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Reciprocity and Polynomial Properties for Even Flows and Potentials on Directed Graphs

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For Paul Erdős on his 80th birthday

We consider special types of mod-λ flows, called odd and even mod-λ flows, for directed graphs, and prove that the numbers of such flows can be interpolated by polynomials in λ with the degree given by the cycle rank of the graph. The proofs involve computation of the number of integer solutions in a polyhedral region of Euclidean space using theorems due to Ehrhart. The resulting reciprocity properties of the interpolating polynomials for even flows are considered. The analogous properties of odd and even mod-λ potential differences and their associated potentials are also developed.

1. Introduction

The role of odd and even mod-λ flows in combinatorial approaches to the problems of directed percolation and vicious walker configurations has been considered in previous papers [2, 4].

In Section 1 we review the polynomial properties of mod-λ flows [12] for undirected graphs. By restricting to even flows, new results are obtained for directed graphs in Section 2. The corresponding polynomial properties for even mod-λ potential differences are considered in Section 3.

1.1. Unrooted mod-λ flows

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Each edge $e \in E$, with vertices $i, j \in V$, can be given the orientation (i,j) or (j, i). The arc set A formed by choosing a given orientation for each edge of E together with the vertex set V is said to form a directed graph $H = (V, A)$. The set of all such directed graphs is denoted by $\mathscr{D}(G)$.

Let $\mathscr{C} = \{0, 1, \ldots, \lambda-1\}, \lambda \in \mathbb{Z}^+$. A mod-$\lambda$ flow on $H \in \mathscr{D}(G)$ is a map $\phi: A \to \mathscr{C}$ that satisfies the zero mod-λ condition

$$\phi_i^+ - \phi_i^- = m_i \lambda \tag{1.1}$$

at each vertex i of V. In (1.1), $m_i \in \mathbb{Z}$ is said to be the *charge* on vertex i, ϕ_i^+ is the sum of the $\phi(a)$ over arcs directed into vertex i, and ϕ_i^- is the sum of the $\phi(a)$ over arcs directed out of vertex i. Flows having zero charge everywhere will be called *natural*, and are discussed in [6]. Those flows that satisfy $\phi(a) \neq 0$ for every $a \in A$ are said to be *proper* [12].

The set of mod-λ flows is denoted by $\mathscr{F}_\lambda(H)$, and its cardinality by $F_\lambda(H)$; the set of proper mod-λ flows is denoted by $\mathscr{F}_\lambda^+(H)$, and its cardinality by $F_\lambda^+(H)$.

Let $H, H' \in \mathscr{D}(G)$. Then it is easy to obtain a bijection between the sets of flows $\mathscr{F}_\lambda(H)$ and $\mathscr{F}_\lambda(H')$. Specifically, the correspondence between flows ϕ and ϕ' on H, H' is given by $\phi(a) = \phi'(a')$ (respectively $\phi(a) = \lambda - \phi'(a')$) when the arcs a and a' obtained from the edge $e \in E$ are coherently oriented (respectively oppositely oriented) by H and H'. The correspondence remains one-one when restricted to the subsets of proper flows $\mathscr{F}_\lambda^+(H)$ and $\mathscr{F}_\lambda^+(H')$. Thus the functions $F_\lambda(H)$ and $F_\lambda^+(H)$ can be associated with the graph G, since they are independent of the orientation H, and hence $F_\lambda(G)$ and $F_\lambda^+(G)$ have unambiguous meanings.

A relation between $F_\lambda(G)$ and $F_\lambda^+(G)$ can be obtained by noting that every mod-λ flow on G is given uniquely by a proper mod-λ flow on some subgraph $G' = (V, E')$ of G. Thus the number of all flows can be written as a sum

$$F_\lambda(G) = \sum_{E' \subseteq E} F_\lambda^+(G'). \tag{1.2}$$

Möbius inversion then gives the relation

$$F_\lambda^+(G) = \sum_{E' \subseteq E} (-1)^{|E \setminus E'|} F_\lambda(G'). \tag{1.3}$$

The set of mod-λ flows on G' is obtained by independently allocating a circulation from the set $\mathscr{C}$ to each member of a cycle basis. The circulations are then added mod-λ for each edge of G' to give the flows. Thus the number of flows $F_\lambda(G')$ is just $\lambda^{c(G')}$ [5, 6], where $c(G')$ is the cycle rank of the subgraph G'. We conclude that the values of both $F_\lambda(G)$ and $F_\lambda^+(G)$ may be interpolated by polynomials $F(\lambda, G)$ and $F^+(\lambda, G)$, respectively, in the variable λ and of degree equal to the cycle rank $c(G)$ of G.

