated in Figure 3. The flype leaves the class of reduced alternating knots invariant and hence allows their equivalence to be addressed within the class of alternating knots.

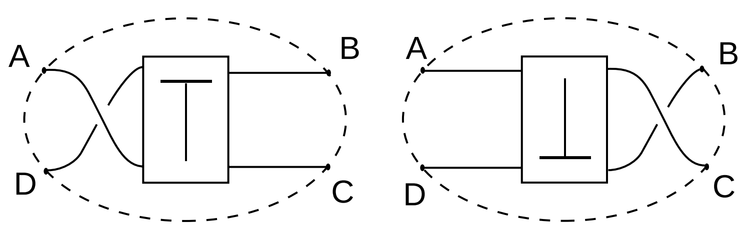

Figure 3. *The flype construction on a (2,2)-tangle of a knot K. The subknot denoted by '$\top$' is rotated by a half-turn, keeping the points A, B, C and D fixed, to form '$\perp$' and the rest of the knot outside the elliptical boundary is left unchanged.*

The important feature of a flype is that the associated Potts models remain unchanged for all choices of the triple λ, u, v and thus the knot bracket $[K](A, B, d)$ is a *regular isotopy* knot invariant of an alternating knot K for all choices of A, B and d. For non-alternating knots, the invariance requires further constraints on A, B and d to ensure invariance under RII and RIII.

4.4.2 Dichromatic polynomial

It should also be noted that if we take $u = 1$ and introduce $\bar{v} = (v-1)$ in the deletion-contraction rule (48) for Z , we obtain

$$Z_\lambda(G,\bar{v}) = Z_\lambda(G_a^\delta,\bar{v}) + \bar{v} Z_\lambda(G_a^\gamma,\bar{v}) \tag{62}$$

The partition function $Z = \lambda^{\frac{\nu}{2}}\hat{Z}$ is the *dichromatic polynomial* [25] which satisfies $Z_\lambda(\bullet,\bar{v}) = \lambda$ for a single vertex and $Z_\lambda(\bullet \cup G,\bar{v}) = \lambda Z_\lambda(G,\bar{v})$. The standard chromatic polynomial can be recovered from the dichromatic polynomial by choosing $\bar{v} = -1$, i.e. $v = 0$ so that

$$Z_\lambda(G,-1) = \lambda^{\frac{\nu}{2}}\hat{Z}_\lambda(G,-1) = \lambda^{\frac{\nu}{2}}\{K(G)\}|_{u=1,v=0}. \tag{63}$$

We conclude that the chromatic polynomial for the graph G can be found by evaluating the knot bracket of the associated knot diagram of $K(G)$.

4.4.3 Signed graphs

The Potts model can be extended to a signed graph G^s of G where $s : E \to \{-,+\}$ denotes an edge-signing of G. This enables us to distinguish different types of crossing (see Fig. 4). Note that in the case of alternating knots, all crossings are of the same type, i.e. either all $-$ or all $+$.

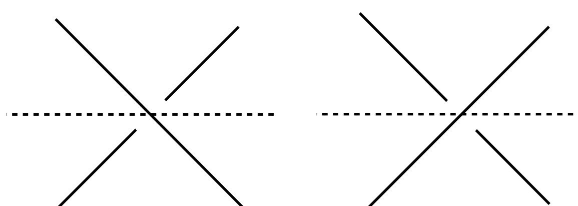

Figure 4. *The signing of knot crossings and edges (dashed) in a knot diagram.*

With the above notation define

$$\hat{Z}_\lambda(G^s,\mathbf{u},\mathbf{v}) = \lambda^{-\frac{\nu}{2}} \sum_c \prod_{e=[i,j]\in E} w_e^{s(e)}(c_i,c_j) \tag{64}$$

where $\mathbf{u} = (u^-,u^+)$, $\mathbf{v} = (v^-,v^+)$ and

$$w_e^\star(c_i,c_j) = u^\star + (v^\star - u^\star)\delta_{c_i,c_j} \tag{65}$$

for $\star = \pm$. We are now able to associate weights dependent on signing of the edges. The signing also gives the partition function a 'deletion-contraction' rule for the edge $e^\star$ as

$$\hat{Z}_\lambda(G^s,\mathbf{u},\mathbf{v}) = u^\star \hat{Z}_\lambda(G^s \backslash e^\star,\mathbf{u},\mathbf{v}) + (v^\star - u^\star)\lambda^{-\frac{1}{2}}\hat{Z}_\lambda(G^s/e^\star,\mathbf{u},\mathbf{v}). \tag{66}$$

To complete the analogous three rules to those of the knot bracket, the partition function of the disjoint union of a single vertex and a graph G is given by

$$\hat{Z}_\lambda(\bullet \cup G^s,\mathbf{u},\mathbf{v}) = \lambda^{\frac{1}{2}}\hat{Z}_\lambda(G^s,\mathbf{u},\mathbf{v}), \tag{67}$$

and for a single vertex

$$\hat{Z}_\lambda(\bullet,\mathbf{u},\mathbf{v})=\lambda^{\frac{1}{2}}. \tag{68}$$

Comparison of coefficients between the reduction formulae for $[K]$ and $\hat{Z}_\lambda(G^s,\mathbf{u},\mathbf{v})$ gives

Lemma 4. *[24] For a knot K, the bracket $[K](A,B,d)$ and the partition function $\hat{Z}_\lambda(G^s_K,\mathbf{u},\mathbf{v})$ are identical when the equations*

$$u^-=(v^+-u^+)\lambda^{-\frac{1}{2}}=A,\quad u^+=(v^--u^-)\lambda^{-\frac{1}{2}}=B,\quad \lambda^{\frac{1}{2}}=d \tag{69}$$

are satisfied.

