MONOTONE PATHS ON POLYTOPES

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ABSTRACT. We investigate the vertex-connectivity of the graph of f-monotone paths on a d-polytope P with respect to a generic functional f. The third author has conjectured that this graph is always $(d-1)$-connected. We resolve this conjecture positively for simple polytopes and show that the graph is 2-connected for any d-polytope with $d \geq 3$. However, we disprove the conjecture in general by exhibiting counterexamples for each $d \geq 4$ in which the graph has a vertex of degree two.

We also re-examine the Baues problem for cellular strings on polytopes, solved by Billera, Kapranov and Sturmfels. Our analysis shows that their positive result is a direct consequence of shellability of polytopes and is therefore less related to convexity than is at first apparent.

1. INTRODUCTION

Let P be a d-dimensional polytope in $\mathbb{R}^d$ and f be a linear functional on $\mathbb{R}^d$ which is generic with respect to P, in the sense that f is nonconstant on every edge of P. An *f-monotone path* γ on P is a sequence of vertices $(v_0, v_1, \ldots, v_m)$ of P such that v_0 and v_m are the unique vertices at which f achieves its minimum and maximum values on P, respectively, v_{i-1} and v_i lie on an edge of P for each i and $f(v_0) < f(v_1) < \cdots < f(v_m)$. Monotone paths have been studied in the context of the Hirsch Conjecture [13, §3.3] [8], and have recently appeared in a directed version of Steinitz's Theorem [9].

The set of all f-monotone paths on P forms the vertex set for a natural graph structure, which we now describe. Each 2-dimensional face F of P is a polytope in its own right, in fact a polygon, and has exactly two f-monotone paths, say γ_F and γ'_F. We say that two f-monotone paths γ and γ' on P differ by a *polygon flip across* F if they agree on vertices not lying on F but differ in that γ restricted to the face F follows the path γ_F, while γ' restricted to F follows γ'_F. The *graph $G(P, f)$ of f-monotone paths on P* is the graph whose vertices are the f-monotone paths on P and whose edges join pairs of f-monotone paths which differ by a polygon flip across some 2-face of P. An example is shown in Figure 1.

The question of connectivity of the graph $G(P, f)$ arises naturally in the work of Billera and Sturmfels [3] and Billera, Kapranov and Sturmfels [2] on the *generalized Baues problem*; see [10] for a survey of this subject. The vertices and edges in $G(P, f)$ index the elements in the bottom two ranks of the *poset of cellular strings on P*. This poset gives rise naturally to a topological space (see Section 5), which was shown to have the homotopy type of a $(d-2)$-dimensional sphere in [2]. It can be deduced from this (see [10, Section 3, p. 20]) that the graph $G(P, f)$ is connected. Furthermore, there is a subset of the vertex set of $G(P, f)$ which is geometrically distinguished, namely the subset of f-monotone paths which are *coherent* (see [3,

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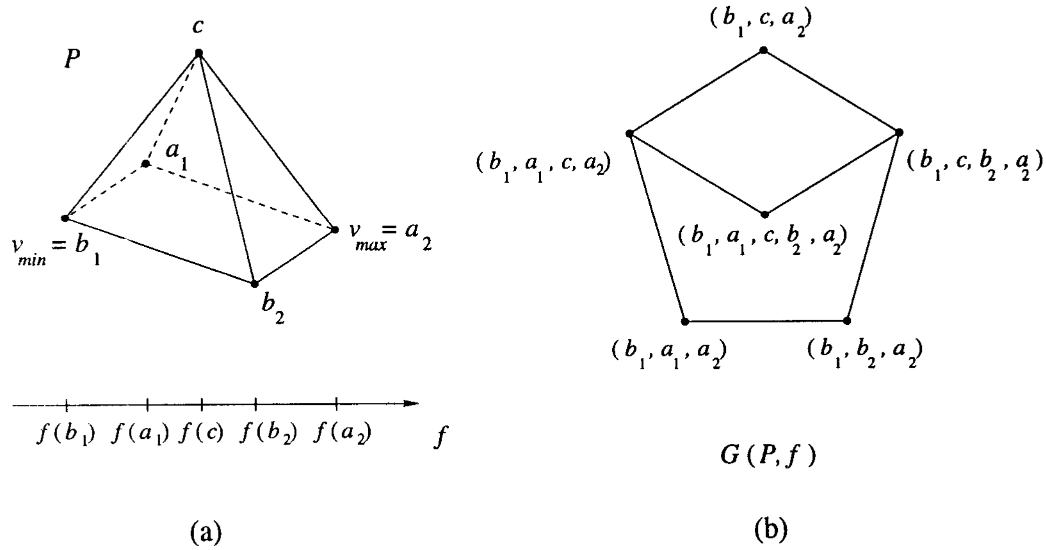

FIGURE 1. (a) A 3-dimensional polytope P and a generic linear functional f. (b) The graph $G(P, f)$ of f-monotone paths on P.

p. 544], [2, p. 552], [10, p. 12] for definitions). Under mild genericity assumptions on P and f, the induced subgraph of $G(P, f)$ on these vertices is the 1-skeleton of a $(d-1)$-dimensional polytope, called the *monotone path polytope*, whose existence is a special case of the general theory of *fiber polytopes* [3]. Recall that Balinski's Theorem [13, §3.5] states that the 1-skeleton of a d-dimensional polytope is d-connected, meaning that any subgraph obtained by removing a set of at most $d-1$ vertices and their incident edges is connected and contains at least two vertices. As a consequence, the subgraph of coherent f-monotone paths on P is $(d-1)$-connected.

The preceding results led the third author to conjecture [10, Conjecture 15] that the entire graph $G(P, f)$ is always $(d-1)$-connected. In Section 3 we exhibit counterexamples which disprove this conjecture for $d \geq 4$. Specifically, for each $d \geq 3$ we exhibit a d-polytope P, linear functional f and f-monotone path γ on P such that γ has degree two in the graph $G(P, f)$.

On the other hand, Sections 2 and 4 contain proofs of the following positive results. Recall that a d-polytope P is *simple* if each vertex is incident to exactly d edges of P.

Theorem 1.1. *If P is a simple d-polytope and f is a generic linear functional on P then the graph $G(P, f)$ is $(d-1)$-connected.*

Theorem 1.2. *If P is any d-polytope with $d \geq 3$ and f is a generic linear functional on P then the graph $G(P, f)$ is 2-connected.*

In Section 5 we re-examine the role played by convexity in the positive answer to the Baues problem for cellular strings, given by Billera, Kapranov and Sturmfels [2]. Specifically, we show that for every shelling of a regular CW-sphere X, there is an associated Baues problem for cellular strings on the sphere X^* which is the polar dual of X and that the shelling of X leads to a positive answer to this problem. This gives a common generalization for the results of [2] and of Björner [4].

