SUMMARY OF RESULTS obtained during Christos visit: (May 1998)

1) Klee-Minty cube was discussed, it does not depend on parameters the MPP. Seems very hard to describe the polytope combinatorially...

2) We proved that any polytope P can be pertubed into a simplicial polytope $\tilde{P}$ that has **more** (coherent)- monotone paths as **P**.

Christos showed me his analysis of MPP of crosspolytopes, they seemed to be related to work by Björner in a conjecture of Lindström that has to do with realizing the poset of intervals of a face lattice by another Polytope.

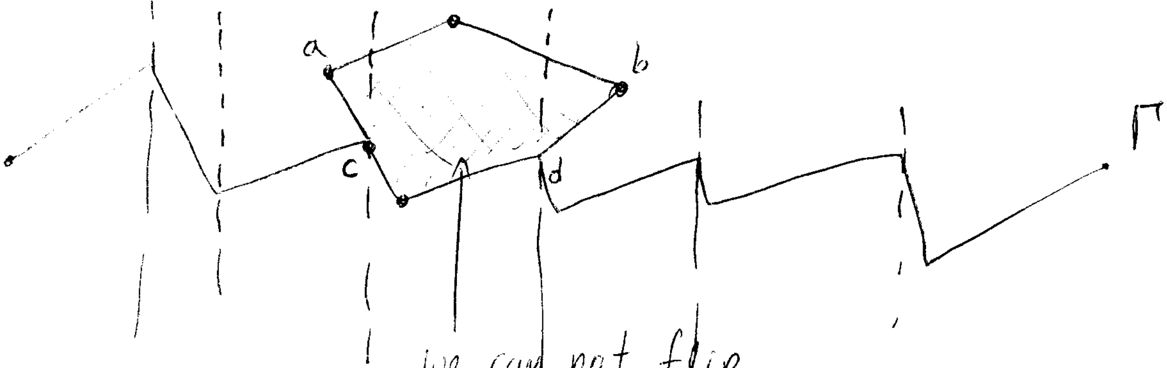
3) Vic suggested looking at monotone paths of $\Delta_n \times \Delta_m$.
It turns out it is a simple polytope whose vertices can be viewed as lattice paths on a **(m+1) x (n+1) grid**
All monotone paths are coherent.

There are $\sum_{k=1}^{n} \sum_{l=1}^{m} \binom{n-1}{k-1}\binom{m-1}{l-1}\binom{k+l}{l}$ = monotone paths.

for $\Delta_2 \times \Delta_m$ = 5, **14**, 37, 94, 232, 560, 1328, 3104, ...

Q: If a polytope Q has only coherent monotone paths for any generic functional, then **is** Q a product of simplices and r-gons?

4) Briefly we looked at the graph of monotone paths by polygonal **moves**.
In dimension 3, every 3-polytope any of its Monotone paths seems to have 2 polygonal moves
idea: Path breaks set of facets bordering the path (touching one edge at least) into two sets. For each of them if the facet (= a 2polygon) is not usable is because one of its edges on path is "beyond" a threshold see figure



we can not flip along this polygon because $a < c$ and $b > d$

c, d are vertices of the path Γ

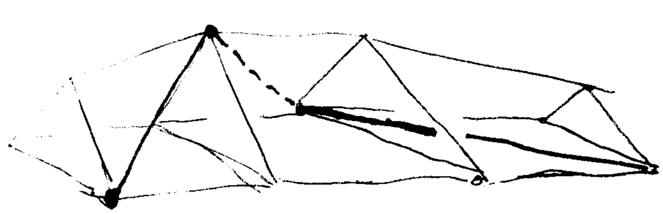
necessary for flip !!

Q: Although this can possibly be fixed in dimension 3 what can be said in dimension arbitrary?

Q: Connectivity of monotone paths follows from general considerations of the Generalized Baues problem (see Vic's survey) Can you write an independent elementary proof?

Q: Can the integration formula for MPP's given by Billera and Sturmfels be of any use in studying products? What is the relevance of these lattice paths that biject with Monotone paths on $\Delta_n \times \Delta_m$?

Finally: False !! The shortest monotone path doesn't necessarily is [redacted] coherent



(?)

← Now I am not so sure anymore...

