SOME REMARKS ON THE UNIMODALITY OF h^*-VECTORS OF LATTICE POLYTOPES

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Abstract. We prove that the h^*-vector $(h_0^*, h_1^*, \dots, h_m^*)$ of an m-dimensional lattice polytope P is unimodal if it is symmetric and P has a regular unimodular triangulation. We also give conditions on P which imply inequalities of the form $h_i^* \geq h_{s-i}^*$ for $1 \leq i \leq \lfloor s/2 \rfloor$ and $h_{\lfloor s/2 \rfloor}^* \geq h_{\lfloor s/2 \rfloor + 1}^* \geq \cdots \geq h_s^*$ and provide an application to stable polytopes of perfect graphs.

1. Introduction

Let P be an m-dimensional convex polytope in $\mathbb{R}^N$ having vertices with integer coordinates. It is a fundamental result due to Ehrhart [5, 6] that the function $i(P, r) = \#(rP \cap \mathbb{Z}^N)$, counting integer points in the r-fold dilate of P, is a polynomial in r of degree m, called the *Ehrhart polynomial* of P. Thus one can write

$$\sum_{r \geq 0} i(P, r)\, t^r = \frac{h_0^* + h_1^* t + \cdots + h_m^* t^m}{(1-t)^{m+1}} \qquad (1)$$

for certain integers h_i^*. Following Stanley [16] we call the vector $(h_0^*, h_1^*, \dots, h_m^*)$ the *h^*-vector* of P and denoted it by $h^*(P)$. It is known that $h^*(P)$ is the h-vector of a semistandard Cohen-Macaulay graded normal domain R_P called the semigroup ring of P; see [3, Chapter 6] and [9, Chapters IX-XI]. In particular the integers h_i^* are nonnegative. A sequence $(a_0, a_1, \dots, a_n)$ of real numbers is said to be *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some $0 \leq j \leq n$. Although h^*-vectors are not always unimodal, various results and conjectures concerning the unimodality of $h^*(P)$ have appeared in the literature. For instance it would follow from [9, Conjecture 1.5] and [15, Conjecture 4a] that $h^*(P)$ is unimodal if the semigroup ring R_P is standard and Gorenstein. Moreover, it was conjectured by Hibi [9, p. 111] that $h^*(P)$ is unimodal whenever it is symmetric, meaning that $h_i^* = h_{m-i}^*$ for $0 \leq i \leq m$. It is the first goal of this paper to provide a partial affirmative answer to this conjecture.

Theorem 1.1. *Let P be an m-dimensional integer polytope with h^*-vector $h^*(P) = (h_0^*, h_1^*, \dots, h_m^*)$. If P has a regular unimodular triangulation then $h_i^* \geq h_{m-i+1}^*$ for $1 \leq i \leq \lfloor (m+1)/2 \rfloor$ and*

$$h_{\lfloor (m+1)/2 \rfloor}^* \geq \cdots \geq h_{m-1}^* \geq h_m^*.$$

In particular, if $h^*(P)$ is symmetric and P has a regular unimodular triangulation then $h^*(P)$ is unimodal.

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An example of a 0/1 polytope P with no regular unimodular triangulations such that R_P is standard was given in [10].

General conditions on an integer polytope P, inspired by the work of V. Reiner and V. Welker on order polytopes of graded posets [12], which guarantee that $h^*(P)$ is unimodal were given in [1]. More precisely it was shown in [1] that if the pulling triangulation Δ_τ of P with respect to an ordering $\tau = (v_1, v_2, \ldots, v_p)$ of the vertices of P is unimodular and for some n any facet of P contains exactly $n-1$ elements of $\{v_1, v_2, \ldots, v_n\}$ then $h^*(P)$ is equal to the h-vector of a simplicial d-dimensional polytope, where $d = m-n+1$, and hence that it is unimodal and satisfies $h^*_i = h^*_{d-i}$ for $0 \le i \le d$ by McMullen's g-theorem. The simplex with vertices $\{v_1, v_2, \ldots, v_n\}$ is called a *special simplex* for P in [1]. If $v_1, v_2, \ldots, v_n$ are any n vertices of P which are affinely independent we call the simplex with these points as vertices a *semispecial simplex* for P if any facet of P contains at least $n-1$ elements of $\{v_1, v_2, \ldots, v_n\}$. It is the second goal of this paper to prove the following more general statement.