2. Odd and even mod-λ flows for odd λ

We now consider two important subclasses $\mathscr{F}_\lambda^{odd}(H)$ and $\mathscr{F}_\lambda^{even}(H)$ of $\mathscr{F}_\lambda(H)$, known as the odd and even mod-λ flows respectively. $\mathscr{F}_\lambda^{odd}(H)$ is obtained by restricting $\phi(a)$, $a \in H$, to be either 0 or an odd integer in $\mathscr{C}$. Similarly, $\mathscr{F}_\lambda^{even}(H)$ is obtained by restricting $\phi(a)$ to be an even integer in $\mathscr{C}$. The numbers of these flows will be denoted by $F_\lambda^{odd}(H)$ and $F_\lambda^{even}(H)$, respectively. We can further restrict odd and even flows to those that are proper. These subsets will be denoted as above, with the addition of a superscript +. The numbers of odd and even flows are expressible in terms of the numbers of proper odd and even flows (*cf.* (1.2)) and *vice-versa*.

For odd λ, simple examples show that the numbers of odd and even flows depend on the choice of directed graph $H \in \mathscr{D}(G)$. By contrast, for even λ, see [8], the bijection given in (1.1) preserves the parity of the flow in each edge, and the number of flows is, therefore, the same for all $H \in \mathscr{D}(G)$. There is a bijection between the sets of even mod-λ and mod-$\lambda/2$ flows. Hence, the number of even flows, $F_\lambda^{even}(H)$ is interpolated by the polynomial $(\lambda/2)^{c(G)}$. A similar result holds for odd flows provided the vertices of the graph are all of even degree, otherwise there can be no such flow. We now restrict attention to the direction-dependent case of λ odd, see also [2, 7].

Let H^ρ be the graph obtained by reversing the orientation of every edge of the directed graph H, and consider the bijection of Section 1.1 with $H' = H^\rho$. For odd λ, the above bijection associates odd mod-λ flows on H with even mod-λ flows on H^ρ, and *vice versa*. Thus

$$F_\lambda^{odd}(H) = F_\lambda^{even}(H^\rho). \tag{2.1}$$

Throughout the rest of Section 2 we therefore consider only even flows for odd λ.

2.1. Polynomial properties of even mod-λ flows for odd λ

The arguments of Section 1.1 do not trivially extend to give the polynomial property for even flows when λ is odd. Although (1.3) holds for $F_\lambda^{even+}(H)$, there is no correspondingly simple formula to that of $F_\lambda(G)$ for the function $F_\lambda^{even}(H)$. Instead, we turn to a theorem due to Ehrhart [9].

Theorem (Ehrhart [9]). *Let P_1 be a d-dimensional polyhedron in $\mathbb{R}^m$, $(1 \leqslant d \leqslant m)$, with integer vertices, and let $P_n = \{n\mathbf{x} \mid \mathbf{x} \in P_1\}$, then:*

(*i*) *the number of points in the relative interior of P_n with integer coordinates is a polynomial $i(n)$ in the variable n, of degree d;*

(*ii*) *the number of integer points when the boundary of P_n is included is also a polynomial $j(n)$ in the variable n, of degree d, and $j(0) = 1$;*

(*iii*) *there is a reciprocity relation between the polynomials given by*

$$i(n) = (-1)^d j(-n). \tag{2.2}$$

We also need the following theorem of Hoffman and Kruskal [11], which ensures the existence of the integer vertices required for Ehrhart's Theorem. A matrix $\mathbf{U}$ is said to be totally unimodular if every square submatrix of $\mathbf{U}$ has determinant ± 1 or 0. Thus $\mathbf{U}$ is an integral matrix.

Theorem (Hoffman and Kruskal [11]). *The $r \times s$ integer matrix $\mathbf{U}$ is totally unimodular if and only if for any integer r-vectors $\mathbf{b}$ and $\mathbf{b}'$ and integer s-vectors $\mathbf{a}$ and $\mathbf{a}'$, each face of the convex polyhedron $\{\mathbf{x} \in \mathbb{R}^s \mid \mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{a}', \mathbf{b} \leqslant \mathbf{U}\mathbf{x} \leqslant \mathbf{b}'\}$ contains an integer point.*

An immediate corollary of the theorem is that the unimodular property of the matrix $\mathbf{U}$ implies that the vertices of the corresponding polyhedron have integer co-ordinates.

Lemma 2.1. *For an even mod-λ flow on H, the charge on any vertex is even. Furthermore, if both the in-degree and out-degree of vertex $i \in V$ are at most two, then $\phi_i^+ - \phi_i^-$ lies between $-(2\lambda - 2)$ and $2\lambda - 2$, and hence $m_i = 0$.*

We will say that a graph $H \in \mathscr{D}(G)$ satisfies condition (A) if H supports a proper flow, and the in-degree and out-degree of every vertex is either one or two.

