Valid Inequalities for Problems with Additive Variable Upper Bounds*

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Abstract. We study the facial structure of a polyhedron associated with the single node relaxation of network flow problems with additive variable upper bounds. This type of structure arises, for example, in network design/expansion problems and in production planning problems with setup times. We first derive two classes of valid inequalities for this polyhedron and give the conditions under which they are facet-defining. Then we generalize our results through sequence independent lifting of valid inequalities for lower-dimensional projections. Our computational experience with large network expansion problems indicates that these inequalities are very effective in improving the quality of the linear programming relaxations.

1 Introduction

The single node fixed-charge flow polyhedron, studied by Padberg et al. [9] and Van Roy and Wolsey [12], arises as an important relaxation of many 0-1 mixed integer programming problems with fixed charges, including lot-sizing problems [4,10] and capacitated facility location problems [1]. The valid inequalities derived for the single node fixed-charge flow polyhedron have proven to be effective for solving these types of problems. Here we study a generalization of the single node fixed-charge flow polyhedron that arises as a relaxation of network flow problems with additive variable upper bounds, such as network design/expansion problems and production planning problems with setup times. We derive several classes of strong valid inequalities for this polyhedron. Our computational experience with network expansion problems indicates that these inequalities are very effective in improving the quality of the linear programming relaxations.

In a network design problem, given a network and demands on the nodes, we are interested in installing capacities on the edges of the network so that the total cost of flows and capacity installation is minimized. If some of the edges already

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have positive capacities, the problem is called a network expansion problem. In many applications capacity is available in discrete quantities and has a cost structure that exhibits economies of scale [5,11]. The constraints of the mixed integer programming formulation of the network expansion problem for a single node are

$$\sum_{i \in M^+} y_i - \sum_{i \in M^-} y_i \le b \tag{1}$$

$$y_i \le u_i + \sum_{j \in N(i)} a_{ij} x_j, \ i \in M. \tag{2}$$

Inequality (1) is the balance constraint of a node with inflow (M^+) and outflow (M^-) edges and demand b. The continuous variable y_i represents the flow on edge i, $i \in M = M^+ \cup M^-$. Inequalities (2) are the *additive variable upper bound* (AVUB) constraints on the flow variables. $N(i)$ is the index set of binary variables x_j representing the availability of resources that increase the capacity of edge i that has capacity u_i. For multi-commodity network expansion problems, it is possible to arrive at this single commodity relaxation by aggregating the balance constraints and the flow variables over commodities for a single node.

Additive variable upper bounds generalize the simple variable upper bounds in three respects. First, several binary variables additively increase the upper bound on the continuous variable. Second, the continuous variable is not necessarily restricted to zero when its additive variable bounds are zero. Third, we allow an overlap of additive variable upper bound variables, i.e. $N(i) \cap N(k) \neq \emptyset$ for $i, k \in M$. This situation typically occurs when capacities are installed on subsets of edges such as on cycles of the network (rings). Note that a simple variable upper bound constraint $y_i \le u_i x_i$ is a special case of (2). We also point out that a variable lower bound constraint $l_i x_i \le y_i$ can be put into a simple form of AVUB, $\bar{y}_i \le (u_i - l_i) + l_i \bar{x}_i$, after complementing the binary variable x_i and the continuous variable y_i assuming that it has a finite upper bound u_i.

Multi-item production planning problems with setup times have the following constraints as part of their MIP formulations

$$\sum_{t' \le t} y_{it'} \ge d_{it}, \qquad \forall i, t \tag{3}$$

$$\sum_i y_{it} \le u_t - \sum_i a_i x_{it}, \ \forall t \tag{4}$$

where d_{it} denotes the demand for item i in period t, u_t the total production capacity in period t and a_i the setup time required for item i if the machine is setup for this item. Aggregating the demand constraints (3) and the production variables y_{it} for each period, we arrive at the same structure as in (1)-(2).

In the next section we introduce four classes of valid inequalities for

$$P = \{(x, y) \in \mathbb{B}^n \times \mathbb{R}^m_+ : \text{subject to (1) and (2)}\}$$

and give conditions under which these inequalities are facet-defining for $conv(P)$. In Section 3 we present a summary of computational results on the use of the new inequalities in a branch-and-cut algorithm for network expansion problems.

2 Valid Inequalities

The proofs of the results given in the sequel are abbreviated or omitted due to space considerations. For detailed proofs and for further results and explanations, the reader is referred to Atamtürk [2].

Let $N = \{1, 2, \ldots, n\}$ be the index set of binary variables and $N(S)$ be the subset of N appearing in the additive variable upper bound constraints associated with $S \subseteq M = \{1, 2, \ldots, m\}$. For notational simplicity we use $N(i)$ for $N(\{i\})$. We define $u(S) = \sum_{i \in S} \left(u_i + \sum_{j \in N(i)} a_{ij}\right)$ for $S \subseteq M$ and $a_j(S) = \sum_{i \in S} a_{ij}$ for $j \in N(S)$. Again for notational simplicity we use $u(i)$ for $u(\{i\})$. Throughout we make the following assumptions on the data of the model:

(A.1) $a_{ij} > 0$ for all $i \in M,\ j \in N(i)$.
(A.2) $u(i) > 0$ for all $i \in M$.
(A.3) $b + u(M^-) > 0$.
(A.4) $u(i) - a_{ij} \geq 0$ for all $i \in M,\ j \in N(i)$.
(A.5) $b + u(M^-) - a_j(M^-) \geq 0$ for all $j \in N$.