Remark. To ensure that $[K]$ is a regular knot invariant we require the further conditions, namely $AB=1$ and $A^2+B^2+d=0$, cf. equation (61) and [24],[25]. This, of course, imposes further constraints on the weights $u^\pm, v^\pm$ of the Potts model. The more typical approach is to produce an invariant which reduces the number of variables in the partition function. A classical example of this is the Jones polynomial which is obtained as a special case of the Kaufmann bracket, and hence of the partition function, for oriented knots, cf. Lemma 4 and [24].

For the case of an alternating knot K, we have observed that the signings are either all $+$ or all $-$. Thus, if required, we can use one of the two signings to replace the variables u,v with either the pair u^+, v^+ or the pair u^-, v^- respectively. Furthermore, the reduction rules for the bracket $[K]$ (respectively the Potts partition function $Z_\lambda(G^s_K,u,v)$) leave the crossing types(signed edges) all of one sign. Finally the flype does not change the type of crossings as the new knot is still alternating. In the following, the weights $w_e(c)$ attached to each edge $e=[i,j]$ can be either all $w^+_e(c_i,c_j)$ or all $w^-_e(c_i,c_j)$. We will denote this choice by $\star$ and assume therefore that the substition $\star\equiv+$ or $\star\equiv-$ is made.

Lemma 5. *Let the knots K and K' differ by a single flype, then the Potts partition functions $Z_\lambda(G^\star_K,u,v)$ and $Z_\lambda(G^\star_{K'},u,v)$ are identical.*

Remark The flype construction does not change either the sign of crossings or their number and so the same Potts models arise from two equivalent alternating knots with the *same* number of crossings. We therefore conclude from the theorem that two alternating knots with the same number of crossings are not equivalent if they have different Potts models.

4.4.4 Chiral partition functions and alternating knot invariants

The idea from the previous section can now be extended to the set of directings $\mathcal{D}(G_K)$ of the undirected graph $G_K=(V,E)$. Typically, $H_K=(V,A)\in\mathcal{D}(G_K)$, where A is an arc set obtained by directing E. Furthermore, define $\mathbf{Z}^{\mathbf{e}}_\lambda(\mathbf{Z}^{\mathbf{o}}_\lambda)\subset\mathbf{Z}_\lambda$ to be the image of the even integers (odd integers) in the integer interval $[1,\lambda-1]$ under the natural projection $p:\mathbf{Z}\to\mathbf{Z}_\lambda$. A chiral partition function is given in [3]. Here we discuss a normalised version given by

$$\hat{\vec{Z}}_\lambda(H_K,\mathbf{u},\bar{\mathbf{u}},\mathbf{v})=\lambda^{-\frac{\nu}{2}}\sum_c\prod_{a=[i,j]\in A}w_a(c). \tag{70}$$

For a given signing s of the directed graph H_K, we consider the partition function

$$\hat{\vec{Z}}_\lambda(H^s, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{v}) = \lambda^{-\frac{\nu}{2}} \sum_c \prod_{a=[i,j]\in A} w_a^{s(a)}(c), \tag{71}$$

where $\mathbf{u} = (u^-, u^+)$, $\bar{\mathbf{u}} = (\bar{u}^-, \bar{u}^+)$, $\mathbf{v} = (v^-, v^+)$ and the weight $w_a^\star$, where the arc $a = [i,j]$ and $\star = \pm$, is defined by

$$w_a^\star(c) = \begin{cases} v^\star & c_j - c_i = 0 \\ u^\star & c_j - c_i = \mathbf{Z}_\lambda^{\mathbf{e}} \\ \bar{u}^\star & c_j - c_i = \mathbf{Z}_\lambda^{\mathbf{o}} \end{cases}. \tag{72}$$

Note that for an alternating knot the signing is the same for all crossings in the knot diagram of K.

Lemma 6. *Let $G_K^\star$ and $G_{K'}^\star$ be as in Lemma 5, then given a directing H_K of G_K, there exists a unique directing $H_{K'}$ of $G_{K'}$ such that*

$$\hat{\vec{Z}}_\lambda(H_K^\star, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{v}) = \hat{\vec{Z}}_\lambda(H_{K'}^\star, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{v}). \tag{73}$$

The existence of a bijection of the directed graphs $H_{K'}$ and H_K can be used to obtain an invariant for knots by considering the totality of directings of G_K and $G_{K'}$. Given an alternating knot K, define

$$\mathbf{D}_K = \{\hat{\vec{Z}}_\lambda(H_K, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{v}) | H_K \in \mathcal{D}(G_K)\}$$

to denote the class of chiral partition functions obtained from all directings of G_K.

Proposition 2 *If K and K' are equivalent alternating knots then $\mathbf{D}_K = \mathbf{D}_{K'}$.*

Finally, we observe that the standard Potts partition function, and therefore the Jones polynomial, can be obtained from the chiral version.

Proposition 3 *For a given knot K, the Potts partition function can be obtained from the chiral Potts model of a unique directing H_K of G_K by the relation*

$$\hat{Z}_\lambda(G_K, \mathbf{u}, \mathbf{v}) = \hat{\vec{Z}}_\lambda(H_K, \mathbf{u}, \mathbf{u}, \mathbf{v}). \tag{74}$$

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