Theorem 1.2. *Let P be an m-dimensional integer polytope with h^*-vector $h^*(P) = (h^*_0, h^*_1, \ldots, h^*_m)$. If $\tau = (v_1, v_2, \ldots, v_p)$ is an ordering of the vertices of P such that*

(i) *the pulling triangulation Δ_τ is unimodular,*
(ii) *$\{v_1, v_2, \ldots, v_n\}$ is the vertex set of a semispecial simplex for P*

and $d = m-n+1$, then $h^*_i \ge h^*_{d-i}$ for $0 \le i \le \lfloor d/2 \rfloor$, $h^*_{\lfloor d/2 \rfloor} \ge h^*_{\lfloor d/2 \rfloor +1} \ge \cdots \ge h^*_d$ and $h^*_i = 0$ for $d < i \le m$.

The next corollary follows essentially from the case $n = 1$ of Theorem 1.2.

Corollary 1.3. *Let P be an m-dimensional integer polytope with h^*-vector $h^*(P) = (h^*_0, h^*_1, \ldots, h^*_m)$. If P has a unimodular pulling triangulation then $h^*_i \ge h^*_{m-i}$ for $1 \le i \le \lfloor m/2 \rfloor$ and $h^*_{\lfloor m/2 \rfloor} \ge h^*_{\lfloor m/2 \rfloor +1} \ge \cdots \ge h^*_m$.*

This paper is structured as follows. Section 2 includes the necessary definitions and background on convex polytopes and their triangulations and h^*-vectors as well as the proof of Theorem 1.1. The notion of triangulation of a polytope P we will use does not require that all vertices of the triangulation are necessarily vertices of P, unless the contrary is explicitly stated. In Section 3 we introduce the concept of a semispecial simplex for P and prove a slight generalization of Theorem 1.2. Specifically we drop the assumption that all vertices of a (special or) semispecial simplex and the elements of the sequence τ which appears in Theorem 1.2 are vertices of P. Our proofs are based on a result of Stanley (Lemma 2.2) on the h-vector of a Cohen-Macaulay subcomplex of the boundary complex of a simplicial polytope. Section 4 includes an application of Theorem 1.2 to stable polytopes of perfect graphs.

2. Triangulations and h^*-vectors

Before proving Theorem 1.1 we review some basic definitions and background on simplicial complexes and convex polytopes. For undefined terminology and more information on these topics we refer the reader to [7, 9, 17, 18, 19]. Recall that the h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a finite (geometric) simplicial complex Δ of dimension $d-1$ is defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i} \tag{2}$$

where f_i is the number of i-dimensional simplices (faces) of Δ for $0 \le i \le d-1$ and $f_{-1} = 1$. We say that Δ is a *triangulation* of a convex polytope $P \subseteq \mathbb{R}^N$ if the union of the faces of Δ is equal to P. In particular we do not require that all vetrices of Δ are vertices of P. Such a triangulation is *regular* if the faces of Δ are the projections, under the map $\mathbb{R}^{N+1} \to \mathbb{R}^N$ which forgets the last coordinate, of the *lower* faces of a convex polytope $Q \subseteq \mathbb{R}^{N+1}$, meaning those faces of Q which are visible from any point in $\mathbb{R}^{N+1}$ with sufficiently large negative last coordinate. A convex polytope $P \subseteq \mathbb{R}^N$ is called an *integer polytope* if all vertices of P have integer coordinates. A triangulation Δ of such a polytope P is called *unimodular* if all vertices of Δ have integer coordinates and the vertex set of any maximal simplex of Δ is a basis of the affine integer lattice $A \cap \mathbb{Z}^N$, where A is the affine span of P in $\mathbb{R}^N$. In the special case of pulling triangulations (which we discuss in the sequel) the following lemma appeared as [13, Corollary 2.5].

Lemma 2.1. ([2]) *If P is an integer polytope and Δ is any unimodular triangulation of P then $h^*(P) = h(\Delta)$.*

The *boundary complex* $\mathcal{F}(\partial P)$ of a convex polytope P is the collection of proper faces of P. The polytope P is said to be *simplicial* if all its proper faces are simplices, so that $\mathcal{F}(\partial P)$ is a simplicial complex. The next lemma is a consequence of [16, Lemma 2.2] and the note following that lemma in [16].

