OPTIMAL TRIANGULATIONS

Introduction. A *triangulation* of a given set S of n points in the Euclidean plane is a maximal set of non-crossing line segments (called *edges*) which have both endpoints in S. Any triangulation of S partitions the interior of the convex hull of S into triangles. Triangulations are used in many areas of engineering and scientific applications such as finite element methods, approximation theory, numerical computation, computer-aided geometric design, computational geometry, etc. Many applications are surveyed in [5], [8], [17], [56].

A triangulation of S can be viewed as a planar graph whose vertex set is S and whose edge set is a subset of $S \times S$. The Eulerian relation for planar graphs implies that the number $e(S)$ of edges, and the number $t(S)$ of triangles, do not depend on the way of triangulating S. In particular

$$\begin{aligned} e(S) &= 3n-3-h \\ t(S) &= 2n-2-h \end{aligned}$$

where h denotes the number of edges bounding the convex hull of S. The number of different triangulations of S is, however, an exponential function of n. The currently best lower and upper bounds are roughly $\Omega(2^{3n})$ [32] and $O(2^{8n})$ [18], respectively.

The problem of automatically generating *optimal triangulations* for a point set S has been a subject of research since decades. Enumerating all possible triangulations and selecting an optimal one (exhaustive search) is too time-consuming even for small n. In fact, constructing optimal triangulations in polynomial time is a challenging task. This becomes more apparent as greedy methods, such as deleting candidate triangles or edges from worst to best, are doomed to fail by the NP-completeness of the following problem; see Lloyd [53]: Given S and some set E of edges, decide whether E contains a triangulation of S.

Results on optimizing *combinatorial properties* of triangulations, such as their degree (Jansen [37]) or connectivity (Dey et al. [19] and Dillencourt [23]) are rare. Most optimization criteria for which efficient algorithms are known concern *geometric* properties of the edges and triangles.

Delaunay triangulations. The most commonly constructed and maybe the most famous triangulation for a point set S is the *Delaunay triangulation*, $DT(S)$. See [5], [25], [31] [43] for extensive treatments and surveys. $DT(S)$ contains – for each triple of points in S – the corresponding triangle provided its circumcircle is empty of points in S. Sibson [63] proved that $DT(S)$ can be constructed from any triangulation T of S by applying a sequence of *good edge flips*. These are exchanges of diagonals in one of T's convex quadrilaterals Q such that – after the flip – the two new triangles are locally Delaunay, i.e., have circumcircles empty of vertices of Q.

Various global optimality properties of $DT(S)$ can be proved by observing that every good flip gives a *local improvement* of the respective optimality measure. Lawson [45] observed that *equiangularity* of a triangulation, which is the sorted list of its angles, increases lexicographically in this way. $DT(S)$ thus maximizes the minimum angle. *Coarseness* of a triangulation is measured by the largest circumcircle that arises for its triangles. D'Azevedo and Simpson [16] showed that $DT(S)$ minimizes coarseness in this sense, and also if smallest enclosing circles are taken rather than circumcircles. The latter property – unlike others – generalizes to higher-dimensional Delaunay triangulations; see Rajan [59]. Similarly, *fatness* may be defined as the sum of triangle inradii. Lambert [44] pointed out that $DT(S)$ maximizes fatness or, equivalently, the mean inradius. Triangular surfaces obtained from lifting $DT(S)$ to 3-space minimize *roughness*, which is the integral of the gradient squared; see Rippa [60]. It is also known that a variant of $DT(S)$ minimizes the minimum angle; see Eppstein [29].

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combinatorial properties
Delaunay triangulation
local improvement

The Delaunay triangulation is a special instance of *regular triangulations*, which are obtained by projecting the lower part of a convex hull in 3-space; see e.g. Edelsbrunner and Shah [26]. Some more optimality properties of $DT(S)$ can be derived from this fact; see Musin [55]. $DT(S)$ is the geometric dual of the famous *Voronoi diagram* of a point set S and can be computed in $O(n \log n)$ time and $O(n)$ space by various different approaches; see e.g. [5], [25], [31]. Su and Drysdale [64] gave a thorough experimental comparison of available Delaunay triangulation algorithms.