1,,
Mail-from: From athana@math.upenn.edu Mon Jan 26 14:08:29 1998
Received: from hans.math.upenn.edu (HANS.MATH.UPENN.EDU [130.91.49.156])
by geom.umn.edu (8.8.6.Beta4/8.8.6.Beta4) with ESMTP id OAA21977
for <deloera@geom.umn.edu>; Mon, 26 Jan 1998 14:08:28 -0600 (CST)
Received: (from athana@localhost)
by hans.math.upenn.edu (8.8.5/8.8.5) id PAA04196;
Mon, 26 Jan 1998 15:08:28 -0500 (EST)
Date: Mon, 26 Jan 1998 15:08:28 -0500 (EST)
From: athana@math.upenn.edu (Christos Athanasiadis)
Message-Id: <199801262008.PAA04196@hans.math.upenn.edu>
To: deloera@geom.umn.edu
Subject: Re: r_d
Cc: athana@math.upenn.edu
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Content-Type: text/plain; charset=us-ascii
Content-Transfer-Encoding: 7bit
Content-MD5: MHc6fb9qLZbM69noE3/N+w==
Status: O

*** EOOH ***
Date: Mon, 26 Jan 1998 15:08:28 -0500 (EST)
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Hola Jesus,

I was sure you would have plenty of choices in the end, you deserve it! I vote for UC Davis or California Politechnic. Boston College is still good since it is pretty close and it is in Boston.

Last time I announced that

$$r_4(n) <= O(n^4) \text{ and } r_5(n) <= O(n^6)$$

but now I claim that

$$r_4(n) <= O(n^2) \text{ and } r_5(n) <= O(n^4).$$

The proof is easy again. The normal fan of the MPP of P is the common refinement of the normal fans of the (n-1 at most) fibers. The extreme rays in the normal fan correspond to facets of P and these are at most O(n^2) by the UBT. So the MPP has at most O(n^2) facets and by the UBT again (in the polar version) the MPP has at most O(n^2) vertices if it is 3-dim (d=4) and at most O(n^4) if it is 4 dim (d=5).

The same reasoning gives a bound of n^{[d/2][(d-1)/2]} in general but this not any good for d => 12 or so.

My reasoning last time was the following: the number of nonparallel edges e of the fibers is at most the number of 2-faces of P, which is at most O(n^3) for d => 6 but O(n^2) for d = 4, 5. There is a result of Gritzmann and Sturmfels (SIAM J. Discrete Math. 6 (1993), 246-269) which bounds the # of vertices of a Minkowski some given e, the maximum occurs when you have zonotopes with a total of e generators in general position.
Unfortunately, the bound obtained is exactly the one I got in out paper,

possibly larger by a factor of 2, I forget. But for d = 4, 5 it gave the first small improvement.

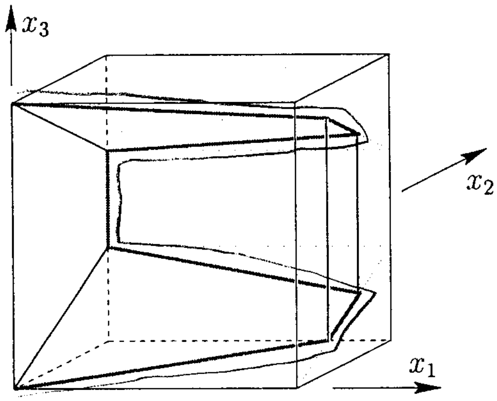
The problem is that the fibers won't be zonotopes when e = O(n^3), i.e. P is essentially 3-neighborly.

In any case, I may get stuck with this soon and any help would be much appreciated. I think it is an interesting problem. There is something special about normal fans of MPP's that have to be used..

Have fan flying.

-Christos

(for $\varepsilon = \frac{1}{3}$). If the Klee-Minty cube was a projective image of a regular cube, then the four "horizontal" edges would have to meet in one point (or be parallel), which is clearly not the case.



Theorem 4.3 *For $n > d \geq 0$:*

$$H_{\rm Bl}(d+1, n+2) \quad \geq \quad 2\, H_{\rm Bl}(d, n).$$

Analogous inequalities are true for the maximal numbers of vertices $H(d,n)$ and $H_{\rm Da}(d,n)$ on arbitrary increasing paths resp. for paths according to Dantzig's rule. Hence for $d \geq 0$ we have

$$H(d, 2d) \quad \geq \quad H_{\rm Da}(d, 2d) \quad \geq \quad H_{\rm Bl}(d, 2d) \quad \geq \quad 2^d.$$