2. $G(P, f)$ IS $(d-1)$-CONNECTED FOR SIMPLE d-POLYTOPES

The goal of this section is to prove Theorem 1.1, namely that for a generic functional f on a simple d-polytope P, the graph $G(P, f)$ is $(d-1)$-connected.

We prove this in a slightly more general form. Let P be a finite, 2-dimensional regular CW-complex (see [6, §4.7], [5] for definitions and background on regular CW-complexes). Let G be the 1-skeleton of P and $\mathcal{O}$ be an acyclic orientation of G. In the motivating special case, P is the 2-skeleton of a simple d-polytope and $\mathcal{O}$ is the acyclic orientation induced by a generic linear functional f. Motivated by this special case, we further assume the following:

(i) the entire graph G has a unique source v_{min} and sink v_{max} with respect to $\mathcal{O}$, as does its restriction to every 2-face of P (such acyclic orientations $\mathcal{O}$ are called *facial* in Section 5),
(ii) the degree of v_{min} in G is at least d,
(iii) any two faces of P intersect in a unique common face of each; in particular, the 1-skeleton of P is a graph with no multiple edges and different 2-faces can share at most one edge, and
(iv) any two directed edges of G having common initial vertex u span a 2-face whose 1-skeleton has source u with respect to $\mathcal{O}$.

We refer to the 2-faces of P as the *polygons* of P. Let $G(P, \mathcal{O})$ be the graph on the vertex set of all directed paths in G from v_{min} to v_{max}, with adjacency defined by the flips across the polygons of P. The following theorem is the main result of this section.

Theorem 2.1. *If P and $\mathcal{O}$ are as before then the graph $G(P, \mathcal{O})$ is $(d-1)$-connected.*

Given two vertices γ_1, γ_2 of $G(P, \mathcal{O})$, we first construct a path $\gamma_1 * \gamma_2$ in $G(P, \mathcal{O})$ from γ_1 to γ_2 as follows. Let u be the first vertex of either path from which γ_1 and γ_2 leave through distinct edges e_1 and e_2, respectively. We proceed by induction on the position of u with respect to the partial order on the vertices of G induced by $\mathcal{O}$. Let F be the polygon of P spanned by e_1 and e_2, v be the unique sink of the boundary of F and p be a directed path in G from v to v_{max}. We choose p as the part of γ_1 from v to v_{max} if v is a vertex of γ_1. For $i = 1, 2$ we let γ_i^F be the path in G which follows γ_1 and γ_2 up to u, then follows the boundary of F through e_i up to v, and finally follows the path p up to v_{max}. The paths γ_1^F and γ_2^F differ by a flip across F. We define $\gamma_1 * \gamma_2$ to be the path in $G(P, \mathcal{O})$ whose successive vertices are the ones of $\gamma_1 * \gamma_1^F$, followed by those of $\gamma_2^F * \gamma_2$. Note that each vertex of $\gamma_1 * \gamma_2$ is a path in G whose initial edge is that of γ_1 or γ_2.

Lemma 2.2. *If γ, γ_1 and γ_1' are vertices of $G(P, \mathcal{O})$ with pairwise distinct initial edges then the paths $\gamma * \gamma_1$ and $\gamma * \gamma_1'$ in $G(P, \mathcal{O})$ are vertex-disjoint, except for their initial vertex γ.*

Proof. Let e_1, e_1' be the initial edges of γ_1, γ_1' and γ_2, γ_2' be vertices of the paths $\gamma * \gamma_1$ and $\gamma * \gamma_1'$ other than γ, respectively. To prove that $\gamma_2 \neq \gamma_2'$ we show that $e_2 \neq e_2'$, where e_2 and e_2' are the first edges of γ_2, γ_2' which are not edges of γ, respectively. Indeed, by construction of $\gamma * \gamma_1$, there is an alternating sequence $(\epsilon_0, F_1, \epsilon_1, F_2, \dots, F_r, \epsilon_r)$ of edges ϵ_i of G through vertices v_i of γ and polygons F_i of P such that $\epsilon_0 = e_1$, $\epsilon_r = e_2$ and that F_i contains $\epsilon_{i-1}, \epsilon_i$, as well as the part of γ between v_{i-1} and v_i. Similarly, there is an analogous sequence with initial edge e_1' and terminal edge e_2'. By our assumption (iii) on P, the two sequences are uniquely determined by γ and e_2, or γ and e'_2, respectively. Since $e_1 \neq e'_1$, we should also have $e_2 \neq e'_2$. □

We now prove Theorem 2.1.

Proof of Theorem 2.1. Let Γ be a subset of the vertex set of $G(P, \mathcal{O})$ with at most $d-2$ elements. We will show that $G(P, \mathcal{O}) - \Gamma$ is connected. Let E be the set of initial edges of the elements of Γ and let Δ be the set of vertices of $G(P, \mathcal{O})$ (that is, paths in G) with initial edge not in E. Note that the induced subgraph of $G(P, \mathcal{O})$ on Δ is connected since if γ_1, γ_2 are in Δ, so is every vertex of the path $\gamma_1 * \gamma_2$.

It thus suffices to show that any path γ not in Γ with initial edge $e \in E$ can be connected in $G(P, \mathcal{O}) - \Gamma$ to some element of Δ. Let k be the number of elements of Γ with initial edge e. Since Γ has at most $d-2$ elements and because of the assumption (ii) on P, there are at least $k+1$ edges of G through v_{min} not in E. Hence we may choose elements γ_i of Δ, for $1 \leq i \leq k+1$, with pairwise distinct initial edges. By Lemma 2.2, the paths $\gamma * \gamma_i$ are $k+1$ vertex-disjoint paths in $G(P, \mathcal{O})$, except for their initial vertex, which connect γ to paths in Δ. At least one of them avoids all k elements of Γ with initial edge e, and hence connects γ to Δ in $G(P, \mathcal{O}) - \Gamma$, as desired. □

The following corollary is an immediate consequence of Theorem 2.1 and generalizes Theorem 1.1.

Corollary 2.3. *If P is a simple polytope, f is a generic linear functional on P and v is a vertex of P with f-outdegree j then the graph of partial f-monotone paths on P from v to v_{max} and polygon flips among them is $(j-1)$-connected.*

3. A monotone path on a d-polytope with only 2 flips

The goal of this section is to prove the following theorem.