Lemma 2.2. *Let $H \in \mathscr{D}(G)$ satisfy condition (A). Then the function $F_\lambda^{even}(H)$ is interpolated by a polynomial $F^{even}(\lambda, H)$ in λ of degree equal to the cycle rank $c(G)$ of G.*

Proof. We have seen that condition (A) implies that the charge on every vertex is zero, and in this case the set of even mod-λ flows is just the set of natural flows with $\phi(a_j) = 2n_j$, for some n_j satisfying $0 \leqslant n_j \leqslant n$, where $\lambda = 2n+1$. The algebraic conditions (1.1) with $m_i = 0$, for all $i \in V$, can be written in the form (see [3]),

$$\mathbf{D}\mathbf{x} = \mathbf{0}, \tag{2.3}$$

where $\mathbf{x} \in \mathbb{R}^{|E|}$, with $0 \leqslant x_j \leqslant n$, and $\mathbf{D} = [d_{ij}]$ is the incidence matrix of the arcs for H. The matrix $\mathbf{D}$ is defined by $d_{ij} = 1$ or -1 according as there is an arc a_j incident into or out of vertex i and zero otherwise and $\mathbf{D}$ is totally unimodular [5]. Thus the set of flows $\mathscr{F}_\lambda^{even}(H)$ can be seen as a restricted set of integer solutions of a linear system of equations.

Let P_1 be the set of points $\{\mathbf{x} \in \mathbb{R}^{|E|} \mid \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{1}, \mathbf{D}\mathbf{x} = \mathbf{0}\}$, where $\mathbf{1}$ is the vector with every component 1. The solution set P_1 is a subset of a linear space of dimension $|E| - \operatorname{rank}(\mathbf{D}) = |E| - r(G) = c(G)$, the cycle rank of G [5]. We now use condition (A) to show P_1 is also a polyhedron of dimension $c(G)$, see [4]. Clearly, the linear hull of P_1, $\operatorname{Lin}(P_1) \subseteq \{\mathbf{x} \in \mathbb{R}^{|E|} \mid \mathbf{D}\mathbf{x} = \mathbf{0}\}$.

Conversely, by condition (A), let $\mathbf{y} \in \{\mathbf{x} \in \mathbb{R}^{|E|} \mid \mathbf{D}\mathbf{x} = \mathbf{0}\}$ be a proper solution vector of the equations such that each of its components is positive. It follows that there exists a basis $\{\mathbf{y}_1, \ldots, \mathbf{y}_h\}$, where $h = c(G)$, of the space $\operatorname{Ker}(\mathbf{D})$ consisting of vectors with entries 0 or 1. These basis vectors represent unit natural flows on directed cycles of the graph H. Thus $\mathbf{y} \in \operatorname{Lin}(P_1)$, and hence $\{\mathbf{x} \in \mathbb{R}^{|E|} \mid \mathbf{D}\mathbf{x} = \mathbf{0}\} \subseteq \operatorname{Lin}(P_1)$, and hence P_1 is $c(G)$ dimensional.

Moreover, it follows from the theorem of Hoffman and Kruskal that P_1 has integer vertices. The polynomial dependence of the number of even flows on λ follows from the observation that

$$F_\lambda^{even}(H) = j(n), \tag{2.4}$$

together with Ehrhart's Theorem. □

It should also be noted that for H satisfying the conditions of Lemma 2.2, the number of proper even flows is given by the number of integer solutions of (2.3) with $1 \leqslant x_j \leqslant n$, and hence

$$F_\lambda^{even+}(H) = i(n+1). \tag{2.5}$$

Note. If the graph H fails to have a proper flow required for condition (A), Lemma 2.2 can be modified to accommodate this more general case. The degree of the polynomial will not be $c(H)$, but $c(H')$, where H' is the maximal subgraph of H that supports a proper flow. Such a subgraph exists from the observation that if two natural proper flows exist on two subgraphs on H, then, by adding these flows, a proper natural flow is obtained on the subgraph obtained by taking the union of the arcs of the two subgraphs. Clearly, the maximal subgraph is found by taking the union of all the subgraphs that support proper flows. In this situation, there is a bijection between the flows on H' and the flows on H, because every flow on H is zero on those arcs not in H'.

We now extend Lemma 2.2 to include all directed graphs. This will be done in two stages, the first of which allows for some of the arcs to have a fixed flow of $2n$.