On the negative side, $DT(S)$ fails to fulfill optimization criteria similar to those mentioned above, such as minimizing the maximum angle, or minimizing the longest edge. Edelsbrunner et al. [28], [27] gave $O(n^2 \log n)$ time and $O(n^2)$ time algorithms, respectively, for computing triangulations optimal in these respects. The former algorithm is based on an *edge insertion paradigm* which is shown in Bern et al. [7] to lead – in polynomial time – to triangulations with maxmin triangle height, minmax triangle eccentricity, and minmax gradient surface, respectively.

Minimum weight triangulations. Most longstanding open is another optimal triangulation problem, the *minimum weight triangulation*, in which the criterion is *weight*, which is defined as the sum of edge lengths. The complexity of computing a minimum weight triangulation, $MWT(S)$, for arbitrary planar point sets S is still open since 1975 when it was mentioned in Shamos and Hoey [61]. Minimum weight triangulation is included in Garey and Johnson's [33] list of problems neither known to be NP-complete, nor known to be solvable in polynomial time. Attempts to prove the problem NP-hard have resulted in some related NP-completeness results; they are listed in [36]. Several *heuristic algorithms* have been proposed to solve this problem; see Lingas [50], Plaisted and Hong [58], and Heath and Pemmaraju [36]. None of these is known to produce even a constant approximation in weight.

It is well known that the Delaunay triangulation $DT(S)$ may exceed $MWT(S)$ in weight by a factor of $\Theta(n)$; see Kirkpatrick [40]. Another popular triangulation, the *greedy triangulation*, $GT(S)$, also may fail to approximate $MWT(S)$ well. $GT(S)$ is obtained by a greedy algorithm intended to yield small weight: Edges are inserted in increasing length order unless previously inserted edges are crossed and until the triangulation is completed. Several fast implementations have been proposed, e.g. by Dickerson et al. [20] who also give a brief history. Levcopoulos [46] showed a lower bound of $\Omega(\sqrt{n})$ on the approximation factor. Recently, a matching upper bound, along with an interesting modification of $GT(S)$ that achieves a constant weight approximation for $MWT(S)$ in polynomial time, has been given in Levcopoulos and Krznaric [47].

For uniformly distributed point sets S, both triangulations $GT(S)$ and $DT(S)$ yield satisfactory approximations for $MWT(S)$; see e.g. Lingas [51]. In fact, $GT(S)$ seems to perform slightly better, as reported in Dickerson et al. [21]. $GT(S)$ can be constructed in $O(n)$ expected time in this case, by an algorithm in Drysdale et al. [24] or by a modification of the algorithm in Levcopoulos and Lingas [49].

The weight of a triangulation may decrease when additional points (so-called *Steiner points*) are admitted. Eppstein [30] showed that the weight of a *minimum weight Steiner triangulation* for S may be $\Omega(n)$ times smaller than the weight of $MWT(S)$. The same paper gives efficient triangulation algorithms that approximate the weight of the former within a constant factor, thus improving over previous results in Bern et al. [9]. No polynomial-time algorithms are known for the exact minimum weight Steiner triangulation problem.

regular triangulations
edge insertion paradigm
minimum weight triangulation
heuristic algorithms
greedy triangulation
minimum weight Steiner triangulation

Dynamic programming is a powerful tool to deal with some discrete optimization problems, and leads to polynomial-time solutions for some restricted instances of the minimum weight triangulation problem. For example, if S is the vertex set of a convex polygon then $MWT(S)$ can be computed in $O(n^3)$ time and $O(n^2)$ space. The basic observation used is that – once some triangle of the triangulation has been fixed – the problem splits into subproblems (subpolygons) whose solutions can be found recursively, thereby avoiding recomputation of common subproblems. The triangulation method, first proposed by Gilbert [34] and Klincsek [42], does not really exploit convexity. It works as well for non-convex polygons, and in fact for any interior face of a planar straight line graph. It is worth mentioning that, in the convex polygon case, $MWT(S)$ is approximated by $GT(S)$ up to a constant factor; see Levcopoulos and Lingas [48]. Anagnostou and Corneil [4] consider the case where S gives rise to a small number k of convex layers (nested convex hulls). Their dynamic programming approach works in time $O(n^{3k+1})$ and thus is polynomial for constant k. Meijer and Rappaport [54] later improved the bound to $O(n^k)$ when S is restricted to lie on k non-intersecting line segments.