Lemma 2.2. ([16]) *If Δ is a Cohen-Macaulay subcomplex of the boundary complex of a d-dimensional simplicial polytope and $h(\Delta) = (h_0, h_1, \dots, h_d)$ then $h_i \ge h_{d-i}$ for $0 \le i \le \lfloor d/2 \rfloor$ and $h_{\lfloor d/2 \rfloor} \ge h_{\lfloor d/2 \rfloor + 1} \ge \cdots \ge h_d$.*

To any polytope Q one can associate a simplicial polytope Q' having the same dimension and number of vertices as Q by *pulling the vertices* of Q in some order [7, p. 80]. More specifically let $\mathrm{pull}_v(P)$ denote the polytope obtained from a polytope P by moving its vertex v beyond the hyperplanes supporting exactly those facets of P which contain v (see [19, Section 3.1]). If $\tau = (v_1, v_2, \dots, v_p)$ is any linear ordering of the vertices of Q and $Q_i = \mathrm{pull}_{v_i}(Q_{i-1})$ for $1 \le i \le p$, with $Q_0 = Q$, then $Q' = Q_p$ is simplicial of the same dimension as Q and the vertices of Q' can be labeled as $v'_1, v'_2, \dots, v'_p$ so that if $v_{i_1}, v_{i_2}, \dots, v_{i_j}$ are the vertices of a $(j-1)$-dimensional simplex which is a face of Q then $v'_{i_1}, v'_{i_2}, \dots, v'_{i_j}$ are the vertices of a $(j-1)$-dimensional face of Q'.

Proof of Theorem 1.1. Let Δ be a regular unimodular triangulation of P. Being regular, Δ is combinatorially isomorphic to the complex of lower faces of an $(m+1)$-dimensional polytope Q. Pulling the vertices of Q in an arbitrary order produces an $(m+1)$-dimensional simplicial polytope Q' such that Δ is combinatorially isomorphic to a subcomplex Δ' of the boundary complex of Q'. Then $h(\Delta) = h(\Delta')$ and $h^*(P) = h(\Delta)$ by Lemma 2.1. Moreover Δ' is topologically a ball, being homeomorphic to Δ, and hence Cohen-Macaulay. Lemma 2.2 implies that $h(\Delta') = h^*(P)$ satisfies $h^*_{\lfloor (m+1)/2 \rfloor} \ge \cdots \ge h^*_m \ge h^*_{m+1} = 0$ and $h^*_i \ge h^*_{m-i+1}$ for $1 \le i \le \lfloor (m+1)/2 \rfloor$. $\square$

We conclude this section with the background on pulling triangulations of polytopal complexes needed for the proof of Theorem 1.2. A *polytopal complex* $\mathcal{F}$ [19, Section 8.1] is a finite, nonempty collection of convex polytopes in $\mathbb{R}^N$ such that (i) any face of a polytope in $\mathcal{F}$ is also in $\mathcal{F}$ and (ii) the intersection of any two polytopes in $\mathcal{F}$ is either empty or a face of both. The elements of $\mathcal{F}$ are its *faces* and those of dimension 0 are its *vertices*. The *dimension* of $\mathcal{F}$ is the maximum dimension of a face. The complex $\mathcal{F}$ is *pure* if all maximal (with respect to inclusion) faces of $\mathcal{F}$ have the same dimension. The collection $\mathcal{F}(P)$ of all faces of a polytope P is a pure polytopal complex, called the *face complex*, as is the boundary complex $\mathcal{F}(\partial P)$. A *triangulation* of a polytopal complex $\mathcal{F}$ is a geometric simplicial complex Δ with underlying space equal to the union $\|\mathcal{F}\|$ of the faces of $\mathcal{F}$, such that every face of Δ is contained in a face of $\mathcal{F}$. For any set of points σ in $\mathbb{R}^N$ we denote by $\mathcal{F}\backslash\sigma$ the subcomplex of faces of $\mathcal{F}$ which do not contain any of the points in σ and write $\mathcal{F}\backslash v$ for $\mathcal{F}\backslash\sigma$ if σ consists of a single point v. Given a sequence $\tau = (v_1, v_2, \ldots, v_p)$ of points containing the vertices of $\mathcal{F}$ we define the *reverse lexicographic triangulation* or *pulling triangulation* $\Delta(\mathcal{F}) = \Delta_\tau(\mathcal{F})$ with respect to τ [13] [18, Chapter 8] as $\Delta(\mathcal{F}) = \{v\}$ if $\mathcal{F}$ consists of a single vertex v and