Lemma 2.3. *Let $H \in \mathscr{D}(G)$ satisfy condition (A) and let A' be a subset of the arc set $A(H)$, the complement of which contains a spanning forest. The number of even mod-λ flows (proper or otherwise) in which the flow in the arcs corresponding to A' is fixed at $2n$ is a polynomial in λ of degree $c(G)-|A'|$.*

Proof. In the proof of the previous theorem we must impose the further condition $n_j = n$, for $a_j \in A'$, which may be accommodated by setting $x_j = 1$ in the definition of P_1. This is the same as replacing the lower limit of zero on x_j by one, and Hoffman and Kruskal's Theorem still applies. Since the arcs of A' belong to a cotree, the additional constraint reduces the dimension of the solution space by $|A'|$, so P_1 is a polyhedron of dimension $c(G)-|A'|$ and the result follows from Ehrhart's Theorem as before. □

Theorem 2.1. *Let $H \in \mathscr{D}(G)$ support a proper even mod-λ flow. Then the function $F_\lambda^{even}(H)$ is interpolated by a polynomial $F^{even}(\lambda, H)$ in the variable λ of degree equal to the cycle rank $c(H)$ of H.*

Proof. We will partition the flows by charge distribution and establish the polynomial property for each class separately. The case where the charge at every vertex is zero is proved in the same way as for Lemma 2.2. So we now consider a non-trivial charge distribution of the m_i's that supports at least one flow on the directed graph H. Let $m_1, \ldots, m_k$ be the set of positive charges at the vertices $x_1, \ldots, x_k$ respectively and $m'_1, \ldots, m'_l$ be the set of negative charges at the vertices $y_1, \ldots, y_l$. Note that for each flow defined by (1.1), $m_i\lambda$ and $m'_i\lambda$, must be even, since they are the sums of the flows at x_i and y_i, which are even. Also we are assuming that λ is odd, and hence m_i and m'_i must be even. Let M be the sum of the positive charges. Then there is a net outflow of $M\lambda$ from the vertices with positive charge that is equal to the net inflow at the vertices with negative charge. This follows by summing the signed flows over all vertices of the graph H.

Consider the extension of H by the addition of vertices u and v and $m_i/2$ parallel arcs directed from u to $x_i, i \in \{1, \ldots, k\}$, and $|m'_j|/2$ parallel arcs directed from $y_j, j \in \{1, \ldots, l\}$, to v. Now impose the condition that the flow in each of the added arcs be 2λ, and the flow in each arc of H be at most $\lambda - 1$. The flows on H with the given charge distribution correspond to the natural flows on the extended graph K with a flow in of $M\lambda$ at u and a flow out of $M\lambda$ at v. Every cutset of the graph K separating u and v contains at least $M\lambda/2\lambda = M/2$ arcs oriented from u to v (at least M if the cut is in H). It follows from Menger's theorem that there exist $M/2$ arc-disjoint paths from u to v. This is also the number of arcs added to each of the vertices u and v and therefore the paths use *all* of these arcs. Thus there is a natural one-one correspondence between the arcs adjacent to u and v respectively. By removing the vertices u and v and identifying these arcs by the above correspondence, we create a new directed graph K', which has $M/2$ arc-disjoint cycles, each containing one of the new arcs. Subtraction of $\lambda + 1$ from the flow in each of these cycles reduces the flow from 2λ in each of the added arcs to $2n$, and the range of the flow variables on the arcs of H that belong to one of the new cycles becomes $\{-(\lambda-1), -(\lambda-3), \ldots, -2\}$ instead of $\{2, 4, \ldots, \lambda-1\}$. Thus the required flows are in one-one correspondence with the even flows on K', with a fixed flow of $2n$ on the added arcs, and where those arcs of H contained in the cycles defined above are reversed. We now apply Lemma 2.3 to K', with $A' = A(K')\backslash A(H)$, noting that $c(K') = c(H)+M/2$ and $|A'| = M/2$. □

2.2. Ehrhart reciprocity for even flows

It is straightforward to obtain implications from applying the reciprocity relation (2.2) to (2.4) and (2.5).

Lemma 2.4. *Let* $H \in \mathscr{D}(G)$ *be a directed graph satisfying condition* (*A*). *Then*

(*i*) $F^{even+}(\lambda, H) = (-1)^{c(H)} F^{even}(-\lambda, H)$,

(*ii*) $|F^{even+}(-1, H)| = 1$, *and*

(*iii*) $|F^{even+}(-3, H)|$ *is the total number of subgraphs of H that are directed cycles, including the trivial cycle on the null subgraph.*

In the above lemma, a *directed cycle* is a subgraph of H with an equal number of arcs oriented into and out of each of its vertices.

Proof.

(i) Condition (A) implies that every even flow on H is a natural flow and from (2.5), $F^{even+}(2n+1, H) = i(n+1)$, and from (2.4), $F^{even}(2n+1, H) = j(n)$. Result (i) then follows by using the reciprocity rule (2.2) to obtain

$$\begin{aligned} F^{even+}(2n+1, H) = i(n+1) &= (-1)^d j(-(n+1)) \\ &= (-1)^d F^{even}(-2(n+1)+1, H) \\ &= (-1)^d F^{even}(-2n+1), H), \end{aligned} \tag{2.6}$$

where $d = c(H)$, the dimension of the polyhedron P_1 in the proof of Lemma 2.2.