The minimum weight triangulation problem can also be formulated as a *linear programming* problem. To this end, a variable x_i is assigned to each of the $\binom{n}{2}$ edges e_i defined by S. The objective is to minimize

$$\sum_{e_i} x_i |e_i|$$

subject to the constraints

$$\begin{aligned} 0 \leq x_i &\leq 1 \\ x_i + x_j &\leq 1 \text{ for } e_i \cap e_j \neq \emptyset \\ x_i + \sum_{e_j \cap e_i \neq \emptyset} x_j &\geq 1 \end{aligned}$$

The last two constraints express the property that a triangulation is a maximal set of non-crossing edges. An integer solution of this linear program yields a minimum weight tringulation: For each edge e_i, inclusion into, or exclusion from $MWT(S)$ is indicated by $x_i = 1$ or $x_i = 0$, respectively. An optimal solution need not be integer, however, and insisting on an integer solution results in an *integer programming* problem whose general version is known to be NP-complete [33]. Using a modified version of this approach, Ono et al. [57] were able to compute $MWT(S)$ for up to a hundred points.

The above-mentioned polynomial-time results for triangulating polygonal domains give motivation for the following *subgraph approach* to compute $MWT(S)$, proposed e.g. in Xu [66] and Cheng et al. [12]. Find a (suitable) subgraph G of $MWT(S)$. If G contains k connected components, try all possibilities to add $k-1$ edges to make it a connected graph C. Complete each of these graphs C to a triangulation by optimally triangulating its faces, and select a triangulation with minimum weight, which gives $MWT(S)$. This approach, which basicly is exhaustive search, can be implemented to run in $O(n^{k+2})$ time. The problem, of course, is to find candidates for G with k small.

The subgraph approach should be distinguished from the heuristic approaches in [50], [58], [36] which also first fix some graph G for S (for example, the minimum spanning tree) and then triangulate out its faces. The difference is that G will *not* be a subgraph of $MWT(S)$ in general, and thus does not lead to an exact solution.

Locally defined subgraphs. Many efforts have been put into the study of subgraphs of $MWT(S)$. Gilbert [34] pointed out that the *shortest edge* defined by S always belongs to $MWT(S)$. Another simple observation is that *unavoidable edges*, which are edges not being crossed by any other edge in $S \times S$, have to appear in any triangulation of S and thus are in

Dynamic programming
linear programming
subgraph approach
unavoidable edges

$MWT(S)$. For example, all edges of the convex hull of S are unavoidable. The number of unavoidable edges does not exceed $2n-2$, see Xu [67], and usually is very small.

Only in recent years, several less trivial subgraphs of $MWT(S)$ have been identified. One of them arises from a class of *empty neighborhood graphs* introduced by Kirkpatrick and Radke [41]. Let $p, q \in S$ and $\beta \geq 1$. The edge $\overline{pq}$ is included in the *β-skeleton*, $\beta(S)$, if the two discs of diameter $\beta|pq|$ and passing through both p and q are empty of points in S. It is not hard to see that $\beta(S)$ always is a subgraph of the Delaunay triangulation $DT(S)$. In fact, $\beta(S)$ can be constructed from $DT(S)$ in $O(n)$ time; see Lingas [52] or Jaromczyk et al. [38].

Interestingly, $\beta(S)$ is a subgraph of $MWT(S)$ for β large enough. The original bound $\beta \geq \sqrt{2}$ in Keil [39] has been improved later in Cheng and Xu [14] to $\beta > 1.1768$, which is close to the largest value $2/\sqrt{3}$ for which a counterexample is available. Only for point sets S in convex position it is known that $\beta > 2/\sqrt{3}$ always implies inclusion of $\beta(S)$ in $MWT(S)$; see Hainz et al. [35]. Whereas $\beta(S)$ is a connected graph for $\beta = 1$ (known as the *Gabriel graph* of S), it may be highly disconnected for larger β-values. Cheng et al. [12] prove that – for uniformly distributed point sets S – the number of components is expected to be $\Theta(n)$.

Yang et al. [68] formulated and proved a different inclusion region: If the union of the two disks centered at p and q with radius $|pq|$ is empty of points in S, then $\overline{pq}$ is an edge of $MWT(S)$. The skeleton generated in this way and the β-skeleton do not contain each other for $\beta > 2/\sqrt{3}$, but for smaller β-values the β-skeleton contains the former as a subgraph. Note that both graphs are defined via a symmetric and local condition. A sufficient asymmetric condition can be found in Wang et al. [65].