Assumptions (A.2-A.5) are made without loss of generality. If $u(i) < 0$ or $b + u(M^-) < 0$, then $P = \emptyset$. If $u(i) = 0$ ($b + u(M^-) = 0$), then $y_i = 0$ ($y_i = 0,\ i \in M^+$) in every feasible solution and can be eliminated. Similarly if $u(i) - a_{ij} < 0$ or $b + u(M^-) - a_j(M^-) < 0$, then $x_j = 1$ in every feasible solution and can be eliminated. Note that given (A.1), if $N(i) \neq \emptyset$ for all $i \in M$, then (A.4) implies (A.2) and (A.5) implies (A.3). Assumption (A.1) is made for convenience. Results presented in the sequel can easily be generalized to the case with $a_{ij} < 0$. Note that, for a particular $j \in N$ if $a_{ij} < 0$ for all $i \in M$, then x_j can be complemented to satisfy (A.1). If there is no overlap of additive variable upper bounds, i.e. $N(i) \cap N(k) = \emptyset$ for all $i, k \in M$, then $M(j)$ is singleton for all $j \in N$ and (A.1) can be satisfied by complementing the binary variables when $a_{ij} < 0$.

Proposition 1. *$Conv(P)$ is full-dimensional.*

2.1 Additive Flow Cover Inequalities

For $C^+ \subseteq M^+$ and $C^- \subseteq M^-$, (C^+, C^-) is said to be a *flow cover* if $\lambda = u(C^+) - b - u(C^-) > 0$. For a flow cover (C^+, C^-), let $L^- \subseteq M^- \setminus C^-$ be such that $\gamma = \sum_{i \in L^-} u_i < \lambda$ and $K = M^- \setminus (C^- \cup L^-)$. Then, the *additive flow cover inequality* is

$$\sum_{i \in C^+} y_i + \sum_{j \in N(C^+)} (a_j(C^+) - \lambda + \gamma)^+ (1 - x_j) - \sum_{j \in N(L^-)} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \leq b + u(C^-) + \gamma. \quad (5)$$

Proposition 2. *The ad*

Proof. Let $(\bar{x}, \bar{y}) \in P$ an $N(C^+) : a_j(C^+) > \lambda -$ $N(L^-)^- = N(L^-) \setminus N(L$

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$$lhs \leq u(C^+) - \sum_{j \in N(C} \quad \sum_{j \in N(C^+)^+ \cap T} (\gamma \quad \leq u(C^+) - \lambda + \gamma \quad \leq b + u(C^-) + \gamma$$

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$y_i \leq b + u(C^-) + \gamma$. (5)

Proposition 2. *The additive flow cover inequality (5) is valid for P.*

Proof. Let $(\bar{x}, \bar{y}) \in P$ and $T = \{j \in N : \bar{x}_j = 0\}$. Also define $N(C^+)^+ = \{j \in N(C^+) : a_j(C^+) > \lambda - \gamma\}$, $N(L^-)^+ = \{j \in N(L^-) : a_j(L^-) > \lambda - \gamma\}$, and $N(L^-)^- = N(L^-) \setminus N(L^-)^+$. For $(\bar{x}, \bar{y})$ the left hand side of (5), *lhs*, equals

$$\sum_{i \in C^-} \bar{y}_i + \sum_{j \in N(C^+) \cap T} (a_j(C^+) - \lambda + \gamma)^+ - \sum_{j \in N(L^-) \setminus T} \min\{a_j(L^-), \lambda - \gamma\} - \sum_{i \in K} \bar{y}_i.$$

If $(N(C^+)^+ \cap T) \cup (N(L^-)^+ \setminus T) = \emptyset$, then

$$lhs \leq \sum_{i \in C^+} \bar{y}_i - \sum_{j \in N(L^-)^- \setminus T} a_j(L^-) - \sum_{i \in K} \bar{y}_i \leq b + u(C^-) + \gamma.$$

To see that the second inequality is valid, observe that $\sum_{i \in C^+} y_i - \sum_{i \in L^-} y_i - \sum_{i \in K} y_i \leq b + u(C^-)$ is valid for P and $\sum_{i \in L^-} y_i \leq u(L^-) - \sum_{j \in N(L^-)^+} a_j(L^-) - \sum_{j \in N(L^-)^- \cap T} a_j(L^-)$ is valid for $(\bar{x}, \bar{y})$ since $N(L^-)^+ \subseteq T$. Adding these two inequalities gives the result. Now, suppose $(N(C^+)^+ \cap T) \cup (N(L^-)^+ \setminus T) \neq \emptyset$. Then

$$\begin{aligned} lhs &\leq u(C^+) - \sum_{j \in N(C^+) \cap T} a_j(C^+) + \sum_{j \in N(C^+)^+ \cap T} a_j(C^+) + \\ &\quad \sum_{j \in N(C^+)^+ \cap T} (\gamma - \lambda) - \sum_{j \in N(L^-)^+ \setminus T} (\lambda - \gamma) - \sum_{j \in N(L^-)^- \setminus T} a_j(L^-) \\ &\leq u(C^+) - \lambda + \gamma - (\lambda - \gamma)[|N(C^+)^+ \cap T| + |N(L^-)^+ \setminus T| - 1] \\ &\leq b + u(C^-) + \gamma. \quad \square \end{aligned}$$

Remark 1. For the single node fixed-charge flow model, where (2) is replaced with $y_i \leq u_i x_i$, the additive flow cover inequality reduces to the flow cover inequality [12]

$$\sum_{i \in C^+} y_i + \sum_{i \in C^+} (u_i - \lambda)^+ (1 - x_i) - \sum_{i \in L^-} \min\{u_i, \lambda\} x_i - \sum_{i \in K} y_i \leq b + \sum_{i \in C^-} u_i.$$

Proposition 3. *The additive flow cover inequality (5) is facet-defining for $conv(P)$ if the following five conditions are satisfied.*

1. *$C^- = \emptyset$,*
2. *$\max_{j \in N(C^+)} a_j(C^+) > \lambda - \gamma$,*
3. *$a_j(L^-) > \lambda - \gamma$ for some $j \in N(i)$ for all $i \in L^-$ with $u_i = 0$,*
4. *$u_i \geq 0$ for all $i \in L^-$,*
5. *$N(L^-) \cap N(M \setminus L^-) = \emptyset$.*

Note that if there is no overlap of additive variable upper bounds among continuous variables, then Condition 5 is trivially satisfied.