(ii) For $\lambda = -1$, we obtain

$$F^{even+}(-1, H) = (-1)^{c(H)} F^{even}(1, H), \tag{2.7}$$

and $F^{even}(1, H) = 1$ by part (ii) of Ehrhart's Theorem. Thus $F^{even+}(-1, H) = (-1)^{c(H)}$.

(iii) For $\lambda = -3$, we obtain

$$F^{even+}(-3, H) = (-1)^{c(H)} F^{even}(3, H). \tag{2.8}$$

Every non-trivial flow $\phi \in \mathscr{F}_3^{even}(H)$ is, by condition (A), a proper natural flow on some non-null subgraph H' of H such that $\phi(a) = 2$ for all $a \in A(H')$. The algebraic condition at each vertex implies that each vertex of the graph H' has the same number of arcs oriented in and out. Thus the arcs of H' form a directed cycle. The null directed cycle $H' = \varnothing$ is given by the zero flow.

Conversely, assigning $\phi(a) = 2$ to every arc of a directed cycle subgraph of H and $\phi(a) = 0$ to arcs not in the cycle gives a unique flow in $\mathscr{F}_3^{even}(H)$. It follows that $|F^{even+}(-3, H)|$ gives the number of subgraphs of H, including the null graph, whose arc sets form a directed cycle. □

We now consider the rooted version of the flows described in Section 1.1. Let 1 and 2 be distinguished vertices (or roots) of the directed graph H. An *external* arc (2, 1) is added to the graph H to form the extended graph H^+. For integers $\lambda > 1$, let $\mathscr{F}_{\alpha,\lambda}(H)$ be the set of mod-λ flows (satisfying (1.1)) on the extended graph H^+ such that the 'external flow' in (2, 1) is fixed to be $\alpha \in \mathscr{C}$.

A set of flows that are of particular interest in percolation theory [4] are constructed on condition (A) rooted graphs H that are unions of directed paths from root 1 to root 2. Each of these natural flows can be expressed as a sum of circulations in each circuit formed by a directed path from root 1 to root 2 and the arc (2, 1). Thus, the maximum flow occurs in the external arc (2, 1), and hence the number of flows $F^{even}_{2m,\lambda}(H)$ is independent of λ for $\lambda > 2m$ and constrained only by the maximum flow $2m$.

Lemma 2.5. *Let $H \in \mathscr{D}(G)$ be a directed rooted graph that is the union of directed paths from root 1 to root 2, and let H^+ be the extended graph of H. Also, let $f^+(m, H) = F^{even+}_{2m,\lambda}(H)$ and $f(m, H) = F^{even}_{2m,\lambda}(H)$ for $\lambda > 2m$. If H^+ satisfies condition (A), then*

(*i*) *the number of rooted flows $f^+(m, H)$ and $f(m, H)$ are polynomials in the variable m and satisfy the reciprocity relation*

$$f^+(m, H) = (-1)^{c(H)} f(-m, H); \tag{2.9}$$

(*ii*) $f^+(0, H) = (-1)^{c(H)}$;

(*iii*) $|f^+(-1, H)|$ *is the number of directed paths from 1 to 2 in H;*

(*iv*) *let H have a unique minimal proper flow with external flow $2m'$, then f has symmetry about $m = \frac{1}{2}m'$, i.e.*

$$f(m-m', H) = (-1)^{c(H)} f(-m, H). \tag{2.10}$$

Proof.

(i) The maximal flow property of arc (2, 1) implies $\mathscr{F}^{even}_{2m,\lambda}(H) = \mathscr{F}^{even}_{2m+1}(H^+) \backslash \mathscr{F}^{even}_{2m-1}(H^+)$. Thus, we have

$$\begin{aligned} f(m, H) &= F^{even}(2m+1, H^+) - F^{even}(2m-1, H^+) \\ &= (-1)^{c(H^+)}[F^{even+}(-2m-1, H^+) - F^{even+}(-2m+1, H^+)] \\ &= (-1)^{c(H^+)-1} f^+(-m, H) \\ &= (-1)^{c(H)} f^+(-m, H), \end{aligned} \tag{2.11}$$

where we have used Lemma 2.4(i).

(ii) and (iii) have proofs similar to the corresponding arguments in Lemma 2.4.