A distinct attempt to find a sufficient local condition defines an edge e as a *light edge* if there is no edge in $S \times S$ crossing e and shorter than e. Let $L(S)$ denote the graph formed by the light edges for S. $L(S)$ obviously contains all unavoidable edges, and in fact contains both Cheng and Xu's 1.1768-skeleton and the skeleton in Yang et al. By construction, $L(S)$ is a subgraph of the greedy triangulation $GT(S)$, as light edges cannot be blocked by any edge previously inserted by the greedy strategy. On the other hand, $L(S)$ is no subgraph of $MWT(S)$ in general. However, if $L(S)$ happens to be a full triangulation of S, then $L(S) = MWT(S) = GT(S)$. This allows for a fast computation of $MWT(S)$ for a certain class of point sets S, using greedy triangulation algorithms.

These results are observed in Aichholzer et al. [1] as a consequence of the following matching theorem for planar triangulations, proved in Aichholzer et al. [2] and in Cheng and Xu [13]: For any two triangulations T_1 and T_2 of a fixed point set S, there is a perfect matching between the edge set of T_1 and the edge set of T_2 such that matched edges either cross or are identical. A similar matching exists for the triangle sets of T_1 and T_2, where matched triangles either overlap or are identical. The paper [1] further gives several polynomially-time computable *lower weight bounds* for $MWT(S)$, the simplest one being the weight of $L(S)$. Another bound can be stated as

$$\min_{g \in X(E)} \sum_{e \in E} |g(e)|$$

where E is any set of non-crossing edges, and $X(E)$ is the set of all matchings $g : E \to S \times S$ with the property $g(e) = e$ or $g(e)$ crosses e.

The concept of light edges gives rise to a partition of the greedy triangulation $GT(S)$ into levels $L_1, \ldots, L_k$: Let $L_1 = L(S)$ be the set of edges that are light with respect to $S \times S$, let L_2 be the set of edges that are light with respect to $(S \times S) \setminus C_1$, where C_1 is the set of edges crossing L_1, and so on. Upper bounds on the ratio in weight between $GT(S)$ and $MWT(S)$ that depend on the number k of levels are observed in Levcopoulos and Krznaric [47] and Aichholzer et al. [3]. In particular, $GT(S)$ is a constant-factor approximation of $MWT(S)$ provided $k = O(1)$.

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empty neighborhood graphs
light edge
lower weight bounds

Globally defined subgraphs. We have seen that several subgraphs of $MWT(S)$ can be found from some local conditions. The subgraphs below are defined in a global way, via intersection of triangulations.

Call a triangulation T of S *locally minimal* if every 4-sided and point-empty polygon drawn by T is optimally triangulated. That is, every convex quadrilateral contains the shorter of its two diagonals. It is easy to see that $GT(S)$ is locally minimal and $DT(S)$, in general, is not. Let $LMT(S)$ denote the intersection of all locally minimal triangulations for S. Then $LMT(S)$ is a subgraph of $MWT(S)$, as this triangulation of course is locally minimal, too.

Whereas it is not known how to compute $LMT(S)$ in polynomial time, a surprisingly large subgraph of $LMT(S)$, the so-called *LMT-skeleton* can be identified by a simple method, proposed in Belleville et al. [6] and in Dickerson and Montague [22]: Consider some edge set $E \subset S \times S$. An edge $e \in E$ is called *redundant in* E if there is no convex quadrilateral formed by E that has e as its shorter diagonal. Edge e is called *unavoidable in* E if no other edge in E crosses e. The LMT-skeleton algorithms puts $E = S \times S$ and proceeds in several rounds. Each round first identifies all edges redundant in E and eliminates them from the set, and then includes all edges that are unavoidable in the reduced set E into the LMT-skeleton. The algorithm stops when no more edge in E can be classified as either redundant or unavoidable. The number of rounds (but not the produced LMT-skeleton) depends on the ordering in which the edges are examined. In particular, there always exists an ordering such that one round suffices; see Hainz et al. [35].