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$$\sum_{i \in C^+} y_i + \sum_{j \in N(C^+)} (a_j(C^+) - \lambda + \gamma)^+ (1 - x_j) - \sum_{j \in N(L^-)} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \leq b + u(C^-) + \gamma. \quad (5)$$

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Proof. Let $(\bar{x}, \bar{y}) \in P$ and $N(C^+) : a_j(C^+) > \lambda - \gamma$ $N(L^-)^- = N(L^-) \setminus N(L$

$$\sum_{i \in C^+} \bar{y}_i + \sum_{j \in N(C^+) \cap T} (a_j(C^+$$

If $(N(C^+)^+ \cap T) \cup (N(L^-$

$$lhs \leq \sum_{i \in C^+} \bar{y}_i - \ldots_{j \in N(}$$

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3. *$a_j(L^-) > \lambda - \gamma$ for*
4. *$u_i \geq 0$ for all $i \in L$*
5. *$N(L^-) \cap N(M \setminus L$*

Note that if there is continuous variables, then

2.2 Additive Flow Packing Inequalities

Next we give the second class of valid inequalities for P. For $C^+ \subseteq M^+$ and $C^- \subseteq M^-$, (C^+, C^-) is said to be a *flow packing* if $\mu = -\lambda = b + u(C^-) - u(C^+) > 0$. For a flow packing (C^+, C^-), let $L^+ \subseteq M^+ \setminus C^+$ be such that $\gamma = \sum_{i \in L^+} u_i < \mu$ and $K = M^- \setminus C^-$. Then the *additive flow packing inequality* is

$$\sum_{i \in C^+ \cup L^+} y_i - \sum_{j \in N(L^+)} \min\{a_j(L^+), \mu - \gamma\} x_j + \sum_{j \in N(C^-)} (a_j(C^-) - \mu + \gamma)^+ (1 - x_j) - \sum_{i \in K} y_i \le u(C^+) + \gamma. \quad (6)$$

Proposition 4. *The additive flow packing inequality (6) is valid for P.*

Proof. Let $(\bar{x}, \bar{y}) \in P$ and $T = \{j \in N : \bar{x}_j = 0\}$. Also let $N(C^-)^+ = \{j \in N(C^+) : a_j(C^-) > \mu - \gamma\}, N(L^+)^+ = \{j \in N(C^+) : a_j(L^+) > \mu - \gamma\}$, and $N(L^+)^- = N(L^+) \setminus N(L^+)^+$. For $(\bar{x}, \bar{y})$ the left hand side of (6), lhs, equals

$$\sum_{i \in C^+ \cup L^+} \bar{y}_i - \sum_{j \in N(L^+) \setminus T} \min\{a_j(L^+), \mu - \gamma\} + \sum_{j \in N(C^-)^+ \cap T} (a_j(C^-) - \mu + \gamma) - \sum_{i \in K} \bar{y}_i.$$

If $(N(L^+)^+ \setminus T) \cup (N(C^-)^+ \cap T) = \emptyset$, then

$$lhs = \textstyle\sum_{i \in C^+ \cup L^+} \bar{y}_i - \sum_{j \in N(L^+)^- \setminus T} a_j(L^+) - \sum_{i \in K} \bar{y}_i \le u(C^+) + \gamma.$$

Otherwise,

$$\begin{aligned} lhs &\le b + u(C^-) - \sum_{j \in N(C^-) \cup T} a_j(C^-) - \sum_{j \in N(L^+)^+ \setminus T} (\mu - \gamma) \\ &\quad \sum_{j \in N(L^+)^- \setminus T} a_j(L^+) + \sum_{j \in N(C^-)^+ \cap T} (a_j(C^-) - \mu + \gamma) \\ &\le b + u(C^-) - \mu + \gamma - (\mu - \gamma)[|N(L^+)^+ \setminus T| + |N(C^-)^+ \cap T| - 1] \\ &\le u(C^+) + \gamma. \quad \square \end{aligned}$$

Remark 2. For the single node fixed-charge flow model, the additive flow packing inequality reduces to the flow packing inequality [3]

$$\sum_{i \in C^+ \cup L^+} y_i - \sum_{i \in L^+} \min\{u_i, \mu\} x_i + \sum_{i \in C^-} (u_i - \mu)^+ (1 - x_i) - \sum_{i \in K} y_i \le \sum_{i \in C^+} u_i.$$

Proposition 5. *The additive flow packing inequality (6) is facet-defining for $conv(P)$ if the following five conditions are satisfied.*

1. $C^+ = \emptyset$,
2. $u(M^-) + b > a_j(C^-) > \mu - \gamma$ *for some* $j \in N(C^-)$,
3. $a_j(L^+) > \mu - \gamma$ *for some* $j \in N(i)$ *for all* $i \in L^+$ *with* $u_i = 0$.
4. $u_i \ge 0$ *for all* $i \in L^+$,
5. $N(L^+) \cap N(M \setminus L^+) = N(C^-) \cap N(K) = \emptyset$.

