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ON SPERNER'S LEMMA

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In memory of my grandmother Anna Bukova-Atanassova

Let $p(A_1, A_2, \dots, A_n)$ be a convex n-gon ($n \geqq 3$) with vertices marked by $A_1, A_2, \dots, A_n$, which is covered with triangles in Sperner's way [1] (see also e.g. [2]): if two triangles have a common point, then the set of their common points is either a common vertex or a common side of the two triangles. Let every vertex of these triangles be marked by the symbols $A_1, A_2, \dots, A_n$ in the following way: the vertices which are on the side A_iA_{i+1} ($1 \leqq i \leqq n$; A_{n+1} coincides with A_1) can be marked with A_i or A_{i+1}; the ones which are inside of $p(A_1, A_2, \dots, A_n)$ — with any one of the symbols $A_1, A_2, \dots, A_n$.

Let $b(p(A_1, A_2, \dots, A_n))$ be the number of the triangles in $p(A_1, A_2, \dots, A_n)$ marked with three different symbols (we shall denote these triangles by $T3$s). By $[A_i/A_j]p(A_1, A_2, \dots, A_n)$ we shall denote the replacement of the symbol A_j of all vertices of $p(A_1, A_2, \dots, A_n)$ by the symbol A_i.

LEMMA 1. *For every two natural numbers i, j:*

$$b([A_i/A_j]p(A_1, A_2, \dots, A_n)) \leqq b(p(A_1, A_2, \dots, A_n)). \tag{1}$$

PROOF. Obviously, the inequality (1) is valid for every i, j, for which $1 \leqq i = j \leqq n$; for every $i > n$; for every $j > n$.

Let $1 \leqq i$, $j \leqq n$ and $i \neq j$. The triangles of $p(A_1, A_2, \dots, A_n)$ can be divided into three groups:

triangles which have not a vertex marked with A_j;
triangles which have only one vertex marked with A_j;
triangles which have two or three vertices marked with A_j.

The number of $T3$s from the first and third groups will not change after replacing A_j by A_i while the number of $T3$s from the second group will decrease with the number of these triangles which initially contain the symbols A_i and A_j simultaneously (because after the substitution they will contain two vertices marked by A_i). Therefore the inequality (1) is valid.

LEMMA 2. *If A and B are two neighbourly vertices which take part in a marking of $p(A_1, A_2, \dots, A_n)$, then they take part jointly in at least one $T3$.*

PROOF. Let us assume (without loss of generality) that the symbols A_1 and A_2 which marked two neighbourly vertices in $p(A_1, A_2, \dots, A_n)$ do not

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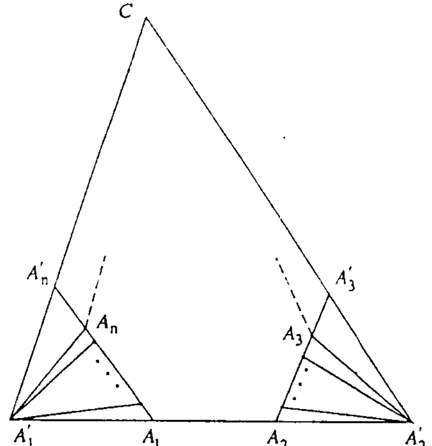

Fig. 1

mark two points of any $T3$. We construct the triangle $t(A'_1, A'_2, C)$ (see Fig. 1) which contains $p(A_1, A_2, \dots, A_n)$ and for which the arc A_1A_2 of $p(A_1, A_2, \dots, A_n)$ lies on the line on which the arc $A'_1A'_2$ of $t(A'_1, A'_2, C)$ lies. Let A'_3 and A'_n be the points of intersection of the lines on which the segments A'_2C and A_2A_3, and A'_1C and A_1A_n lie, respectively. We construct a triangle net in Sperner's way for the triangles $t'(A'_1, A_1, A'_n)$ and $t''(A_2, A'_2, A'_3)$ for which every triangle has as a vertex the points A'_1 and A'_2, respectively. The part of $t(A'_1, A'_2, C)$ which is outside of both last triangles and outside of $p(A_1, A_2, \dots, A_n)$, and which we shall mark by Q, also is covered by triangles by Sperner's way. We mark all vertices of the last figure, except those lying on the boundary of $p(A_1, \dots, A_n)$, by symbol C. Finally, we mark by new symbols the points of $t(A'_1, A'_2, C)$ constructing the triangle