The fact that the LMT-skeleton for S, and thus $LMT(S)$, tend to be connected even for large point sets comes at a surprise. From the practical point of view, the LMT-skeleton almost always nearly triangulates S. On the other hand, a 19-point counterexample to connectedness exists [6], and for uniformly distributed points, the expected number of components is $\Theta(n)$; see Bose et al. [10]. The constant of proportionality is extremely small, however. It is interesting to note that the LMT-skeleton, and the graph of light edges $L(S)$, exhibit a similar behavior of connectedness, but do not contain each other in general. We mention further that the improved LMT-algorithm in [35], that tends to yield some additional edges of $LMT(S)$, indeed exactly constructs $LMT(S)$ provided the connectedness of this structure.

The LMT-skeleton clearly can be constructed in polynomial time, and several variants have been considered in order to gain efficiency [6], [22], [11], [35]. A powerful tool is pre-exclusion of edges before starting the LMT-algorithm, using the *exclusion region* in Das and Joseph [15]: For an edge e, consider the two triangular regions with base e and base angles $\frac{\pi}{8}$. If both regions contain points in S then e cannot be part of $MWT(S)$. If S is drawn from a uniform distribution, reduction to an expected linear number of candidate edges for $MWT(S)$ is achieved [20], and near-linear expected-time implementations of the LMT-algorithm exist [35], [62]. The LMT-skeleton based subgraph approach enables the computation of a minimum weight triangulation for some tenthousand points in reasonable time.

Some open problems. Let us briefly state a few open problems related to minimum weight triangulations. The most important one, of course, is to determine the complexity status of finding a minimum weight triangulation of a general planar point set. An algorithm for computing a minimum weight triangulation of a convex polygon in less than $\Theta(n^3)$ time would have many applications and thus is of practical interest. Finally, minimality in weight may be relaxed to *k-optimality*, meaning that all k-sided and point-empty polygons in a triangulation are optimally triangulated. This is a generalization of

locally minimal
LMT-skeleton
exclusion region
k-optimality

local minimality which constitutes the instance $k = 4$. Whereas it is easy to compute 4-optimal triangulations (the greedy triangulation is one), no results are known on how to compute a 5-optimal triangulation in polynomial time.

References

[1] O.Aichholzer, F.Aurenhammer, S.-W.Cheng, N.Katoh, G.Rote, M.Taschwer, Y.-F.Xu: 'Triangulations intersect nicely', *Discrete&Computational Geometry* **16** (1996) pp. 339-359

[2] O.Aichholzer, F.Aurenhammer, G.Rote, M.Taschwer: 'Triangulations intersect nicely', *Proc. 11th Ann. ACM Symp. on Computational Geometry*, (1995) pp. 220-229

[3] O.Aichholzer, F.Aurenhammer, G.Rote, Y.-F.Xu: 'Constant-level greedy triangulations approximate the MWT well', *Journal of Combinatorial Optimization* (to appear.)

[4] E.Anagnostou, D.Corneil: 'Polynomial-time instances of the minimum weight triangulation problem', *Computational Geometry: Theory and Appications* **3** (1993) pp. 247-259

[5] F.Aurenhammer: 'Voronoi diagrams – a survey of a fundamental geometric data structure', *ACM Computing Surveys* **23** (1991) pp. 345-405

[6] P.Belleville, M.Keil, M.McAllister, J.Snoeyink: 'On computing edges that are in all minimum-weight triangulations', *Proc. 12th Ann. ACM Symp. on Computational Geometry*, (1996) pp. V7-V8

[7] M.Bern, H.Edelsbrunner, D.Eppstein, S.Mitchell, T.S.Tan: 'Edge insertion for optimal triangulations', *Discrete&Computational Geometry* **10** (1993) pp. 47-65

[8] M.Bern, D.Eppstein: 'Mesh generation and optimal triangulation', D.-Z.Du, F.K.Hwang (ed.): *Computing in Euclidean Geometry, Lecture Notes Series in Computing 4*, World Scientific, Singapore (1995) pp. 47-123

[9] M.Bern, D.Eppstein, J.Gilbert: 'Provably good mesh generation', *Proc. 31st IEEE Symp. on Foundations of Computer Science*, (1990) pp. 231-241

[10] P.Bose, L.Devroye, W.Evans: 'Diamonds are not a minimum weight triangulation's best friend', *Proc. 8th Canadian Conf. on Computational Geometry*, (1996) pp. 68-73

[11] S.-W.Cheng, N.Katoh, M.Sugai: 'A study of the LMT-skeleton', *Proc. Int. Symp. on Algorithms and Computation (ISAAC), Lecture Notes in Computer Science 1178*, Springer Verlag (1996) pp. 256-265