2.3 Generalized Add

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Note that inequality
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Since (7) satisfies th
$\lambda - \gamma$ or equivalently $b+$

2.3 Generalized Additive Flow Cover Inequalities

In order to derive more general classes of valid inequalities for P, we fix a subset of the binary variables to zero, derive a valid inequality for the resulting projection, and then lift this inequality with the variables that are fixed to zero. More precisely, let $F \subseteq N$ and consider the projection

$$P_F = \{(x,y) \in \mathbb{B}^{n-|F|} \times \mathbb{R}^m_+ : \sum_{i \in M^+} y_i - \sum_{i \in M^-} y_i \leq b, \quad y_i \leq u_i + \sum_{j \in N(i)\setminus F} a_{ij} x_j, \ \ i \in M\}$$

of P obtained by fixing $x_j = 0$ for all $j \in F$. We assume that $conv(P_F)$ is full-dimensional.

For $S \subseteq M$, let $\bar{u}(S) = \sum_{i \in S}(u_i + \sum_{j \in N(i)\setminus F} a_{ij})$. Let $C^+ \subseteq M^+$ and $C^- \subseteq M^-$ such that $\lambda = \bar{u}(C^+) - b - \bar{u}(C^-) > 0$ and $L^- \subseteq M^- \setminus C^-$ such that $\gamma = \sum_{i \in L^-} u_i < \lambda$. Then, from Section 2.1 we have the following valid additive flow cover inequality for P_F

$$\sum_{i \in C^+} y_i + \sum_{j \in N(C^+)\setminus F} (a_j(C^+) - \lambda + \gamma)^+(1 - x_j) - \sum_{j \in N(L^-)\setminus F} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \leq b + \bar{u}(C^-) + \gamma. \qquad (7)$$

Note that inequality (7) is not necessarily valid for P. We assume that the conditions of Proposition 3 are satisfied and hence (7) is facet-defining for $conv(P_F)$. In order to derive a generalized additive flow cover inequality for P, we lift (7) in two phases. In the first phase we lift the inequality with the variables in $N(L^- \cup C^-) \cap F$. Then in the second phase we lift the resulting inequality with the variables in $N(C^+) \cap F$. When lifting the variables in phases, for convenience, we make the following assumption:

(A.6) $(N(C^+) \cap F) \cap (N(L^- \cup C^-) \cap F) = \emptyset$.

Even if (A.6) is not satisfied, the lifted inequality is still valid for P, but it may not be facet-defining for $conv(P)$. Now, let (7) be lifted first with variable x_l, $l \in N(L^- \cup C^-) \cap F$. Then the lifting coefficient associated with x_l is equal to

$$f(a_l(L^- \cup C^-)) = b + \bar{u}(C^-) + \gamma - \max_{(x,y) \in P_{F \setminus l}, x_l = 1} \left\{ \sum_{i \in C^+} y_i + \sum_{j \in N(C^+)\setminus F} (a_j(C^+) - \lambda + \gamma)^+(1 - x_j) - \sum_{j \in N(L^-)\setminus F} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \right\}.$$

Since (7) satisfies the conditions of Proposition 3, it follows that $\bar{u}(C^+) > \lambda - \gamma$ or equivalently $b + \bar{u}(C^-) + \gamma > 0$. Then the lifting problem has an optimal solution such that $y_i = 0$ for all $i \in (M^+ \setminus C^+) \cup K$. Let $(\bar{x}, \bar{y})$ be such an optimal solution and let $S = \{j \in N(C^+) \setminus F : \bar{x}_j = 0\}$ and $T = \{j \in N(L^-) \setminus F : \bar{x}_j = 1\}$. Clearly, we may assume that $S \subseteq \{j \in N(C^+) \setminus F : a_j(C^+) > \lambda - \gamma\}$ and $T \subseteq \{j \in N(L^-) \setminus F : a_j(L^-) > \lambda - \gamma\}$; otherwise we can obtain a solution with the same or better objective value by considering a subset of S or T satisfying these conditions. There are two cases to consider when determining the value of $f(a_l(L^- \cup C^-))$ depending on how $\sum_{i \in C^+} y_i$ is bounded in an optimal solution. We analyze $f(a_l(L^- \cup C^-))$ separately for each case.

Case 1: $\lambda - \gamma \leq \sum_{j \in S} a_j(C^+) + \sum_{j \in T} a_j(L^-) + a_l(L^- \cup C^-)$.

$$\begin{aligned} f(a_l(L^- \cup C^-)) &= b + \bar{u}(C^-) + \gamma - [\bar{u}(C^+) - \sum_{j \in S} a_j(C^+) + \\ &\qquad \sum_{j \in S} (a_j(C^+) - \lambda + \gamma) - \sum_{j \in T} (\lambda - \gamma)] \\ &= (|S \cup T| - 1)(\lambda - \gamma). \end{aligned}$$

Case 2: $\lambda - \gamma > \sum_{j \in S} a_j(C^+) + \sum_{j \in T} a_j(L^-) + a_l(L^- \cup C^-)$.

$$\begin{aligned} f(a_l(L^- \cup C^-)) &= b + \bar{u}(C^-) + \gamma - [b + \bar{u}(C^-) + \gamma + \sum_{j \in T} a_j(L^-) + \\ &\qquad a_l(L^- \cup C^-) + \sum_{j \in S} (a_j(C^+) - \lambda + \gamma) - \sum_{j \in T} (\lambda - \gamma)] \\ &= |S \cup T|(\lambda - \gamma) - \sum_{j \in S} a_j(C^+) - \sum_{j \in T} a_j(L^-) - a_l(L^- \cup C^-). \end{aligned}$$