$$t^*(A_1, A_2, C) = [A_1/A'_1][A_2/A'_2][C/A_3]\dots[C/A_n]t(A'_1, A'_2, C).$$

Obviously, after the change, in the triangles $t'(A'_1, A_1, A'_n)$ and $t''(A_2, A'_2, A'_3)$ there are not $T3$s, yet, by Lemma 1, in $p(A_1, A_2, \dots, A_n)$ there cannot be generated a new $T3$, because, by condition, in $p(A_1, A_2, \dots, A_n)$ initially there has not been a $T3$ with symbols A_1 and A_2; in figure Q also there are no $T3$s because all points are marked only by symbol C. Therefore there are no $T3$s in $t^*(A_1, A_2, C)$, which is a contradiction with the ordinary Sperner's lemma. Hence, the assumption that A_1 and A_2 do not mark two points in any $T3$ is false. From this it follows that Lemma 2 is valid.

LEMMA 3 (Generalization of Sperner's lemma). *For every natural number $n \geqq 3$ in every convex n-gon $p(A_1, A_2, \dots, A_n)$ there exist at least $n-2$ $T3$s.*

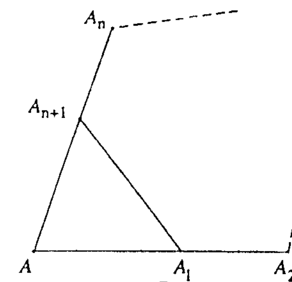

Fig. 2

PROOF. When $n = 3$ we obtain Sperner's lemma. Let us assume that the assertion is valid for some $n \geqq 3$ and let $p'(A_1, A_2, \dots, A_n, A_{n+1})$ be a convex $(n+1)$-gon. With the exception of the case that $p'(A_1, A_2, \dots, A_{n+1})$ is a parallelogram there exist three sides of $p'(A_1, A_2, \dots, A_{n+1})$ (let them be $A_nA_{n+1}, A_{n+1}A_1$ and A_1A_2; see Fig. 2) for which there exists a point A common to the rays A_nA_{n+1} and A_2A_1. This follows from the convexity of $p'(A_1, A_2, \dots, A_{n+1})$. Now we shall consider the convex n-gon $p''(A, A_2, \dots, A_n)$ with the triangulation of Fig. 2 or Fig. 3 (we retain the edges in the triangulation of $p'(A_1, A_2, \dots, A_{n+1})$ and there are possibly still new edges from A to interior points of A_1A_{n+1}; see e.g., Fig. 3, where the points $B_1, B_2, \dots, B_s$ are vertices of the triangulation of $p'(A_1, A_2, \dots, A_{n+1})$ and

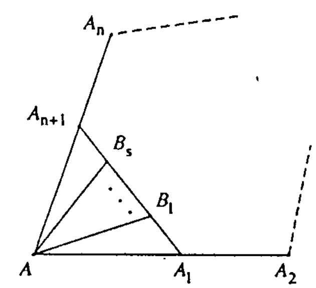

Fig. 3

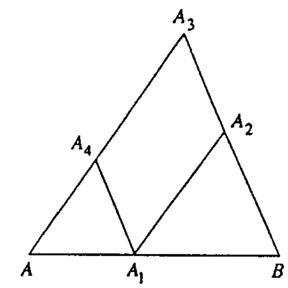

Fig. 4

$$p(A_1, A_2, \dots, A_n) = [A_1/A][A_1/A_{n+1}]p''(A, A_2, \dots, A_n).$$

By induction, $b(p(A_1, A_2, \dots, A_n)) \geqq n-2$.