Theorem 3.1. *For each $d \geq 3$ there is a d-polytope P, linear functional f and f-monotone path γ on P such that γ has degree two in the graph $G(P, f)$.*

Proof. To construct P, start with a $(d-2)$-simplex in $\mathbb{R}^{d-2}$. Form the $(d-1)$-polytope which is the prism over this simplex, that is, its Cartesian product with the line segment $[-1, 1]$, and then let P be the d-polytope which is the pyramid over this prism.

To be somewhat more explicit, choose real numbers $t_1 < t_2 < \cdots < t_{d-1}$ symmetrically about 0, i.e. so that $t_{d-i} = -t_i$. Then for $i = 1, \ldots, d-1$ let

$$\begin{array}{rcl} a_i & = & (t_i,\ t_i^2,\ \ldots,\ t_i^{d-2},\ \ 1,\ 0), \\ b_i & = & (t_i,\ t_i^2,\ \ldots,\ t_i^{d-2},\ -1,\ 0), \\ c & = & (0,\ 0,\ \ldots,\ 0,\ \ 0,\ 1) \end{array}$$

and let P be the convex hull of $\{a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}, c\}$.

Our requirement on the functional f is that it is generic and that it orders the vertices so that

$$f(b_1) < f(b_2), \ldots, f(b_{d-2}), f(a_1) < f(c) < f(b_{d-1}), f(a_2), \ldots, f(a_{d-2}) < f(a_{d-1}).$$

One can achieve this by choosing an f of the form $f(\mathbf{x}) = \alpha x_1 + \beta x_{d-1}$ for some real constants α, β. To see that such a choice is possible, note that the projection of the vertices a_i, b_i, c onto the (x_1, x_{d-1})-plane will look as in Figure 2 and one need only choose the constants α, β so that the level set $f = 0$ is as the line depicted

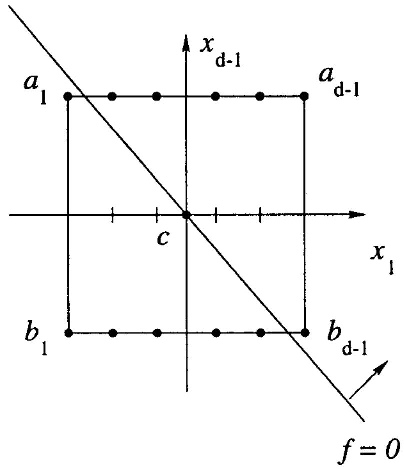

FIGURE 2. The functional $f(\mathbf{x}) = \alpha x_1 + \beta x_{d-1}$ should be chosen so that the level set $f = 0$ looks as shown.

there and f increases to the northeast. For example, the polytope and functional shown in Figure 1(a) are the case $d = 3$ of our construction.

With P and f as described, the f-monotone path γ is the sequence of vertices $(b_1, a_1, c, b_{d-1}, a_{d-1})$. It is straightforward to check that the only two polygon flips applicable to γ are across the triangular 2-face having vertices b_1, a_1, c and its symmetric partner having vertices c, b_{d-1}, a_{d-1}. In checking that these are the only flips, one uses the fact that the 2-faces of P can be listed as

$$\begin{aligned}
\text{triangles:} & \{a_i, a_j, a_k\}_{1 \le i < j < k \le d-1} \\
& \{b_i, b_j, b_k\}_{1 \le i < j < k \le d-1} \\
& \{c, a_i, a_j\}_{1 \le i < j \le d-1} \\
& \{c, b_i, b_j\}_{1 \le i < j \le d-1} \\
& \{c, a_i, b_i\}_{1 \le i \le d-1}, \\
\text{quadrangles:} & \{a_i, a_j, b_i, b_j\}_{1 \le i < j \le d-1}.
\end{aligned}$$

This completes the proof. □

We conclude this section with a series of remarks about the counterexample just constructed.

Remark 3.2. The fact that our counterexample is a d-polytope with $2d-1$ vertices raises the following question.

Question 3.3. *For a given dimension d, what is the minimum number of vertices required to construct a d-polytope and generic functional f having an f-monotone path with fewer than $d-1$ flips? What about two flips? Is either of these two numbers less than $2d-1$?*

For $d = 4$, the counterexample of Theorem 3.1 achieves the minimum number of vertices possible, which is 7. Indeed, a 4-polytope with five vertices is a 4-simplex and all its f-monotone paths have exactly three flips, while if P is a 4-polytope with six vertices, then a quick enumeration of *Gale diagrams* (see, e.g., [13, §6.5]) shows that there are only four possibilities for the combinatorial type of P. For each of these four types, one can check by computer that no matter how f orders the vertices, every f-monotone path on P has at least three flips.

The counterexample of Theorem 3.1 for $d = 4$ is tight in another sense, in connection to Theorem 1.1. It has a single vertex (the apex c) of degree 6 and the rest of its vertices simple, i.e. of degree 4.

Remark 3.4. In light of Theorem 1.1, it would be interesting to investigate the question of connectivity of $G(P, f)$ within other natural classes of polytopes.

For example, one might ask whether there exist monotone paths on *simplicial* polytopes having few polygon flips. Starting with the counterexample in Theorem 3.1 for $d = 4$, one can produce a simplicial counterexample by *pulling* (see [6, p. 410]) the vertices b_1 and a_3 of P in either order, and using the same functional f and f-monotone path γ. We do not know if this technique can be used to produce simplicial counterexamples in higher dimensions.

A class of particular interest with regard to monotone paths is that of *zonotopes*; see [6, §2.2]. It is possible to construct a 4-zonotope Z with 6 zones and a vertex v such that the vertex figure of Z at v is a prism over a 2-simplex, as in the $d = 4$ case of our polytopal counterexample P. Furthermore, it is also possible to choose a functional f and an f-monotone path on Z having only two flips. However, the analogous construction for $d = 5$ fails to give any paths with fewer than four flips. These observations suggest the following questions.

Question 3.5. *If Z is a d-zonotope and $d \geq 5$, is $G(Z, f)$ always $(d-1)$-connected? If Z is a d-zonotope with at least 7 zones, is $G(Z, f)$ always $(d-1)$-connected?*

4. Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2, which states that the graph $G(P, f)$ is 2-connected for $d \geq 3$. We use ideas analogous to those in the first proof of homotopy-sphericity of the poset of cellular strings on P, given in [2, Theorem 1.2]. We construct the graph $G(P, f)$ as a certain inverse limit in the category of graphs and simplicial maps. This construction is an iteration of a simpler "pullback" construction, which will be shown to behave sufficiently well with respect to connectivity and 2-connectivity.