Observe that in Case 2, i.e. if the balance constraint is tight, $S = T = \emptyset$ since $a_l(L^- \cup C^-) \geq 0$ and by assumption $a_j(C^+) > \lambda - \gamma$ for all $j \in S$ and $a_j(L^-) > \lambda - \gamma$ for all $j \in T$. Also in Case 1, $f(a_l(L^- \cup C^-))$ is minimized when $S = T = \emptyset$. Then, we conclude that $f(a_l(L^- \cup C^-)) = -\min\{a_l(L^- \cup C^-), \lambda - \gamma\}$. It is easy to see that f is superadditive on $\mathbb{R}_-$, which implies that the lifting is sequence independent, that is the lifting function f remains unchanged as the projected variables in $N(L^- \cup C^-) \cap F$ are introduced to inequality (7) sequentially [7,13]. Therefore,

$$\begin{aligned} &\sum_{i \in C^+} y_i + \sum_{j \in N(C^+) \setminus F} (a_j(C^+) - \lambda + \gamma)^+ (1 - x_j) - \\ &\qquad \sum_{j \in N(L^- \cup C^-) \cap F} \min\{a_j(L^- \cup C^-), \lambda - \gamma\} x_j - \\ &\qquad\qquad \sum_{j \in N(L^-) \setminus F} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \leq b + \bar{u}(C^-) + \gamma \end{aligned} \tag{8}$$

is a valid inequality for $P_{N(C^+) \cap F}$.

In the second phase. w
coefficient of x_l equals

$g(a_l(C^+)) = b + \bar{u}(C^-$

$\sum_{j \in N(C^+) \setminus F} (a_j(C^+) - \lambda$

$\sum_{j \in N(L^- \cup C^-) \cap F} \min\{a$

The lifting problem f
$(M^+ \setminus C^+) \cup K$, $x_j = 1$ fc
for all $j \in N(L^- \cup C^-) \cap$
$j \in N(L^-) \setminus F$ such that
Let $R = \{j \in N(L^- \cup C^-$
$T = \{j \in N(L^-) \setminus F : \bar{x}$
the value of $g(a_l(C^+))$ d
solution.

Case 1: $\lambda - \gamma \leq \sum_{j \in S} a_j(C$

$g(a_l(C^+)) = b + \bar{u}(C^-)$ -
$\sum_{j \in S} a_j(C^+)$
$= (|R \cup S \cup T$

Case 2: $\lambda - \gamma > \sum_{j \in S} a_j(C$

$g(a_l(C^+)) = b + \bar{u}(C^-) +$
$\sum_{j \in T} a_j(L^-)$ -
$= |R \cup S \cup T|($

Now, let

$v_j = \{$

In the second phase, we lift inequality (8) with $x_l, l \in N(C^+) \cap F$. The lifting coefficient of x_l equals

$$g(a_l(C^+)) = b + \bar{u}(C^-) + \gamma - \max_{(x,y) \in P_{N(C^+) \cap F \setminus l, x_l = 1}} \left\{ \sum_{i \in C^+} y_i + \sum_{j \in N(C^+) \setminus F} (a_j(C^+) - \lambda + \gamma)^+ (1 - x_j) - \sum_{j \in N(L^-) \setminus F} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{j \in N(L^- \cup C^-) \cap F} \min\{a_j(L^- \cup C^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \right\}.$$

The lifting problem has an optimal solution such that $y_i = 0$ for all $i \in (M^+ \setminus C^+) \cup K$, $x_j = 1$ for all $j \in N(C^+) \setminus F$ such that $a_j(C^+) \leq \lambda - \gamma$, $x_j = 0$ for all $j \in N(L^- \cup C^-) \cap F$ such that $a_j(L^- \cup C^-) \leq \lambda - \gamma$, and $x_j = 0$ for all $j \in N(L^-) \setminus F$ such that $a_j(L^-) \leq \lambda - \gamma$. Let $(\bar{x}, \bar{y})$ be such an optimal solution. Let $R = \{j \in N(L^- \cup C^-) \cap F) : \bar{x}_j = 1\}$, $S = \{j \in N(C^+) \setminus F : \bar{x}_j = 0\}$, and $T = \{j \in N(L^-) \setminus F : \bar{x}_j = 1\}$. Again, there are two cases when determining the value of $g(a_l(C^+))$ depending on how $\sum_{i \in C^+} y_i$ is bounded in an optimal solution.

Case 1: $\lambda - \gamma \leq \sum_{j \in S} a_j(C^+) + \sum_{j \in R} a_j(L^- \cup C^-) + \sum_{j \in T} a_j(L^-) - a_l(C^+)$.

$$\begin{aligned} g(a_l(C^+)) &= b + \bar{u}(C^-) + \gamma - [\bar{u}(C^+) - \\ &\quad \sum_{j \in S} a_j(C^+) + a_l(C^+) + \sum_{j \in S} (a_j(C^+) - \lambda + \gamma) - \sum_{i \in R \cup T} (\lambda - \gamma)] \\ &= (|R \cup S \cup T| - 1)(\lambda - \gamma) - a_l(C^+). \end{aligned}$$

Case 2: $\lambda - \gamma > \sum_{j \in S} a_j(C^+) + \sum_{j \in R} a_j(L^- \cup C^-) + \sum_{j \in T} a_j(L^-) - a_l(C^+)$.