[12] S.-W.Cheng, M.J.Golin, J.C.F.Tsang: 'Expected-case analysis of β-skeletons with applications to the construction of minimum-weight triangulations', *Proc. 7th Canadian Conf. on Computational Geometry*, (1995) pp. 279-283

[13] S.-W.Cheng, Y.-F.Xu: 'Constrained independence system and triangulations of planar point sets', *Computing and Combinatorics, Proc. 1st Ann. Int. Conf. COCOON, Lecture Notes in Computer Science 959*, Springer Verlag (1995) pp. 41-50

[14] S.-W.Cheng, Y.-F.Xu: 'Approaching the largest β-skeleton within a minimum-weight triangulation', *Proc. 12th Ann. ACM Symp. on Computational Geometry*, (1996) pp. 196-203

[15] G.Das, G.Joseph: 'Which triangulations approximate the complete graph?', *Optimal Algorithms, Lecture Notes in Computer Science 401*, Springer Verlag (1989) pp. 168-192

[16] E.F.D'Azevedo, R.B.Simpson: 'On optimal interpolation triangle incidences', *SIAM J. Sci. Statist. Comput.* **10** (1989) pp. 1063-1075

[17] L.De Floriani: 'Surface representation based on triangular grids', *The Visual Computer* **3** (1987) pp. 27-50

[18] M.Denny, C.Sohler: 'Encoding a triangulation as a permutation of its point set', (Manuscript, Univ. des Saarlandes, Saarbruecken, Germany, 1997.)

[19] T.K.Dey, M.B.Dillencourt, S.K.Ghosh: 'Triangulating with high connectivity', *Proc. 6th Canadian Conf. on Computational Geometry*, (1994) pp. 339-343

[20] M.T.Dickerson, R.L.S Drysdale, S.A.McElfresh, E.Welzl: 'Fast Greedy Triangulation Algorithms', *Computational Geometry: Theory and Applications* **8** (1997) pp. 67-86

[21] M.T.Dickerson, S.A.McElfresh, M.H.Montague: 'New algorithms and empirical findings on minimum weight triangulation heuristics', *Proc. 11th Ann. ACM Symp. on Computational Geometry*, (1995) pp. 238-247

[22] M.T.Dickerson, M.H.Montague: 'A (usually?) connected subgraph of the minimum weight triangulation', *Proc. 12th Ann. ACM Symp. on Computational Geometry*, (1996) pp. 204-213

[23] M.B.Dillencourt: 'Toughness and Delaunay triangulations', *Discrete&Computational Geometry* **5** (1990) pp. 575-601

[24] R.L.S Drysdale, G.Rote, O.Aichholzer: 'A simple linear time greedy triangulation algorithm for uniformly distributed points', (Rep. 408, IIG, Graz Univ. of Technology, Graz, Austria, 1995.)

[25] H.Edelsbrunner: 'Algorithms in Combinatorial Geometry, EATCS Monographs on Theoretical Computer Science 10', Springer-Verlag, Heidelberg (1987)

[26] H.Edelsbrunner, N.R.Shah: 'Incremental topological flipping works for regular triangulations', *Algorithmica* **15** (1996) pp. 223-241

[27] H.Edelsbrunner, T.S.Tan: 'A quadratic time algorithm for the minmax length triangulation', *SIAM J. Comput.* **22** (1993) pp. 527-551

[28] H.Edelsbrunner, T.S.Tan, R.Waupotitsch: 'An $O(N^2 \log N)$ time algorithm for the minmax angle triangulation', *SIAM J. Sci. Statist. Comput.* **13** (1992) pp. 994-1008

[29] D.Eppstein: 'The farthest point Delaunay triangulation minimizes angles', *Computational Geometry: Theory and Applications* **1** (1992) pp. 143-148

[30] D.Eppstein: 'Approximating the minimum weight steiner triangulation', *Discrete&Computational Geometry* **11** (1994) pp. 163–194

[31] S.Fortune: 'Voronoi Diagrams and Delaunay Triangulations', D.-Z.Du, F.K.Hwang (ed.): *Computing in Euclidean Geometry, Lecture Notes Series in Computing 4*, World Scientific, Singapore (1995) pp. 225-265

[32] A.Garcia, M.Noy, J.Tejel: 'Lower bounds for the number of crossing-free subgraphs of K_n', *Proc. 7th Canadian Conf. on Computational Geometry*, (1995) pp. 97-102

[33] M.Garey, D.Johnson: 'Computers and Intractability. A Guide to the Theory of NP-completeness', W.H.Freeman (1979)

[34] P.D.Gilbert: 'New results in planar triangulation', (M.S. thesis, Coordinated Science Laboratory, University of Illinois, Urbana, 1979.)