Let v_{min} and v_{max} be the vertices of P at which f achieves its minimum and maximum on P, respectively, and

$$f(v_{min}) = c_0 < c_1 < \cdots < c_{m-1} < c_m = f(v_{max})$$

be the set of values of f taken at the vertices of P. The fiber $P_i := f^{-1}(c_i)$ is a $(d-1)$-polytope for $1 \leq i \leq m-1$. We denote by G_i the 1-skeleton of P_i. Similarly, for each $0 \leq i \leq m-1$ we denote by $G_{i,i+1}$ the 1-skeleton of the fiber $P_{i,i+1} := f^{-1}(c)$ for some value c with $c_i < c < c_{i+1}$. The facial structure of $P_{i,i+1}$, and hence the graph $G_{i,i+1}$, are independent of the choice of c; see Figure 3.

Since every vertex or edge in $P_{i,i+1}$ degenerates to some vertex or edge of P_i as c approaches c_i, there is a map $\beta_i : G_{i,i+1} \to G_i$ for each i. Similarly, there is a map $\alpha_i : G_{i-1,i} \to G_i$. Thus one obtains a diagram of graphs and maps as follows:

$$(1) \qquad G_{0,1} \xrightarrow{\alpha_1} G_1 \xleftarrow{\beta_1} G_{1,2} \xrightarrow{\alpha_2} G_2 \xleftarrow{\beta_2} \cdots \xrightarrow{\alpha_{m-1}} G_{m-1} \xleftarrow{\beta_{m-1}} G_{m,m-1}.$$

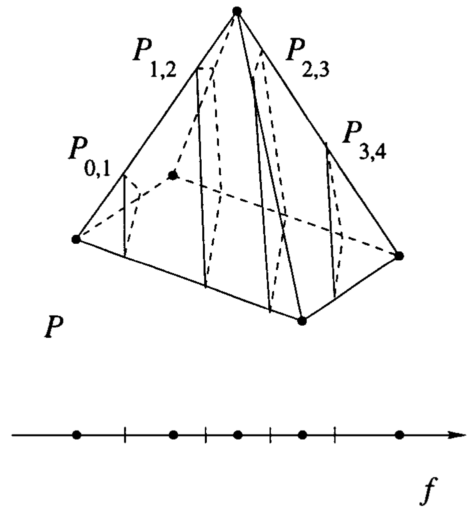

FIGURE 3. The fibers $P_{i,i+1}$ over intermediate values of f.

If we consider graphs as 1-dimensional simplicial complexes, then the maps in (1) are all *simplicial*, meaning that they send vertices to vertices and edges to either edges or vertices. We define the *inverse limit* G of a diagram of graphs and simplicial maps, such as (1), to be a graph whose vertices and edges are certain ordered m-tuples

$$F = (F_{0,1}, F_{1,2}, \ldots, F_{m-1,m}),$$

where $F_{i,i+1}$ is a vertex or an edge of $G_{i,i+1}$ for each i, $F_{i-1,i}$ and $F_{i,i+1}$ have the same image in G_i under the maps α_i and β_i, respectively, and moreover, this image is an edge of G_i if both $F_{i-1,i}$ and $F_{i,i+1}$ are edges. We call F a *vertex* of G if each $F_{i,i+1}$ is a vertex of the graph $G_{i,i+1}$ and an *edge* of G if the set of indices i for which $F_{i,i+1}$ is an edge of $G_{i,i+1}$ forms a sequence of consecutive integers. The vertex F of G is defined to be *incident* to the edge $F' = (F'_{0,1}, F'_{1,2}, \ldots, F'_{m-1,m})$ if $F_{i,i+1} \subseteq F'_{i,i+1}$ for each i. One can check that each edge of G has exactly two vertices incident to it, so that G indeed defines a graph with no loops or multiple edges, that is, a 1-dimensional simplicial complex.

Each f-monotone path γ on P corresponds to a vertex in the inverse limit G, namely the vertex $F = (F_{0,1}, F_{1,2}, \ldots, F_{m-1,m})$ for which $F_{i,i+1}$ is the intersection of the fiber $P_{i,i+1}$ of f with the union of the edges of P which γ traverses. This correspondence is in fact a bijection. Similarly, a polygon flip between two f-monotone paths γ_1, γ_2 corresponds to an edge of G, namely the edge $(F'_{0,1}, F'_{1,2}, \ldots, F'_{m-1,m})$ for which $F_{i,i+1}$ is the intersection of the fiber $P_{i,i+1}$ of f with the union of the edges of P which the paths traverse and the 2-face where the flip occurs. Hence we can deduce the following proposition.

Proposition 4.1. (cf. [2, Lemma 1.3]) *If diagram* (1) *comes from the family of graphs* $\{G_i, G_{i,i+1}\}$ *of fibers of* f *on* P, *then the inverse limit* G *is isomorphic to the graph* $G(P, f)$ *of* f*-monotone paths on* P. □

A special case of the inverse limit construction, called the pullback, occurs when $m = 2$. In this case we have a diagram of simplicial maps of graphs $A \overset{\alpha}{\to} C \overset{\beta}{\leftarrow} B$ whose inverse limit G has the following simpler description. Vertices of G are ordered pairs (a, b), where a, b are vertices of A, B, respectively, and $\alpha(a) = \beta(b)$. The vertices (a, b) and (a', b') are connected by an edge in G if either

- $a = a'$ and $\{b, b'\}$ is an edge of B or
- $b = b'$ and $\{a, a'\}$ is an edge of A or
- $\{a, a'\}$ and $\{b, b'\}$ are edges of A and B, respectively, which both map homeomorphically onto the same edge of C.

In this case, we have the following commutative diagram of simplicial maps

$$\begin{array}{ccc} G & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & {\scriptstyle \beta}\downarrow \\ A & \xrightarrow{\alpha} & C \end{array} \tag{2}$$

where $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$ are the usual projections. We call such a diagram a *pullback diagram* and G the *pullback* of α, β [1]. For future use, we denote by $\pi : G \to C$ the composite map $\alpha \circ \pi_A = \beta \circ \pi_B$.

The pullback is a simpler construction than that of the general inverse limit of diagram (1). However, the inverse limit can be recovered by iterating pullbacks as follows. Let H_i denote the inverse limit of the subdiagram of (1) which consists of all graphs and maps weakly to the left of $G_{i,i+1}$, so that $H_0 = G_{0,1}$, H_1 is the pullback of the diagram $G_{0,1} \to G_1 \leftarrow G_{1,2}$ and H_{m-1} is the full inverse limit G. There is a natural map $H_{i-1} \to G_i$ obtained by first projecting a tuple $(F_{0,1}, \ldots, F_{i-1,i})$ in H_{i-1} onto its last entry $F_{i-1,i}$, which lies in $G_{i-1,i}$, and then applying the map $G_{i-1,i} \to G_i$. The following proposition is straightforward to verify.