$$\begin{aligned} g(a_l(C^+)) &= b + \bar{u}(C^-) + \gamma - [b + \bar{u}(C^-) + \gamma + \sum_{j \in R} a_j(L^- \cup C^-) + \\ &\quad \sum_{j \in T} a_j(L^-) + \sum_{j \in S} (a_j(C^+) - \lambda + \gamma) - \sum_{j \in R \cup T} (\lambda - \gamma)] \\ &= |R \cup S \cup T|(\lambda - \gamma) - \sum_{j \in S} a_j(C^+) - \sum_{j \in R} a_j(L^- \cup C^-) - \sum_{j \in T} a_j(L^-). \end{aligned}$$

Now, let

$$v_j = \begin{cases} a_j(C^+), & \text{if } j \in N(C^+) \setminus F, \\ a_j(L^- \cup C^-), & \text{if } j \in N(L^- \cup C^-) \cap F, \\ a_j(L^-), & \text{if } j \in N(L^-) \setminus F, \end{cases}$$

$$x'_j = \begin{cases} 1 - x_j, & \text{if } j \in N(C^+) \setminus F, \\ x_j, & \text{if } j \in N(L^-) \cup (N(C^-) \cap F). \end{cases}$$

and $\{j_1, j_2, \ldots, j_r\} = \{j \in (N(C^+) \setminus F) \cup N(L^-) \cup (N(C^-) \cap F) : v_j > \lambda - \gamma\}$ such that $v_{j_k} \geq v_{j_{k+1}}$ for $k = 1, 2, \ldots r-1$. We also define the partial sums $w_0 = 0$, $w_k = \sum_{i=1}^{k} v_{j_i}$ for $k = 1, 2, \ldots, r$.

It is not hard to show that there is a monotone optimal solution to the lifting problem. That is, there exists an optimal solution such that $\bar{x}'_{j_k} \geq \bar{x}'_{j_{k+1}}$ for $k = 1, 2, \ldots, r-1$. Therefore $g(a_l(C^+))$ can be expressed in a closed form as follows:

$$g(a_l(C^+)) = \begin{cases} k(\lambda - \gamma) - a_l(C^+), & w_k < a_l(C^+) \leq w_{k+1} - \lambda + \gamma, \\ & \qquad k = 0, 1, \ldots, r-1, \\ k(\lambda - \gamma) - w_k, & w_k - \lambda + \gamma < a_l(C^+) \leq w_k, \quad k = 1, 2, \ldots, r, \\ r(\lambda - \gamma) - w_r, & w_r < a_l(C^+). \end{cases}$$

It can be shown that g is superadditive on $\mathbb{R}_-$, which implies that the lifting function g remains unchanged as the projected variables in $N(C^+) \cap F$ are introduced to inequality (8) sequentially [7,13]. Hence we have the following result.

Proposition 6. *The generalized additive flow cover inequality*

$$\sum_{i \in C^+} y_i + \sum_{j \in N(C^+) \setminus F} (a_j(C^+) - \lambda + \gamma)^+ (1 - x_j) +$$
$$\sum_{j \in N(C^+) \cap F} \alpha_j x_j - \sum_{j \in N(L^- \cup C^-) \cap F} \min\{a_j(L^- \cup C^-), \lambda - \gamma\} x_j - \qquad (9)$$
$$\sum_{j \in N(L^-) \setminus F} \min\{a_j(L^-), \lambda - \gamma\} x_j - \sum_{i \in K} y_i \leq b + \bar{u}(C^-) + \gamma$$

with

$$\alpha_j = \begin{cases} k(\lambda - \gamma) - a_j(C^+), & w_k < a_j(C^+) \leq w_{k+1} - \lambda + \gamma, \ k = 0, 1, \ldots, r-1, \\ k(\lambda - \gamma) - w_k, & w_k - \lambda + \gamma < a_j(C^+) \leq w_k, \quad k = 1, 2, \ldots, r, \\ r(\lambda - \gamma) - w_r, & w_r < a_j(C^+). \end{cases}$$

is valid for P.

Proposition 7. *The generalized additive flow cover inequality (9) is facet-defining for conv(P) if (7) is facet-defining for conv(P_F).*

2.4 Generalized Additive Flow Packing Inequalities

Here we generalize the additive flow packing inequalities with the same approach taken in Section 2.3 for the additive flow cover inequalities. Consider the projection P_F of P introduced in Section 2.3. Let $C^+ \subseteq M^+$ and $C^- \subseteq M^-$ such that $\mu = b + \bar{u}(C^-) - \bar{u}(C^+) > 0$ and $L^+ \subseteq M^+ \setminus C^+$ such that $\gamma = \sum_{i \in L^+} u_i < \mu$.

Then from Section 2.2 w
ity for P_F

$\sum_{i \in C^+ \cup L^+} y_i - \sum_{j \in N(}$

$\sum_{j \in N(C^-) \setminus F} (a_j$

We assume that the
is facet-defining for con
(10), we lift (10) in two
$N(C^+ \cup L^+) \cap F$. Then
variables in $N(C^-) \cap F$
convenience, we assume

(A.7) $(N(L^+ \cup C^+)$

The lifting of inequalit
Therefore, we only give

Proposition 8. *The g*

$\sum_{i \in C^+ \cup L^+} y_i - \sum_{j \in N(L^+}$

$\sum_{j \in N(C^+ \cup L^+) \cap F}$

$\sum_{j \in N(C}$

with

$\alpha_j = \begin{cases} k(\mu - \gamma) - a_j(\\ k(\mu - \gamma) - w_k, \\ r(\mu - \gamma) - w_r, \end{cases}$

is valid for P.