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F}\backslash v_1) \cup \bigcup_F \{\mathrm{conv}(\{v_1\} \cup G) : G \in \Delta(\mathcal{F}(F)) \cup \{\emptyset\}\}$$

otherwise, where the union runs through the facets F not containing v_1 of the maximal faces of $\mathcal{F}$ which contain v_1 and $\Delta(\mathcal{F}\backslash v_1)$ and $\Delta(\mathcal{F}(F))$ are defined with respect to $(v_2, \ldots, v_p)$. Equivalently for $i_0 > i_1 > \cdots > i_t$ the set $\{v_{i_0}, v_{i_1}, \ldots, v_{i_t}\}$ is the vertex set of a maximal simplex of $\Delta_\tau(\mathcal{F})$ if there exists a maximal flag $F_0 \subset F_1 \subset \cdots \subset F_t$ of faces of $\mathcal{F}$ such that v_{i_j} is the first element of τ in F_j for all j and v_{i_j} is not in F_{j-1} for $j \geq 1$.

If $\mathcal{F}$ is the boundary complex of a polytope Q then $\Delta_\tau(\mathcal{F})$ is combinatorially isomorphic to the boundary complex of a simplicial polytope Q' of the same dimension, obtained from Q by a *sequence of pullings* with respect to τ. More precisely we can let $Q' = Q_p$ where $Q_i = \mathrm{pull}_{v_i}(Q_{i-1})$ for $1 \leq i \leq p$, $Q_0 = Q$ and $\mathrm{pull}_v(P)$ is either the convex hull of the set of vertices of P and the point obtained by moving v beyond the hyperplanes supporting exactly those facets of P which contain v, if v lies on the boundary of P, or $\mathrm{pull}_v(P) = P$ if not. Clearly Q' is the polytope obtained by pulling the vertices of Q in the order τ, as discussed earlier, if τ is a linear ordering of the vertices of Q.

3. Semispecial simplices

Throughout this section P denotes an m-dimensional convex polytope in $\mathbb{R}^N$. We call an $(n-1)$-dimensional simplex Σ a *special simplex* for P if each facet of P contains exactly $n-1$ vertices of Σ. This definition is less restrictive than the one given originally in [1] since we do not require that all vertices of Σ are vertices of P. We call Σ a *semispecial simplex* if each facet of P contains at least $n-1$ vertices of Σ. Thus a semispecial simplex for P is special if and only if it is not contained in the boundary of P. Two examples are shown in Figure 1.

Remark 3.1. Any set σ of n points in $\mathbb{R}^N$ having the property that any facet of P contains at least $n-1$ elements of σ must be affinely independent and hence it is the vertex set of a semispecial $(n-1)$-simplex for P. Indeed, if $v \in \sigma$ were in the

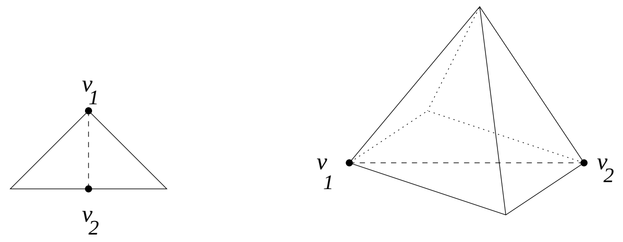

FIGURE 1. A special simplex for a triangle and a semispecial simplex for a square piramid.

affine span of $\sigma\backslash v$ then the affine span of any facet of P would have to contain v, which is impossible.

If V is any linear subspace of $\mathbb{R}^N$ then the *quotient polytope* $P/V \subseteq \mathbb{R}^N / V$ is the image of P under the canonical surjection $\mathbb{R}^N \to \mathbb{R}^N / V$. This is a convex polytope in $\mathbb{R}^N / V$ linearly isomorphic to the image $\pi(P)$ of P under any linear surjection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-\dim V}$ with kernel V. Recall that the *simplicial join* $\Delta_1 * \Delta_2$ of two geometric simplicial complexes Δ_1 and Δ_2 in $\mathbb{R}^N$ is defined as the simplicial complex whose facets are the simplices of the form $\mathrm{conv}(F_1 \cup F_2)$ where F_1 and F_2 are facets of Δ_1 and Δ_2, respectively, provided that the affine spans of F_1 and F_2 are skew affine subspaces of $\mathbb{R}^N$ for each such pair of facets (F_1, F_2). The simplicial join satisfies $h(\Delta_1 * \Delta_2, x) = h(\Delta_1, x)\, h(\Delta_2, x)$, where $h(\Delta, x) = \sum_{i=0}^{d} h_i x^{d-i}$ if $h(\Delta) = (h_0, h_1, \ldots, h_d)$. In particular $h(\Delta_1 * \Delta_2) = h(\Delta_2)$ if Δ_1 is a simplex.