Proposition 4.2. *We have a pullback diagram*

$$\begin{array}{ccc} H_i & \longrightarrow & G_{i,i+1} \\ \downarrow & & {\scriptstyle \beta_i}\downarrow \\ H_{i-1} & \xrightarrow{\alpha} & G_i. \end{array} \tag{3}$$

□

If the graphs G_i and $G_{i,i+1}$ are the 1-skeleta of the corresponding fibers of f on the polytope P, it is natural to think of H_i as the graph of partial f-monotone paths ending in fiber $P_{i,i+1}$.

Next we collect for future use a few propositions about pullbacks, whose proofs are completely straightforward. We use the following convention throughout: if $\beta : B \to C$ is a simplicial map of graphs, then for any vertex c in C, $\beta^{-1}(c)$ refers to the vertex-induced subgraph of B on the set of vertices of B which are mapped to c by β. The simplicial map β is *surjective* if any vertex or edge in C is the image of some vertex or edge, respectively, of B. The special case of a pullback in which C is a single vertex is called the *Cartesian product* $A \times B$ (see, e.g., [12, p. 175]).

Proposition 4.3. *If* (2) *is a pullback diagram then:*

[1]Note that what we are calling inverse limits and pullbacks do not satisfy the usual universal existence and uniqueness properties in the category of graphs and simplicial maps. For example, if A, B are both graphs consisting of a single edge and map to a graph C which is a single vertex then the pullback, as we have defined it, is a cycle of length 4, while the categorical pullback is a complete graph on 4 vertices.

(i) *the subgraph* $\pi_B^{-1}(b)$ *of* G *is isomorphic to the subgraph* $\alpha^{-1}(\beta(b))$ *of* A *for any vertex* b *of* B,

(ii) *the subgraph* $\pi^{-1}(c)$ *of* G *is isomorphic to the Cartesian product* $\alpha^{-1}(c) \times \beta^{-1}(c)$ *for any vertex* c *of* C *and*

(iii) *surjectivity of* α *implies surjectivity of* π_B. □

Proposition 4.4. *If* (2) *is a pullback diagram and*

(i) B *is nonempty and connected,*

(ii) α *is surjective and*

(iii) *the fiber* $\alpha^{-1}(c)$ *is connected for every vertex* c *in* C

then G *is nonempty and connected.* □

Proposition 4.5. *The Cartesian product* $G \times H$ *of* 1*-connected graphs* G, H *is* 2*-connected.* □

Remark 4.6. Although we will not need this fact here, the previous proposition generalizes to the statement that if G is k-connected and H is l-connected, then $G \times H$ is $(k + l)$-connected.

One property of the polytopal degeneration maps in diagram (1) which will be used repeatedly is the following.

Proposition 4.7. (cf. [2, Lemma 1.4]) *The maps*

$$\alpha_i : G_{i-1,i} \to G_i$$
$$\beta_i : G_{i,i+1} \to G_i$$

are surjective. Moreover, for any vertex v *of* G_i *the fibers* $\alpha_i^{-1}(v)$ *and* $\beta_i^{-1}(v)$ *are connected.*

Proof. The normal fan of the polytope $P_{i-1,i}$ refines the normal fan of P_i. Furthermore, the adjacency graph for the full-dimensional cones in the normal fan of $P_{i-1,i}$ is exactly $G_{i-1,i}$. For each vertex v of G_i, the subgraph $\alpha_i^{-1}(v)$ is the adjacency graph for the subset of cones which refine the unique cone corresponding to v in the normal fan of P_i. The assertions of the proposition for α_i follow from this description. The case of the map β_i is analogous. □

Theorem 1.2 will be deduced from the following proposition.

Proposition 4.8. *If* (2) *is a pullback diagram and*

(i) A, B *are both* 2*-connected,*

(ii) α, β *are both surjective and*

(iii) *the fibers* $\alpha^{-1}(c), \beta^{-1}(c)$ *are connected for every vertex* c *in* C

then the maps π_A, π_B *are both surjective and* G *is* 2*-connected.*

Proof. Surjectivity of π_A and π_B is immediate from Proposition 4.3 (iii). Note also that since π_B is surjective and B has at least three vertices, by virtue of its 2-connectivity, G must have at least three vertices. Hence removing at most one vertex from G leaves at least 2 vertices. It remains to show that if u, u' and v are distinct vertices of G then there exists a path from u to u' in $G - v$. We distinguish three cases.

Case 1: $\pi_A(v) \neq \pi_A(u), \pi_A(u')$. By 2-connectivity of A, there exists a path γ from $\pi_A(u)$ to $\pi_A(u')$ in $A - \pi_A(v)$. Since π_A is surjective, we can lift each edge of γ to an edge in G. By Proposition 4.3 (i), the fibers of π_A are connected since so are the fibers of β. Hence we can "sew" together the previously lifted edges to form a path $\tilde{\gamma}$ from u to u' in G, whose image under π_A is γ. This path $\tilde{\gamma}$ must lie in $G - v$, since γ lies in $A - \pi_A(v)$.

Case 2: $\pi_B(v) \neq \pi_B(u), \pi_B(u')$. This is symmetric to the previous case, interchanging A and B.

Case 3: $\pi_A(v) \in \{\pi_A(u), \pi_A(u')\}$ and $\pi_B(v) \in \{\pi_B(u), \pi_B(u')\}$. Since u, u' and v are distinct and any vertex of G is determined by its two projections under π_A and π_B, we may assume that

$$\pi_A(v) = \pi_A(u),$$
$$\pi_B(v) = \pi_B(u').$$

It follows that $\pi(v) = \pi(u) = \pi(u') =: c$, so all three vertices u, u' and v lie in $\pi^{-1}(c)$. By Proposition 4.3 (ii), $\pi^{-1}(c)$ is isomorphic to the Cartesian product $\alpha^{-1}(c) \times \beta^{-1}(c)$. Since $\alpha^{-1}(c)$ and $\beta^{-1}(c)$ are connected by hypothesis, they are also 1-connected as they have at least two vertices (e.g. $\alpha^{-1}(c)$ contains $\pi_A(v) = \pi_A(u)$ and $\pi_A(u')$). Hence $\pi^{-1}(c)$ is 2-connected by Proposition 4.5, so there exists a path from u to u' in $\pi^{-1}(c)$ which avoids v. □

In order to apply Proposition 4.8 to the pullback diagram of Proposition 4.2, we need to identify the fibers of the map $H_{i-1} \xrightarrow{\alpha} G_i$. For a fixed vertex v of G_i, the fiber $\alpha^{-1}(v)$ is, by definition, the induced subgraph H_v of H_{i-1} on the vertex set of partial f-monotone paths which end in fiber $P_{i-1,i}$ and whose last entries map to v under $\alpha_i : G_{i-1,i} \to G_i$. One should think of this subgraph H_v of H_{i-1} as the graph of partial f-monotone paths from v_{min} to v.