Proof. Let $P \subseteq \mathbb{R}^d$ be a simple (d,n)-polytope which achieves the maximum length for an arbitrary increasing path (respectively of a path according to Bland's or Dantzig's rule), and let $\varphi : P \longrightarrow [0,1]$ be a corresponding objective function. Then for some $0 < \varepsilon < \frac{1}{2}$ form the deformed product

$$(P, \varphi) \bowtie ([0,1], [\varepsilon, 1-\varepsilon])$$

and use the same proof as for the Klee-Minty cubes. □

4.2 The Goldfarb-Sit Cubes

Goldfarb & Sit [16] constructed linear programs — rescaled Klee-Minty cubes tailored to fool the steepest increase rule — as follows. They analyzed the programs

$$\max \sum_{i=1}^{d} \beta^{i-1} x_i \colon \; x \in \mathrm{GS}_d$$

where $\mathrm{GS}_d \subseteq \mathbb{R}^d$ is the polytope given by

$$\begin{array}{rcccl} 0 & \leq & x_1 & \leq & 1 \\ \beta x_{j-1} & \leq & x_j & \leq & \delta_j - \beta x_{j-1} \quad \text{for } 2 \leq j \leq d, \end{array}$$

where $\beta \geq 2$, $\theta > 2$ and $\delta_i := (\theta\beta)^{i-1}$. It is immediate from Theorem 3.4 that we may construct the *Goldfarb-Sit cubes* as deformed products

$$\begin{array}{lcll} \mathrm{GS}_0 & = & \{0\} & \subseteq \mathbb{R}^0 \\ \mathrm{GS}_1 & = & [0,1] & \subseteq \mathbb{R}^1 \\ \mathrm{GS}_{d+1} & = & (\mathrm{GS}_d, x_d) \bowtie ([0, \delta_{d+1}], [\beta, \delta_{d+1} - \beta]) & \subseteq \mathbb{R}^{d+1}. \end{array}$$

and that they are combinatorial d-cubes.

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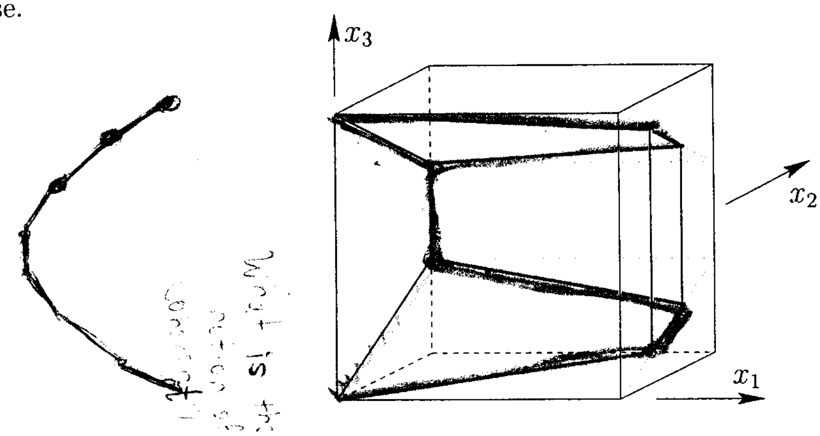