(iv) an arbitrary proper flow on H is formed uniquely by the superposition of the minimum proper flow with source strength $2m'$ and an arbitrary flow of source strength $2(m-m')$. Thus $f^+(m, H) = f(m-m', H)$, which combined with (2.9) gives (2.10). □

Note. The above properties for the polynomials f and f^+ can also be obtained by considering the subset of natural even mod-λ flows on a rooted graph H consisting of directed paths, but which is not necessarily of type (A). There is a bijection of such natural even flows with the natural mod-$(n+1)$ flows, where $\lambda = 2n+1$. Polynomial and reciprocity properties then arise by using results from [3] (see [4] also).

3. λ-potentials and even mod-λ potential differences for odd λ

For a graph G, a λ-potential c is simply a colouring of the vertices of G, using λ colours, and can be described by a map c: $V(G) \to \mathscr{C}$. The number of unconstrained colourings is, therefore, $\lambda^{\nu(G)}$, where $\nu(G) = |V(G)|$. The number of proper colourings $P_\lambda^+(G)$, where adjacent vertices have distinct colours, is given by Whitney's formula [13]

$$P_\lambda^+(G) = \sum_{E' \subseteq E} (-1)^{|E'|} \lambda^{\omega(G')}, \tag{3.1}$$

where $\omega(G')$ is the number of connected components of the graph G', which is a polynomial in λ of degree $\nu(G)$.

We recall that an even λ-colouring or potential c [2] is one for which the potential difference $\delta c(i,j) = c(j) - c(i) \in \mathscr{C}$, when reduced mod-$\lambda$, is even on each arc (i,j). Let $\mathscr{P}_\lambda^{even}(H)$ and $\mathscr{P}_\lambda^{even+}(H)$ denote the sets of even and proper even respectively, λ-colourings of the directed graph H. Let the corresponding cardinalities be $P_\lambda^{even}(H)$ and $P_\lambda^{even+}(H)$.

For even λ, the sets $\mathscr{P}_\lambda^{even}(H)$ and $\mathscr{P}_\lambda^{even+}(H)$ correspond to ordinary colourings of H with $\lambda/2$ colours, and the corresponding enumerators may therefore be interpolated by polynomials of degree $\nu(H)$. A similar enumeration can be carried out for $P_\lambda^{odd}(H)$, provided the graph H is bipartite.

3.1. Polynomial properties of λ-potentials with even potential difference for odd λ

For odd λ, the numbers of even λ-colourings are equal to the corresponding numbers of odd λ-colourings on the reversed di-graph, for if $\delta c(i,j)$ is an even element of $\mathscr{C}$, then $\delta c(j,i)$ is odd. To establish the polynomial property for both types of colourings for odd λ, it is therefore sufficient to prove the following theorem for even colourings.

Theorem 3.1. *Let $H \in \mathscr{D}(G)$ support a λ-potential for which the potential difference is both proper and even mod-λ. For odd λ, the number of λ-potentials (colourings), $P_\lambda^{even}(H)$, for which the potential difference is even on every edge of H may be interpolated by a polynomial $P^{even}(\lambda, H)$ in λ of degree $\nu(G)$ such that $P^{even}(1, H) = 1$.*

The polynomial property for proper colourings then follows from Möbius inversion of the formula

$$P_\lambda^{even}(H) = \sum_{A' \subseteq A} P_\lambda^{even+}(H/H'). \tag{3.2}$$

Note that to obtain (3.2), the λ-colourings of H are subdivided according to the subgraph H' on which they are improperly coloured. The λ-colourings that are improper on H' correspond to proper colourings on the directed graphs H/H' obtained by contracting the arcs of H'.

Proof of Theorem 3.1. Let $\lambda = 2n+1$, where n is a non-negative integer, and write the potential (colour) of vertex i as $c_i = 2n_i (\text{mod}\, \lambda) \in \mathscr{C}$. If we allow each n_i to range over $0 \leqslant n_i \leqslant 2n$, all possible λ-potentials will be generated exactly once. In the space of the

n_i-variables, the even values of c_i are contiguous, as are the odd values, the latter corresponding to the range $n+1,\dots,2n$ of the n_i. The potentials for which the potential differences are even come from the bands (I) $0 \leqslant n_j - n_i \leqslant n$, and (II) $-2n \leqslant n_j - n_i < -n$. The first range corresponds to algebraically even values of the potential difference $c_j - c_i$ on the arc (i,j), and the second corresponds to even values when reduced mod-λ. We can therefore write

$$P_\lambda^{even}(H) = \sum_{A_2 \subseteq A} \mu_\lambda(A_2), \tag{3.3}$$

where $\mu_\lambda(A_2)$ is the number of integer points in the hypercube $[0, 2n]^{\nu(G)}$ such that $n_j - n_i$ lies in the second range (II) above if $(i,j) \in A_2 \subseteq A$, and in the range (I) for $(i,j) \in A_1 = A \backslash A_2$. When $A_2 \neq \varnothing$ the Ehrhart Theorem cannot be applied directly to find $\mu_\lambda(A_2)$. The theorem does not apply to constraints with strict inequality at the upper limit of (II). So the approach is to consider the set of potentials that satisfy constraint (I) on the arc set A_1, and constraint (II) on the arc set $A_2 \backslash A'$, where $A' \subseteq A_2$ and $n_j - n_i = -n$ on A'. An inclusion–exclusion argument is then used on the subsets of A_2.