We will need to show that H_v is connected, so we construct it as an iteration of pullbacks. For $j < i$, start with the inverse limit of the subdiagram of (1) containing $G_{j,j+1}$, $G_{i-1,i}$ and all graphs and maps lying between them. Then take the vertex-induced subgraph $H_{j,v}$ of this graph on the vertices whose $G_{i-1,i}$-coordinate maps to v under $\alpha_i : G_{i-1,i} \to G_i$. Note that $H_{i-1,v}$ is the fiber $\alpha_i^{-1}(v)$. One might think of the graph $H_{j,v}$ as the graph of partial f-monotone paths which start in $P_{j,j+1}$ and end in v. The following proposition should be clear.

Proposition 4.9. *We have a pullback diagram*

$$\begin{array}{ccc} H_{j-1,v} & \longrightarrow & H_{j,v} \\ \downarrow & & \downarrow{\scriptstyle\beta} \\ G_{j-1,j} & \xrightarrow{\alpha_j} & G_j \end{array} \tag{4}$$

and $H_{0,v} = H_v$.

Corollary 4.10. *For any vertex v in G_i, the graph H_v of partial f-monotone paths ending at v is nonempty and connected.*

Proof. We wish to apply Proposition 4.4 to the pullback diagram (4), in order to show $H_{j,v}$ is nonempty and connected by descending induction on j for $j < i$. At the base of the induction, we know that $H_{i-1,v}$ is nonempty and connected by Proposition 4.7. For the inductive step, we must check that the hypotheses of Proposition 4.4 are satisfied. Indeed, we know that $H_{j,v}$ is nonempty and connected, by induction, and that α_j is surjective with connected fibers, by Proposition 4.7. □

We can now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We wish to apply Proposition 4.8 to the pullback diagram (3), in order to show that H_i is 2-connected by induction on i. This will suffice, since $G(P,f) = H_{m-1}$. Note that, at the base of the induction, $H_0 = G_{0,1}$ is 2-connected by Balinski's Theorem, since it is the graph of a $(d-1)$-polytope and $d \geq 3$.

To apply Proposition 4.8, we must check that H_{i-1} and $G_{i,i+1}$ are both 2-connected, that the maps α, β_i are surjective and that their fibers are connected. The graph H_{i-1} is 2-connected by the inductive hypothesis. The graph $G_{i,i+1}$ is 2-connected by Balinski's Theorem. Surjectivity of β_i and connectivity of its fibers follow from Proposition 4.7. Surjectivity of α follows from the fact that it is the composition of two maps

$$H_{i-1} \longrightarrow G_{i-1,i} \xrightarrow{\alpha_i} G_i,$$

the first of which is surjective by Proposition 4.3 (iii) applied inductively to diagram (3), and the second being surjective by Proposition 4.7. For any vertex v in G_i, the fiber $\alpha^{-1}(v)$ is the graph H_v which was shown to be connected in Corollary 4.10. □

5. Cellular strings and shellable spheres

In this section we consider a Baues problem associated with cellular strings on the duals of shellable CW-spheres and show that shellability suffices to give a positive answer to this problem. This generalizes the main results of [2, 4]. Theorem 5.3, which is the main result of this section, shows that convexity is not absolutely essential in this special case of the Baues problem, as was hinted by the context and methods of Björner [4], and that it plays a crucial role only in so far as it implies shellability.

We assume some familiarity with the notions of a regular CW-complex, shellability and a recursive (co)atom ordering. An excellent reference for this material is [6, §4.7]; see also [5].

We begin by defining a general context in which monotone paths and cellular strings make sense. Let P be a finite, graded poset with minimum and maximum elements $\hat{0}_P$ and $\hat{1}_P$, and rank function r. Our motivation comes from the special case in which P is the (augmented) face poset of a regular CW-sphere X, so that $\hat{0}_P$ and $\hat{1}_P$ correspond to the "empty face" and the "interior" of the sphere, respectively, and $r(x) - 1$ is the dimension of x for each cell x in X.

We say that P *has a 1-skeleton* if every lower interval $[\hat{0}, x]$ with $r(x) = 2$ has exactly four elements, namely $\hat{0}$, x and two atoms a, a'. In this situation we say that the *1-skeleton* or *graph* $G(P)$ is the graph (with no loops but multiple edges allowed) which has the set of atoms of P as its vertex set and an edge with endpoints a, a' for each element x with $r(x) = 2$, covering a and a'. If P has a 1-skeleton, then any linear ordering of the atoms of P gives rise to an acyclic orientation $\mathcal{O}$ of the graph $G(P)$ which orients the edge corresponding to x from a toward a', if a comes earlier than a' in the ordering.

We call an acyclic orientation $\mathcal{O}$ of $G(P)$ *facial* if for every x in P with $r(x) \geq 2$, the restriction of $\mathcal{O}$ to the vertex-induced subgraph of $G(P)$ on the atoms of $[\hat{0}, x]$ has a unique source $a_{min}(x)$ and a unique sink $a_{max}(x)$. We fix the notation $a_{min} = a_{min}(\hat{1}_P)$ and $a_{max} = a_{max}(\hat{1}_P)$. Given a facial acyclic orientation $\mathcal{O}$ of

$G(P)$, a *cellular string on* P (with respect to $\mathcal{O}$) is a sequence $\mathbf{x} = (x_1, \ldots, x_r)$ of elements of P with $r(x_i) \geq 2$, such that $a_{min}(x_1) = a_{min}$, $a_{max}(x_r) = a_{max}$ and for each i we have $a_{max}(x_i) = a_{min}(x_{i+1})$. We partially order the set of cellular strings on P by defining

$$\mathbf{x} = (x_1, \ldots, x_r) \leq \mathbf{y} = (y_1, \ldots, y_s)$$

if for each $i \leq r$ there exists some $j \leq s$ with $x_i \leq_P y_j$. We denote by $\omega(P, \mathcal{O})$ this partial order on the set of cellular strings on P with respect to $\mathcal{O}$. Note that $\omega(P, \mathcal{O})$ has a maximum element $\hat{1}_\omega$, given by the cellular string $(\hat{1}_P)$.

The next proposition records a few properties of cellular strings. All of them generalize properties which are easily verified in the special case of cellular strings on a polytope with respect to a generic linear functional (see [2, §1]).