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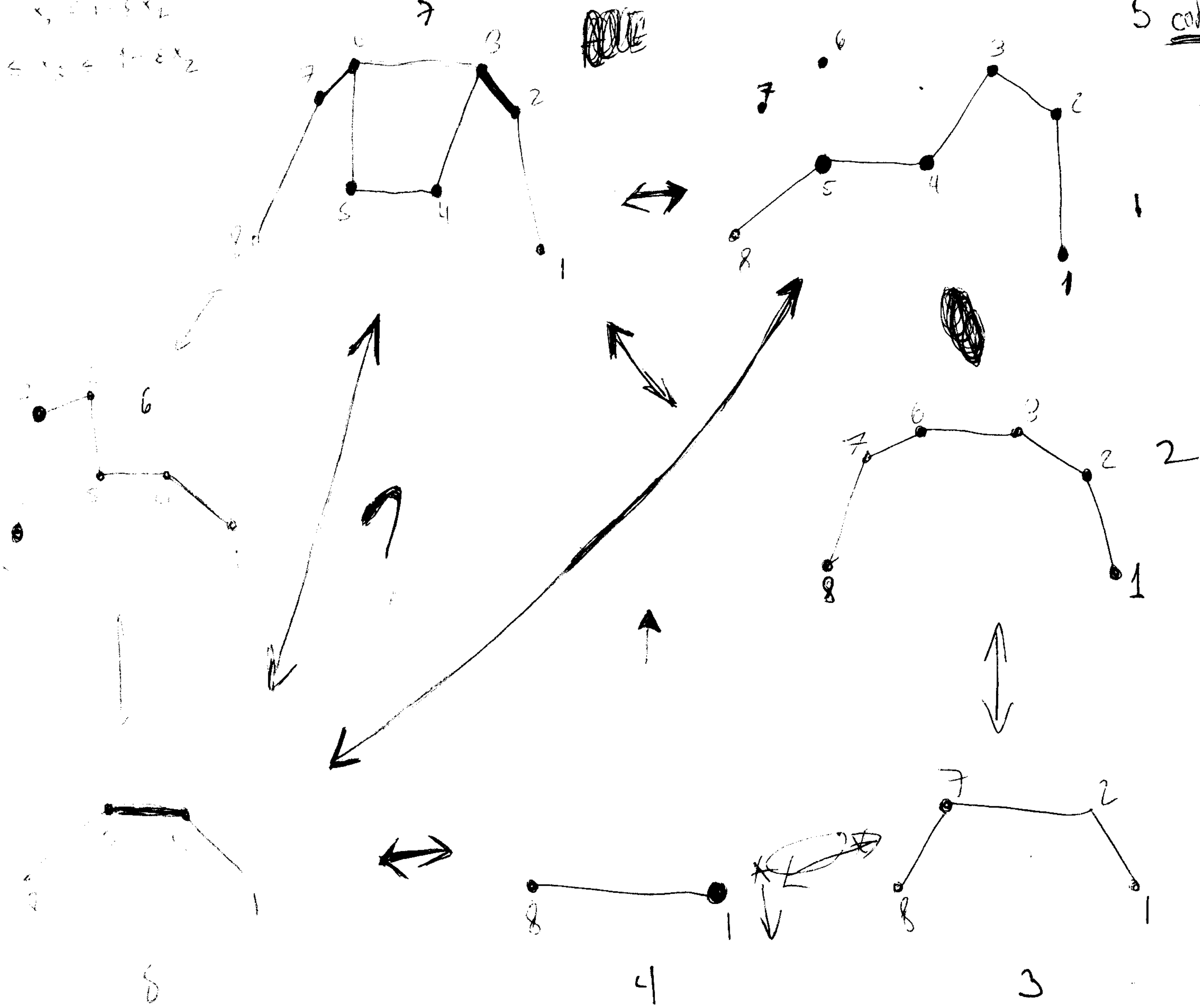
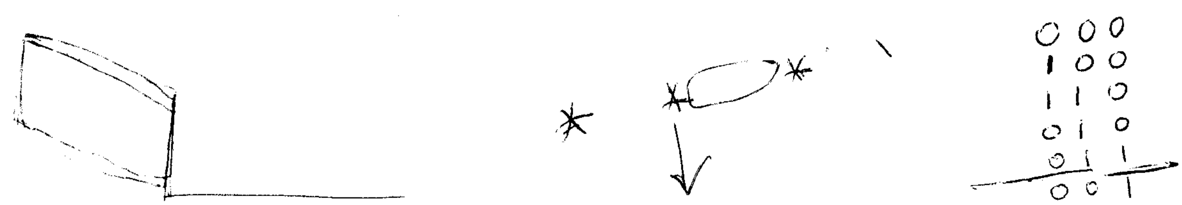
Seven paths but only 5 coherent (?)





$22 ×

C) (1 5 6 2)

D) (1 3 4 2)

A) 1 5 7 8 6 2

B) 1 5 7 3 4 8 6 2

E) (1,2)

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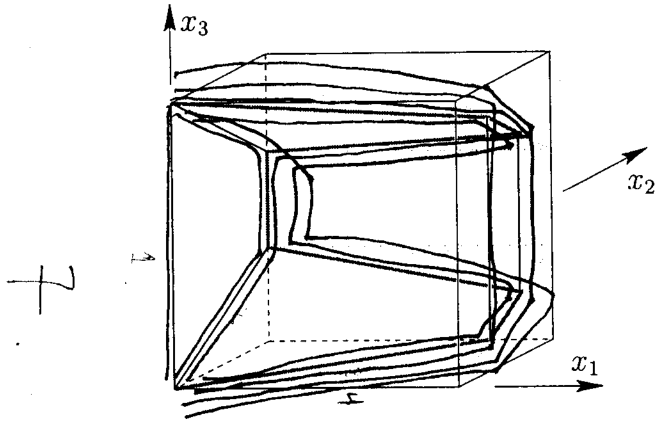

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