Proposition 9. *The g*
defining for conv(P) if

3 Computationa

In this section, we pre
pansion problems with
separation algorithms f
inequalities for the singl

$C^-) \cap F)$,

$(N(C^-) \cap F) : v_j > \lambda - \gamma\}$
ılso define the partial sums

ɔne optimal solution to the
ution such that $\bar{x}'_{j_k} \geq \bar{x}'_{j_{k+1}}$
xpressed in a closed form as

$v_{k+1} - \lambda + \gamma,$
$k = 0, 1, \ldots, r-1,$
$C^+) \leq w_k, \quad k = 1, 2, \ldots, r,$

vhich implies that the lifting
ariables in $N(C^+) \cap F$ are
ence we have the following

r inequality

$\cup C^-), \lambda - \gamma\} x_j -$ (9)

$y_i \leq b + \bar{u}(C^-) + \gamma$
K

$\lambda + \gamma, k = 0, 1, \ldots, r-1,$
$\leq w_k, \quad k = 1, 2, \ldots, r,$

r inequality (9) is facet-defi-
$F)$.

qualities

ities with the same approach
ıalities. Consider the projec-
I^+ and $C^- \subseteq M^-$ such that
ıch that $\gamma = \sum_{i \in L^+} u_i < \mu$.

Then from Section 2.2 we have the following valid additive flow packing inequality for P_F

$$\sum_{i \in C^+ \cup L^+} y_i - \sum_{j \in N(L^+) \setminus F} \min\{a_j(L^+), \mu - \gamma\} x_j + \sum_{j \in N(C^-) \setminus F} (a_j(C^-) - \mu + \gamma)^+ (1 - x_j) - \sum_{i \in K} y_i \leq \bar{u}(C^+) + \gamma. \tag{10}$$

We assume that the conditions of Proposition 5 are satisfied and hence (10) is facet-defining for $conv(P_F)$. To introduce the variables in F into inequality (10), we lift (10) in two phases. First we lift the inequality with the variables in $N(C^+ \cup L^+) \cap F$. Then in the second phase we lift the resulting inequality with variables in $N(C^-) \cap F$. When employing this two phase lifting procedure, for convenience, we assume that

(A.7) $(N(L^+ \cup C^+) \cap F) \cap (N(C^-) \cap F) = \emptyset$.

The lifting of inequality (10) proceeds similar to the lifting of inequality (7). Therefore, we only give the final result here.

Proposition 8. *The generalized additive flow packing inequality*

$$\sum_{i \in C^+ \cup L^+} y_i - \sum_{j \in N(L^+) \setminus F} \min\{a_j(L^+), \mu - \gamma\} x_j - \sum_{j \in N(C^+ \cup L^+) \cap F} \min\{a_j(C^+ \cup L^+), \mu - \gamma\} x_j + \sum_{j \in N(C^-) \cap F} \alpha_j x_j - \sum_{j \in N(C^-) \setminus F} (a_j - \mu + \gamma)^+ (1 - x_j) - \sum_{i \in K} y_i \leq \bar{u}(C^+) + \gamma \tag{11}$$

with

$$\alpha_j = \begin{cases} k(\mu - \gamma) - a_j(C^-), & w_k < a_j(C^-) \leq w_{k+1} - \mu + \gamma, \; k = 0, 1, \ldots, r-1, \\ k(\mu - \gamma) - w_k, & w_k - \mu + \gamma < a_j(C^-) \leq w_k, \quad k = 1, 2, \ldots, r, \\ r(\mu - \gamma) - w_r, & w_r < a_j(C^-), \end{cases}$$

is valid for P.

Proposition 9. *The generalized additive flow packing inequality (11) is facet-defining for $conv(P)$ if (10) is facet-defining for $conv(P_F)$.*

3 Computational Results

In this section, we present our computational results on solving network expansion problems with a branch-and-cut algorithm. We implemented heuristic separation algorithms for the generalized additive flow cover and flow packing inequalities for the single node relaxation of the problem. We also used the lifted cover inequalities [6] for surrogate 0-1 knapsack relaxations of the single node relaxation, where the continuous flow variables are replaced with either their 0-1 additive variable upper bound variables or with their lower bounds. The branch-and-cut algorithm was implemented with MINTO [8] (version 3.0) using CPLEX as LP solver (version 6.0). All of the experiments were performed on a SUN Ultra 10 workstation with a one hour CPU time limit and a 100,000 nodes search tree size limit.

We present a summary of two experiments. The first experiment is performed to test the effectiveness of the cuts in solving a set of randomly generated network expansion problems with 20 vertices and 70% edge density. The instances were solved using MINTO first with its default settings and then with the above mentioned cutting planes generated throughout the search tree. In Table 1, we report the number of AVUB variables per flow variable (`avubs`) and the average values for the LP relaxation at the root node of the search tree (`zroot`), the best lower bound (`zlb`) and the best upper bound (`zub`) on the optimal value at termination, the percentage gap between `zlb` and `zub` (`endgap`), the number of generalized additive flow cover cuts (`gafcov`), generalized additive flow packing cuts (`gafpack`), surrogate knapsack cover cuts (`skcov`) added, the number of nodes evaluated (`nodes`), and the CPU time elapsed in seconds (`time`) for five random instances. While none of the problems could be solved to optimality without adding the cuts within 100,000 nodes, all of the problems were solved easily when the cuts were added. We note that MINTO does not generate any flow cover inequalities for these problem, since it does not recognize that additive variable upper bounds can be relaxed to simple variable upper bounds. Observe that the addition of the cuts improves the lower bounds as well as the upper bounds significantly, which leads to much smaller search trees and overall solution times. Table 1 clearly shows the effectiveness of the cuts.