Proposition 5.1. *Given cellular strings* $\mathbf{x} = (x_1, \ldots, x_r)$ *and* $\mathbf{y} = (y_1, \ldots, y_s)$ *on* P *with* $\mathbf{x} \leq \mathbf{y}$, *we have:*

(i) $(\hat{0}, x_i] \cap (\hat{0}, x_{i+1}] = \{a_{max}(x_i)\} = \{a_{min}(x_{i+1})\}$ *for each* $1 \leq i \leq r-1$,
(ii) $(\hat{0}, x_i] \cap (\hat{0}, x_j] = \emptyset$ *if* $|i - j| \geq 2$,
(iii) *for each* $1 \leq i \leq r$ *the index* $j(i)$ *satisfying* $x_i \leq y_{j(i)}$ *is unique and the function* $i \mapsto j(i)$ *is increasing, that is,* $i \leq i'$ *implies* $j(i) \leq j(i')$,
(iv) *for each* $j \leq s$, *there exist unique indices* i, i' *such that* $a_{min}(x_i) = a_{min}(y_j)$ *and* $a_{max}(x_{i'}) = a_{max}(y_j)$.

Proof. For (i), note that if the intersection is larger then it contains at least one atom $a \neq a_{max}(x_i) = a_{min}(x_{i+1})$. The atom a has a directed path in $G(P) \cap [\hat{0}, x_i]$ to the unique sink $a_{max}(x_i)$, and there is also a directed path in $G(P) \cap [\hat{0}, x_{i+1}]$ from the unique source $a_{min}(x_{i+1})$ to a. Since $a_{max}(x_i) = a_{min}(x_{i+1})$, this contradicts acyclicity of $\mathcal{O}$.

The remaining assertions follow by similar arguments. □

One way in which facial acyclic orientations arise is from a *shelling order* on the maximal faces of a regular CW-sphere X. Recall [6, Lemma 4.7.18, Theorem 4.7.24] that a shelling order of the maximal faces of X is equivalent to a *recursive coatom ordering* on its face poset P. This gives rise to a *recursive atom ordering* on the opposite poset P^{op}, which is the face poset of the polar dual CW-sphere X^* [6, Proposition 4.7.26]. The poset P^{op} has a 1-skeleton because it is the face poset of a sphere. The recursive coatom ordering on P induces an acyclic orientation $\mathcal{O}$ of $G(P^{op})$.

Proposition 5.2. *The acyclic orientation* $\mathcal{O}$ *of* $G(P^{op})$ *induced from a shelling order of a regular* CW*-sphere with face poset* P *is facial.*

Proof. We must show that for any $x \in P^{op}$ with $r(x) \geq 2$, the restriction of the orientation $\mathcal{O}$ to the atoms of $[\hat{0}, x]_{P^{op}}$ has a unique source and sink. By definition of a recursive atom ordering, the recursive atom ordering on P restricts to one on its lower interval $[\hat{0}, x]_{P^{op}}$ and this interval is also opposite to the face poset of a (shellable) regular CW-sphere (see [6, Proposition 4.7.24]). Thus we may assume without loss of generality that $x = \hat{1}_P$ and show that $\mathcal{O}$ has exactly one source and sink on the *entire* 1-skeleton $G(P^{op})$. It has at least one source and sink, namely the first and last atoms in the ordering. There can be no other sources or sinks because $\mathcal{O}$ comes from a shelling of a sphere (see the proof of [6, Proposition 4.7.22]). □

One can associate to every poset Q its *order complex* $\Delta(Q)$, that is the simplicial complex of totally ordered subsets of Q, and the topological space which is the *geometric realization* of $\Delta(Q)$. In what follows, we will abuse notation and make no distinction between Q, the order complex $\Delta(Q)$ and its geometric realization, hoping that no confusion will ensue.

We can now state the main result of this section. The special case of the Baues problem for monotone paths asks about the homotopy type of the poset $\omega(P,\mathcal{O})-\hat{1}_\omega$ of *proper* cellular strings on P (see [2]).

Theorem 5.3. *Let X be a regular CW d-sphere with face poset P. Let $\mathcal{O}$ be the facial acyclic orientation of the 1-skeleton $G(P^{op})$ of P^{op} induced by some fixed shelling order on the maximal faces of X.*

Then $\omega(P^{op},\mathcal{O}) - \hat{1}_\omega$ is homotopy equivalent to the $(d-1)$-sphere triangulated by the open interval $(a_{max},\hat{1})_{P^{op}}$.

Before proving Theorem 5.3, we identify some auxiliary objects and an important hypothesis on $\mathcal{O}$ that arise in the proof.

Given any atom a of P, let $\omega(P,\mathcal{O},a)$ be the poset of partial cellular strings ending at a, i.e. the set of tuples $\mathbf{x} = (x_1,\ldots,x_r)$ of elements of P with

- $r(x_i) \geq 2$ for each i,
- $a_{min}(x_1) = a_{min}$,
- $a_{max}(x_i) = a_{min}(x_{i+1})$ for $1 \leq i \leq r-1$ and
- $a_{max}(x_r) = a$.

Also define the poset $D(P,\mathcal{O},a)$, called the *$\mathcal{O}$-backward vertex figure of P at a*, to be the following induced subposet of P:

$$D(P,\mathcal{O},a) = \{x \in P : a_{max}(x) = a\}.$$

Note that if $a = a_{max}$, then $D(P,\mathcal{O},a_{max})$ is just the half-open interval $(a_{max},\hat{1}_P]$. We say that the facial acyclic orientation $\mathcal{O}$ is *tame* if $D(P,\mathcal{O},a)$ is contractible for every atom a of P.

Lemma 5.4. *Assume that P has a 1-skeleton and $\mathcal{O}$ is a tame, facial acyclic orientation of $G(P)$. Then $\omega(P,\mathcal{O},a)$ is contractible for every atom a in P.*

Proof. We use Babson's Lemma, whose statement we recall next. See [11, Lemma 3.2] for a proof.

Lemma 5.5. (Babson [1]) *Let $f: X \to Y$ be an order preserving map of posets. If*

(i) *$f^{-1}(y)$ is contractible for each y in Y and*
(ii) *$f^{-1}(y) \cap X_{\leq x}$ is contractible for each x in X and y in Y with $f(x) \geq y$,*

then f induces a homotopy equivalence. □

We can assume that $a \neq a_{max}$, since $\omega(P,\mathcal{O},a_{max}) = \omega(P,\mathcal{O})$ has the maximum element $\hat{1}_\omega$ and hence is contractible. We wish to apply Lemma 5.5 to the forgetful map

$$\begin{array}{rccc} f: & \omega(P,\mathcal{O},a) & \rightarrow & D(P,\mathcal{O},a) \\ & (x_1,\ldots,x_r) & \mapsto & x_r \end{array}$$

and use induction on the product of two orderings, one of which is the position of a in the partial ordering of the atoms of P induced by $\mathcal{O}$, and the other being the rank of the poset P.