Lemma 3.2. *If P has a triangulation of the form $\Sigma * \Delta$ for some $(n-1)$-simplex Σ and simplicial complex Δ then Δ is combinatorially isomorphic to a triangulation of a pure $(m-n)$-dimensional shellable subcomplex of the boundary complex of a polytope of dimension $m-n+1$.*

Proof. Let V be the linear $(n-1)$-dimensional subspace of $\mathbb{R}^N$ parallel to the affine span of Σ and let Q be the corresponding $(m-n+1)$-dimensional quotient polytope P/V. If v is the point which is the image of Σ under the canonical surjection $\mathbb{R}^N \to \mathbb{R}^N / V$ then Q inherits a triangulation of the form $v * \Gamma$ for some simplicial complex Γ which is combinatorially isomorphic to Δ and triangulates a subcomplex of the boundary complex $\mathcal{F}(\partial Q)$. If Σ is not contained in the boundary of P then v is an interior point of Q and Γ triangulates $\mathcal{F}(\partial Q)$, which is pure and shellable [19, Section 8.2]. Otherwise v lies on the boundary of Q and Γ triangulates the subcomplex of $\mathcal{F}(\partial Q)$ consisting of all faces of Q which do not contain v. This is the complex of faces of Q which are not visible from a point beyond the hyperplanes supporting exactly those facets of Q which contain v and hence is pure $(m-n)$-dimensional and shellable [19, Theorem 8.12]. □

Lemma 3.3. *Suppose that $P = \mathrm{conv}\{v_1, v_2, \ldots, v_p\}$. Let $\tau = (v_1, v_2, \ldots, v_p)$, $\Sigma = \mathrm{conv}\{v_1, v_2, \ldots, v_n\}$ and Δ be the pulling triangulation of $\mathcal{F}(P)\backslash \{v_1, v_2, \ldots, v_n\}$ with respect to $(v_{n+1}, \ldots, v_p)$. If Σ is a semispecial $(n-1)$-simplex for P then*

(i) *the pulling triangulation Δ_τ of P is combinatorially isomorphic to the simplicial join $\Sigma * \Delta$ and*
(ii) *Δ is combinatorially isomorphic to a Cohen-Macaulay subcomplex of the boundary complex of a simplicial polytope of dimension $m-n+1$.*

Proof. Part (i) can be proved exactly as part (i) of [1, Lemma 3.4], where Σ is assumed to be a special simplex for P. Lemma 3.2 implies that Δ is combinatorially isomorphic to a triangulation Γ of a pure $(m-n)$-dimensional shellable subcomplex of the boundary complex of a convex polytope Q of dimension $m-n+1$. Clearly Γ is homeomorphic to a ball or a sphere and hence Cohen-Macaulay. Let the isomorphism between Δ and Γ be induced by the map $v_i \to v_i'$ for $n+1 \leq i \leq p$ and Q' be the simplicial $(m-n+1)$-dimensional polytope obtained from Q by any sequence of pullings starting with $(v_{n+1}', \ldots, v_p')$. Since Δ is a pulling triangulation with respect to $(v_{n+1}', \ldots, v_p')$ Γ is combinatorially isomorphic to a subcomplex of the boundary complex of Q'. This proves part (ii). $\square$

Let $h^*(P) = (h_0^*, h_1^*, \ldots, h_m^*)$ be the h^*-vector of P. Theorem 1.2 is a special case of the following corollary.

Corollary 3.4. *Suppose that $P = \mathrm{conv}\{v_1, v_2, \ldots, v_p\}$ and $v_i \in \mathbb{Z}^N$ for $1 \leq i \leq p$. Let $d = m-n+1$ and $\tau = (v_1, v_2, \ldots, v_p)$. If Δ_τ is a unimodular triangulation of P and $\Sigma = \mathrm{conv}(\{v_1, v_2, \ldots, v_n\})$ is a semispecial $(n-1)$-simplex for P then $h_i^* \geq h_{d-i}^*$ for $0 \leq i \leq \lfloor d/2 \rfloor$,*

$$h^*_{\lfloor d/2 \rfloor} \geq h^*_{\lfloor d/2 \rfloor + 1} \geq \cdots \geq h_d^*$$

and $h_i^* = 0$ for $d < i \leq m$. Moreover $h_d^* = 0$ if Σ is not special.