Table 1. Effectiveness of cuts: 20 vertices.

	avubs	zroot	zlb	zub	endgap	gafcov	gafpack	skcov	nodes	time
without cuts	2	9.49	10.22	16.80	39.00	0	0	0	100,000	1386
	4	12.74	16.99	25.60	33.86	0	0	0	100,000	1018
	8	2.77	11.51	58.40	79.15	0	0	0	100,000	1859
with cuts	2	15.30	15.40	15.40	0.00	46	42	15	6	1
	4	23.73	25.20	25.20	0.00	55	33	47	73	5
	8	36.36	38.40	38.40	0.00	47	34	41	121	9

In the next experiment, we solved larger instances of the network expansion problem with 20% edge density to find out the sizes of instances that can be solved with the branch-and-cut algorithm. The results of this experiment are summarized in Table 2, where we present the number of AVUB variables per flow variable (`avubs`), the average values for the percentage difference between the initial LP relaxation and `zub` (`initgap`), the percentage difference between the LP relaxation after the cuts are added at the root node and `zub` (`rootgap`),

in addition to endgap, gaf
instances with 50, 100 anc
larger than ones for whic
Although all of the instan
the larger instances the ga
bound could not be comple
within an hour of CPU ti
is significant, ranging betv

Table 2. Perf

vertices	avubs	initga
50	1	25.0
	2	20.1
	4	36.3
	8	92.0
100	1	14.8
	2	12.2
	4	38.0
	8	92.7
150	1	16.5
	2	10.8
	4	43.6
	8	93.0

For most of the unsolve
bound were found at the rc
was observed later in the
the initial LP relaxation w
LP relaxation improved to
in the search tree, in 617 se
the root node by a simple l
feasible for the flow on eac
the unsolved problems it i
LP relaxations are much si

More detailed experime
ent classes of cuts revealed
were the most effective, an
more effective than the ger
the use of all three classes
From these computational
from the single node relax
for network design/expansi

.axations of the single node
' replaced with either their
h their lower bounds. The
NTO [8] (version 3.0) using
ments were performed on a
e limit and a 100.000 nodes

rst experiment is performed
of randomly generated net-
edge density. The instances
ıgs and then with the above
search tree. In Table 1, we
ɔle (avubs) and the average
ıe search tree (zroot), the
.ub) on the optimal value at
ıb (endgap), the number of
ılized additive flow packing
:ov) added, the number of
l in seconds (time) for five
ıld be solved to optimality
f the problems were solved
TO does not generate any
ıoes not recognize that ad-
ple variable upper bounds.
.ower bounds as well as the
ller search trees and overall
ıs of the cuts.

vertices.

pack	skcov	nodes	time
0	0	100,000	1386
0	0	100,000	1018
0	0	100.000	1859
12	15	6	1
33	47	73	5
34	41	121	9

's of the network expansion
's of instances that can be
ılts of this experiment are
ɔer of AVUB variables per
centage difference between
centage difference between
t node and zub (rootgap),

in addition to endgap, gafcov, gafpack, skcov, nodes, and time for five random instances with 50, 100 and 150 vertices. We note that these problems are much larger than ones for which computations are provided in the literature [5,11]. Although all of the instances with 50 vertices could be solved to optimality, for the larger instances the gap between the best lower bound and the best upper bound could not be completely closed for most of the problems with 4 or 8 avubs within an hour of CPU time. Nevertheless, the improvement in LP relaxations is significant, ranging between 50% and 98%.

Table 2. Performance of the branch-and-cut algorithm.

vertices	avubs	initgap	rootgap	endgap	gafcov	gafpack	skcov	nodes	time
50	1	25.09	1.05	0.00	48	56	16	24	2
	2	20.12	2.20	0.00	115	102	38	110	11
	4	36.34	3.31	0.00	192	139	106	721	49
	8	92.06	3.72	0.00	416	216	164	7712	960
100	1	14.80	0.94	0.00	182	148	41	44	39
	2	12.27	4.44	0.00	590	324	191	1236	961
	4	38.04	3.74	2.37	707	382	448	2437	2171
	8	92.75	9.15	8.75	828	408	509	1717	3600
150	1	16.56	0.29	0.00	480	307	57	346	586
	2	10.81	5.39	4.34	586	395	318	813	2965
	4	43.69	12.22	12.22	862	527	672	711	3600
	8	93.08	15.02	15.02	659	449	503	477	3600

For most of the unsolved problems, the best lower bound and the best upper bound were found at the root node in a few minutes; no improvement in the gap was observed later in the search tree. For instance nexp.100.8.5 the value of the initial LP relaxation was 18.23. After adding 916 cuts in 42 rounds the root LP relaxation improved to 220.81, which was in fact the best lower bound found in the search tree, in 617 seconds. The best upper bound 259 was again found at the root node by a simple heuristic which installs the least cost integral capacity feasible for the flow on each edge provided by the LP relaxation. Therefore, for the unsolved problems it is likely that the actual duality gaps of the improved LP relaxations are much smaller.

More detailed experiments to compare the relative effectiveness of the different classes of cuts revealed that the generalized additive flow cover inequalities were the most effective, and that the lifted surrogate knapsack inequalities were more effective than the generalized additive flow packing inequalities. However, the use of all three classes of cuts delivered the best performance in most cases. From these computational results, we conclude that the valid inequalities derived from the single node relaxations are very effective in improving the LP bounds for network design/expansion problems.

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Abstract. We estab
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to be *TDI*, where H
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covering cycles, toge
the feedback vertex
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Key words. feedbac
tegrality, min-max re

AMS subject class

1 Introduction

The basic theme of polyh
gramming duality theory
is often a combinatorial m
theoretical interest, a co
rithmic solvability of the
the totally dual integral (
Giles [2,3] serves as a gen

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