Define $P_1(A_2, A')$, $A' \subseteq A_2$, in Ehrhart's Theorem by

$$P_1(A_2, A') = \{\mathbf{x} \in \mathbb{R}^\nu \mid \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{2}, \mathbf{b} \leqslant \mathbf{U}\mathbf{x} \leqslant \mathbf{b}'\},$$

where $\mathbf{0}$ and $\mathbf{2}$ are vectors with all components equal to 0 and 2, respectively, and $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^{|A|}$. The components of $\mathbf{b}, \mathbf{b}'$ are chosen such that for $a \in A'$, $b_a = b'_a = -1$, for $a \in A_2 \backslash A'$, $b_a = -2$, $b'_a = -1$ and thirdly for $a \in A_1$, $b_a = 0$ and $b'_a = 1$. The $|A| \times \nu(G)$ matrix $\mathbf{U}$ is the transpose of the incidence matrix $\mathbf{D}$ (defined in the proof of Lemma 2.2) and has rank equal to the cocycle rank $r(H)$ (see Biggs [5], Propositions 4.3 and 5.3). The same is therefore true of $\mathbf{U}$, and if $P_1(A_2, A')$ is non-empty, it is a convex polyhedron with integer vertices, by Hoffman and Kruskal's Theorem. Thus, by Ehrhart's Theorem, $j_{A_2, A'}(n)$, the number of integer points in $P_n(A_2, A')$, is a polynomial in n of degree $\nu(G)$.

For $A_2 \neq \varnothing$, $j_{A_2, \varnothing}(n)$ overcounts $\mu_\lambda(A_2)$, since $P_n(A_2, \varnothing)$ includes $n_j - n_i = -n$ for arcs of A_2. Applying an inclusion–exclusion argument gives

$$\mu_\lambda(A_2) = \sum_{\varnothing \subseteq A' \subseteq A_2} (-1)^{|A'|} j_{A_2, A'}(n). \tag{3.4}$$

This formula is also valid for $A_2 = \varnothing$, and it follows that $\mu_\lambda(A_2)$, and hence $P_\lambda^{even}(H)$, is a polynomial in n, and therefore also in λ. From (ii) of Ehrhart's Theorem, it follows that $\mu_1(\varnothing) = 1$ and for $A_2 \neq \varnothing$

$$\mu_1(A_2) = \sum_{\varnothing \subseteq A' \subseteq A_2} (-1)^{|A'|} = 0, \tag{3.5}$$

and hence $P_1^{even}(H) = 1$. This is consistent with the fact that when $\lambda = 1$, we must have $c_i = 0$ for all i, and this is a valid 1-potential with even potential difference. □

3.2. Ehrhart reciprocity for even mod-λ potential differences

The mod-λ *potential difference* $\delta c\colon A(H) \to \mathscr{C}$ associated with the λ-potential c on the directed graph $H \in \mathscr{D}(G)$ is unchanged by adding a constant to the potential on any component of H. Thus there are $\lambda^{\omega(H)}$ potentials that give rise to a given potential difference, and therefore

$$P_\lambda^{even}(H) = \lambda^{\omega(H)} D_\lambda^{even}(H), \tag{3.6}$$

where $D_\lambda^{even}(H)$ is the number of distinct even mod-λ potential differences, and $\omega(H)$ is the number of connected components of H. The relation (3.6) also restricts to proper even potentials and potential differences, so it follows that the numbers of odd and even mod-λ potential differences for odd λ may be interpolated by polynomials of degree at most $r(H) = \nu(H) - \omega(H)$, the cocycle rank of H. All mod-λ potential differences have the defining property that they sum to zero when evaluated mod-λ on cycles of H. *Natural* potential differences are those that sum to zero in the integers on cycles of H.

The graph $H \in \mathscr{D}(G)$ is said to satisfy condition (B) if H supports a proper potential difference, and if it has a cycle basis such that each cycle contains at most 4 edges and has either one or two edges of each orientation.