There exists the following particular case when the construction of the point A is not possible: $p'(A_1, A_2, A_3, A_4)$ is a parallelogram ($n = 3$). Then we construct the points A and B (see Fig. 4) and the triangle $p''(A, B, A_3)$ and then

$$p(A_1, A_2, A_3) = [A_1/A][A_1/A_4][A_2/B]p''(A, B, A_3).$$

By Sperner's lemma it follows that $b(p(A_1, A_2, A_3)) \geqq 1$.

By Lemma 2 the symbols A_1 and A_{n+1} mark two of the vertices of at least one $T3$. Since the same triangle is no $T3$ in $p(A_1, A_2, \dots, A_n)$, and $p(A_1, A_2, \dots, A_n)$ does not contain a $T3$ outside $p'(A_1, A_2, \dots, A_{n+1})$, thus the number of $T3$s in $p'(A_1, A_2, \dots, A_{n+1})$ will be greater than the number of $T3$s in $p(A_1, A_2, \dots, A_n)$ with at least one, i.e., the inequality (1) is strict. Then

$$b(p'(A_1, A_2, \dots, A_{n+1})) > b(p(A_1, A_2, \dots, A_n)) \geqq n-2,$$

by which Lemma 3 is proved.

Observe that any simple n-gon is topologically equivalent to a convex n-gon. Further both the proof of Sperner's lemma [1] and the proofs of Lemmas 1–3 of the present paper only use the combinatorial structure of the triangulations. Thus these proofs apply to any triangulation of any domain that is topologically equivalent to a triangulation of a convex n-gon by topological arcs, in Sperner's way. Thus, in particular, we have the following

THEOREM. *Every simple n-gon, which is marked in Sperner's way, has at least $n-2$ $T3$s.*

Finally, we shall formulate the following

HYPOTHESIS. *For every n-vertex polytope P in a k-dimensional space $(n \geqq k+1; k \geqq 1)$ which is covered by k-dimensional simplices in the n-dimensional analogue of Sperner's way, and every marking of the vertices of this triangulation by the vertices of P, for which the vertices of the triangulation lying on an i-face of P $(0 \leqq i \leqq k-1)$ are marked by the vertices of this i-face, there exist at least $n-k$ k-dimensional simplices which are marked with $k+1$ different symbols.*

The author formulated the above described generalization of Sperner's lemma in the summer of 1970, but its first proof was published in the preprint [3] in 1989.

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SOME DISTRIBUTION RESULTS ON TWO-SAMPLE RANK STATISTICS FOR UNEQUAL SAMPLE SIZES

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Abstract

This paper deals with the derivation of the null joint and marginal probab butions of some rank order statistics by using the extended Dwass technique Aneja [1] and Očka [8]. The rank order statistics considered include the numbe reflections, the index of the i^{th} positive reflection and the interval between the l^{th} positive reflections.

1. Introduction

Suppose $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ $(m \geqq n)$ are two in dent random samples from populations with unknown continuous di tion functions $F(x)$ and $G(x)$, respectively. Let $F_m(x)$ and $G_n(x)$ corresponding empirical distribution functions. Let $\{Z_k\}$, $(k=1,2,\dots$ $+n)$ denote the combined set of these $m+n$ values arranged in an in order of magnitude and let $Z_0 = -\infty$. Since the variables X's and independent and their distribution functions are continuous, the pro that any two values are equal is zero. Therefore, ties between two va ruled out and we have $Z_1 < Z_2 < \dots < Z_{m+n}$. If we replace X's by and Y's by (-1)'s in this ordered set, we obtain a sequence of rank o dicators whose suitable functions called rank order statistics can be in terms of

$$H_{m,n}(u) = mF_m(u) - nG_n(u), \qquad -\infty < u < \infty.$$

With every sequence of rank order indicators one can associate a walk of a particle performing m upward and n downward steps. Fo Dwass [2] developed an alternate method based on the simple rand with independent steps for obtaining the distributions of two-sam order statistics. Aneja [1] and Očka [8] have extended the Dwass te when $m \neq n$ and derived the distributions of quite a few rank order s

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