Proof. Let Δ denote the pulling triangulation of $\mathcal{F}(P) \setminus \{v_1, v_2, \ldots, v_n\}$ with respect to $(v_{n+1}, \ldots, v_p)$. Lemma 2.1 guarantees that $h^*(P) = h(\Delta_\tau)$. Then part (i) of Lemma 3.3 implies that $h(\Delta_\tau) = h(\Sigma * \Delta) = h(\Delta)$ and the result follows from part (ii) of the same lemma and Lemma 2.2. If Σ is not special then Δ is homeomorphic to a $(d-1)$-dimensional ball and hence $h_d^* = 0$. $\square$

Proof of Corollary 1.3. It follows from the case $n = 1$ of Corollary 3.4 since a singleton $\{v_1\}$ is always a semispecial 0-dimensional simplex for P. If all vertices of Δ are vertices of P then $\{v_1\}$ is not special and hence $h_m^* = 0$. $\square$

4. Stable polytopes of perfect graphs

In this section we apply Theorem 1.1 to stable polytopes of graphs. We consider simple graphs G (with no loops or multiple edges) on the finite set of nodes $[N] := \{1, 2, \ldots, N\}$. For $W \subseteq [N]$ we denote by $\rho(W)$ the 0/1 vector $\sum_{j \in W} e_j \in \mathbb{R}^N$, where e_j is the jth unit coordinate vector in $\mathbb{R}^N$, and call W a *stable set* for G if no two elements of W are joined with an edge in G. Observe that the empty set and all singleton subsets of $[N]$ are stable. The *stable polytope* of G, introduced in [4] and denoted by $P(G)$, is the convex hull of all vectors $\rho(W)$, where W is a stable set for G. The graph G is called *perfect* if for any induced subgraph H of G the chromatic number of H is equal to the cardinality of the largest complete subgraph of H. We call G *semipure* if there exists a positive integer k such that all maximal complete subgraphs of G have size $k-1$ or k. Thus if G is perfect and semipure of chromatic number r then all maximal complete subgraphs of G have size $r-1$ or r. The following theorem applies in particular to all bipartite graphs and all perfect graphs of chromatic number three with no isolated vertices.

Theorem 4.1. *If G is a semipure perfect graph on the vertex set $[N]$ with chromatic number r and $d = N - r + 1$ then the h^*-vector $(h_0^*, h_1^*, \ldots, h_N^*)$ of the stable polytope*

$P(G)$ satisfies $h_i^* \geq h_{d-i}^*$ for $1 \leq i \leq \lfloor d/2 \rfloor$, $h_{\lfloor d/2 \rfloor}^* \geq h_{\lfloor d/2 \rfloor+1}^* \geq \cdots \geq h_{d-1}^*$ and $h_i^* = 0$ for $d \leq i \leq N$.

Proof. It was proved in [11] that all pulling triangulations of stable polytopes of perfect graphs are unimodular. In view of Corollary 3.4 it suffices to prove that there exists a semispecial $(r-1)$-simplex for $P(G)$ with integer vertices which is not special. Consider a coloring of G with colors $1, 2, \ldots, r$ and for $1 \leq i \leq r$ let W_i be the set of vertices of G colored with i, so that W_i is a stable set for G. We claim that $\Sigma = \mathrm{conv}\{\rho(W_i) : 1 \leq i \leq r\} \subset \mathbb{R}^N$ is a semispecial $(r-1)$-simplex for $P(G)$. Since G is perfect, by [4, Theorem 3.1] a facet of $P(G)$ is defined by a linear equality of the form $x_i = 0$ or $\sum_{j \in U} x_j = 1$ for some vertex set U of a maximal complete subgraph of G. A facet of the first form contains exaclty $r-1$ of the points $\rho(W_i)$ since the sets W_i form a partition of $[N]$. Since G is also semipure, a facet of the second form contains either $r-1$ or r points $\rho(W_i)$, where the second case occurs, and the claim follows from Remark 3.1. $\square$

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