To check condition (i) of Lemma 5.5, given $y \in D(P, \mathcal{O}, a)$, the fiber $f^{-1}(y)$ consists of all partial cellular strings of the form $(x_1, \ldots, x_{r-1}, y)$. Using Proposition 5.1, the map which sends such a cellular string to $(x_1, \ldots, x_{r-1})$ gives a poset isomorphism

$$f^{-1}(y) \cong \omega(P, \mathcal{O}, a_{min}(y)).$$

Since $a_{min}(y)$ is strictly less than a in the partial order induced by $\mathcal{O}$, induction implies that $\omega(P, \mathcal{O}, a_{min}(y))$ is contractible, as desired.

To check condition (ii) of Lemma 5.5, given $y \in D(P, \mathcal{O}, a)$ and given any $\mathbf{x} = (x'_1, \ldots, x'_{s-1}, x'_s)$ with $f(\mathbf{x}) \geq y$, the set $f^{-1}(y) \cap \omega(P, \mathcal{O}, a)_{\leq \mathbf{x}}$ consists of all partial cellular strings of the form $(x_1, \ldots, x_{r-1}, y)$ which lie below $\mathbf{x}$. Using Proposition 5.1, any such cellular string must contain some element x_k with $a_{min}(x_k) = a_{min}(x'_s)$, and the map which breaks such a cellular string into two strings

$$((x_1, \ldots, x_{k-1}), (x_k, \ldots, x_{r-1}))$$

gives a poset isomorphism

$$f^{-1}(y) \cap \omega(P, \mathcal{O}, a)_{\leq \mathbf{x}} \cong \omega(P, \mathcal{O}, a_{min}(x'_s)_{\leq (x'_1, \ldots, x'_{s-1})}) \times \omega([\hat{0}, x'_s], \mathcal{O}, a_{min}(y)).$$

The first factor in the latter direct product of posets is contractible because it has $(x'_1, \ldots, x'_{s-1})$ as its maximum element. The second factor is contractible by induction on the rank of P, since $[\hat{0}, x'_s]_P$ has smaller rank than P. □

Corollary 5.6. *Assume that P has a 1-skeleton and $\mathcal{O}$ is a tame, facial acyclic orientation of $G(P)$. Then the poset $\omega(P, \mathcal{O}) - \hat{1}_\omega$ of proper cellular strings on P is homotopy equivalent to the open interval $(a_{max}, \hat{1}_P)$ in P.*

Proof. Consider the special case of the map f in the proof of Lemma 5.4 when $a = a_{max}$. This gives a map

$$f : \omega(P, \mathcal{O}) \rightarrow (a_{max}, \hat{1}_P]$$

which satisfies the hypotheses of Lemma 5.5. As a result, the restriction

$$f : \omega(P, \mathcal{O}) - \hat{1}_\omega \rightarrow (a_{max}, \hat{1}_P)$$

of this map also satisfies the hypotheses of Lemma 5.5. Therefore it induces the desired homotopy equivalence. □

We can now prove Theorem 5.3.

Proof of Theorem 5.3. Assume that P and $\mathcal{O}$ are as in the statement of the theorem. In light of Corollary 5.6, we will show that $\mathcal{O}$ is tame. Thus, for any atom a of P^{op}, we must show that $D(P^{op}, \mathcal{O}, a)$ is contractible. To this end, we rephrase the definition of $D(P^{op}, \mathcal{O}, a)$.

Let $\mathcal{R}$ be the recursive atom ordering of P^{op} induced by the shelling of X. Let I be the set of atoms b of $[a, \hat{1}]_{P^{op}}$ such that a' precedes a in $\mathcal{R}$, where $[\hat{0}, b]_{P^{op}} = \{\hat{0}, a, a', b\}$. The definition of a recursive atom ordering implies that I is an initial segment in the ordering that $\mathcal{R}$ induces on the atoms of $[a, \hat{1}]_{P^{op}}$. Let I^c be the complement of I in the set of atoms of $[a, \hat{1}]_{P^{op}}$. Note that I^c is also an initial segment in a recursive atom ordering of $[a, \hat{1}]_{P^{op}}$, since [7, Lemma 1.2] states that the reverse of a recursive atom ordering is again a recursive atom ordering when X is a sphere. Recall that $D(P^{op}, \mathcal{O}, a)$ is the set of elements $x > a$ of P^{op} for which $a_{max}(x) = a$. This is exactly the set of $x > a$ for which every atom of $[a, x]$

lies in I, or equivalently, the set of $x > a$ which do *not* lie above any atoms in I^c. It follows from [7, Theorem 1.3] (see also [6, Lemma 4.7.28]) that $D(P^{op}, \mathcal{O}, a)$ is contractible, since it is the complement of the filter generated by a set of atoms (namely I^c) which form an initial segment in some recursive atom ordering coming from the dual of a shellable sphere.

Corollary 5.6 now implies that $\omega(P^{op}, \mathcal{O}) - \hat{1}_\omega$ is homotopy equivalent to the interval $(a_{max}, \hat{1})_{P^{op}}$. This interval is homeomorphic to a $(d-1)$-sphere because it is an open interval of corank one in the face poset of a shellable d-sphere (see e.g. [6, Propositions 4.7.19, 4.7.24]). □

Corollary 5.7.

(i) [2, Theorem 1.2] *For a convex d-polytope Q and generic linear functional f, the poset $\omega(Q, s_f, t_f)$ of cellular strings on Q is homotopy equivalent to a $(d-2)$-sphere.*

(ii) [4, Theorem 2] *For an oriented matroid $\mathcal{L}$ of rank r, the poset $Ess(\mathcal{T}(\mathcal{L}, B))$ of essential chains in the poset of topes (with respect to the base tope B) has the homotopy type of an $(r-2)$-sphere.*

Proof. A generic linear functional f on Q gives rise to a shelling order on the boundary $(d-1)$-sphere X of the polar polytope Q^* (see [13, Exercises 8.10, 8.11]). One can then check that the poset of cellular strings $\omega(Q, s_f, t_f)$ from [2] is exactly our $\omega(P^{op}, \mathcal{O})$. Hence the first assertion follows from Theorem 5.3.

For an oriented matroid $\mathcal{L}$ of rank r, [6, Theorem 4.3.3] states that any linear extension of the poset of topes $\mathcal{T}(\mathcal{L}, B)$ gives rise to a shelling of a regular CW $(r-1)$-sphere X. One can then check that the poset of essential chains $Ess(\mathcal{T}(\mathcal{L}, B))$ is exactly our $\omega(P^{op}, \mathcal{O})$. Hence the second assertion follows also from Theorem 5.3. □

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