Lemma 3.1. *Let $H \in \mathscr{D}(G)$ satisfy condition (B), then a mod-λ potential difference on $\delta c \in \mathscr{D}_\lambda^{even}(H)$ is a natural potential difference.*

Proof. Let σ_i be an element of a cycle basis satisfying condition (B). Consider the case when σ_i has two arcs a_1, a_2. We have two even potential differences $\delta c(a_1)$, $\delta c(a_2) \in \mathscr{C}$ such that $\delta c(a_1) - \delta c(a_2) = 0 \pmod \lambda$. This implies $\delta c(a_1) = \delta c(a_2)$, and we see that $\delta c(a_1) - \delta c(a_2) = 0 \in \mathbb{Z}$. The same reductions can be made for cycles of 3 or 4 edges by using the even constraint on the potential differences, together with condition (B). More generally, let σ be a cycle of H. By condition (B), we can write $\sigma = \sum_1^k m_i \sigma_i$, where $\{\sigma_1, \ldots, \sigma_k\}$ is the special basis and $m_i \in \mathbb{Z}$. Then $\delta c(\sigma) = \sum_1^k m_i \delta c(\sigma_i)$, and hence $\delta c(\sigma)$ sums to zero over $\mathbb{Z}$. □

Lemma 3.2. *Let $H \in \mathscr{D}(G)$ be a directed graph satisfying condition (B), then*
(*i*) $D^{even+}(\lambda, H) = (-1)^{r(H)} D^{even}(-\lambda, H)$,
(*ii*) $|D^{even+}(-1, H)| = 1$, *and*
(*iii*) $|D^{even+}(-3, H)|$ *is the number of subgraphs H' of H for which the arc set of H/H' forms a directed cocycle of H.*

A *cocycle* subgraph of H is formed from a partition of the vertex set $[S, S']$ with the property that each cocycle arc has an incident vertex in both S and S' and all other arcs have both incident vertices in either S or S'. The cocycle is *directed* if every directed cycle of H has equal number of arcs oriented from S to S' and S' to S.

Proof.
(i) Condition (B) implies that every even potential difference on H is a natural potential difference. Thus, using the notation of the proof of Theorem 3.1, the only contribution in (3.3) is given by $A_2 = \varnothing$. Hence $P^{even}(2n+1, H) = j_{\varnothing.\varnothing}(n) = j(n)$ and $P^{even+}(2n+1, H) = i_{\varnothing.\varnothing}(n) = i(n+1)$. The result then follows by using the reciprocity relation (2.2) for $j(n)$ and $i(n)$, with $d = \nu(H)$, to obtain

$$\begin{aligned} P^{even+}(2n+1, H) &= i(n+1) = (-1)^{\nu(H)} j(-(n+1)) \\ &= (-1)^{\nu(H)} P^{even}(-2(n+1)+1, H) = (-1)^{\nu(H)} P^{even}(-(2n+1), H). \end{aligned} \tag{3.7}$$

Property (i) is then obtained for potential differences by (3.6).

(ii) For $\lambda = -1$, we obtain

$$P^{even+}(-1, H) = (-1)^{\nu(H)} P^{even}(1, H), \tag{3.8}$$

and $P^{even}(H) = 1$ by part (ii) of Ehrhart's Theorem. Thus $P^{even+}(-1, H) = (-1)^{\nu(H)}$.

(iii) For $\lambda = -3$, we obtain

$$D^{even+}(-3, H) = (-1)^{\nu(H)} D^{even}(3, H). \tag{3.9}$$

Consider a non-trivial potential difference $\delta c \in \mathscr{D}_3^{even}(H)$. Let H' be the subgraph of H formed by the edges on which the potential difference δc is zero. If H/H' is the graph obtained from H by contracting the arcs of H', then $\delta c(a) = 2$ for all arcs of H/H'. The potential difference δc sums to $0 \in \mathbb{Z}$ on cycles of H/H' by Lemma 3.1, so any cycle of H/H' has equal numbers of arcs of positive and negative orientation. Thus the cycles of H/H' are all of even length, so H/H' is bipartite. Let $[S, S']$ be the corresponding partition of the vertices of H/H', then $[\mathscr{S}, \mathscr{S}']$ forms a cocycle of H such that any directed cycle of H has equal numbers of its arcs oriented from both $\mathscr{S}$ to $\mathscr{S}'$ and $\mathscr{S}'$ to $\mathscr{S}$.

Conversely, let $[\mathscr{S}, \mathscr{S}']$ be a directed cocycle of H, and let δc be the function that assigns the value two to the arcs of $[\mathscr{S}, \mathscr{S}']$ and zero to the remaining arcs. The signed sum of δc on any cycle of H is zero, since either it contains no arcs of $[\mathscr{S}, \mathscr{S}']$ or it has equal numbers of arcs directed from $\mathscr{S}$ to $\mathscr{S}'$ and $\mathscr{S}'$ to $\mathscr{S}$. Hence $\delta c \in \mathscr{D}_3^{even}(H)$. □

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