[35] R.Hainz, O.Aichholzer, F.Aurenhammer: 'New results on minimum weight triangulations and the LMT-skeleton (extended abstract)', *Proc. 13th European Workshop on Computational Geometry*, (1997) pp. 4-6

[36] L.S.Heath, S.V.Pemmaraju: 'New results for the minimum weight triangulation problem', *Algorithmica* **12** (1994) pp. 533-552

[37] K.Jansen: 'One strike against the min-max degree triangulation problem', *Computational Geometry: Theory and Applications* **3** (1993) pp. 107-120

[38] J.W.Jaromczyk, M.Kowaluk, F.Yao: 'An optimal algorithm for constructing β-skeletons in the L_p-metric', *SIAM J. Comput.* (to appear.)

[39] M.Keil: 'Computing a subgraph of the minimum weight triangulation', *Computational Geometry: Theory and Applications* **4** (1994) pp. 13-26

[40] D.G.Kirkpatrick: 'A note on Delaunay and optimal triangulations', *Information Processing Letters* **10** (1980) pp. 127-128

[41] D.G.Kirkpatrick, J.D.Radke: 'A framework for computational morphology', G.T.Toussaint (ed.): *Computational Geometry*, Elsevier, Amsterdam (1985) pp. 217-248

[42] G.T.Klincsek: 'Minimal triangulations of polygonal domains', *Annals of Discrete Mathematics* **9** (1980) pp. 127-128

[43] T.Lambert: 'Empty-shape Triangulation Algorithms', (Ph.D. thesis, Dept. of Computer Science, Univ. of Manitoba, Winnipeg, MB, 1994.)

[44] T.Lambert: 'The Delaunay triangulation maximizes the mean inradius', *Proc. 6th Canadian Conf. on Computational Geometry*, (1994) pp. 201-206

[45] C.L.Lawson: 'Software for C^1 surface interpolation', J.R.Rice (ed.): *Mathematical Software III*, Academic Press, New York (1977) pp. 161-194

[46] C.Levcopoulos: 'An $\Omega(\sqrt(n))$ lower bound for non-optimality of the greedy triangulation', *Information Processing Letters* **25** (1987) pp. 247-251

[47] C.Levcopoulos, D.Krznaric: 'Quasi-greedy triangulations approximating the minimum weight triangulation', *Proc. 7th Ann. ACM–SIAM Symp. on Discrete Algorithms*, (1996) pp. 392–401

[48] C.Levcopoulos, A.Lingas: 'On approximation behavior of the greedy triangulation for convex polygons', *Algorithmica* **2** (1987) pp. 175-193

[49] C.Levcopoulos, A.Lingas: 'Fast algorithms for greedy triangulation', *BIT* **32** (1992) pp. 280-296

[50] A.Lingas: 'A new heuristic for the minimum weight triangulation', *SIAM J. Algebraic and Discrete Methods* **8** (1987) pp. 646-658

[51] A.Lingas: 'The greedy and Delaunay triangulations are not bad in the average case', *Information Processing Letters* **22** (1986) pp. 25-31

[52] A.Lingas: 'A linear-time construction of the relative neighborhood graph from the Delaunay triangulation', *Computational Geometry: Theory and Applications* **4** (1994) pp. 199-208

[53] E.L.Lloyd: 'On triangulations of a set of points in the plane', *Proc. 18th IEEE Symp. on Foundations of Computer Science*, (1977) pp. 228-240

[54] H.Meijer, D.Rappaport: 'Computing the minimum weight triangulation for a set of linearly ordered points', *Information Processing Letters* **42** (1992) pp. 35-38

[55] O.R.Musin: 'Properties of the Delaunay triangulation', *Proc. 13th Ann. ACM Symp. on Computational Geometry*, (1997) pp. 424-426

[56] A.Okabe, B.Boots, K.Sugihara: 'Spatial Tesselations: Concepts and Applications of Voronoi Diagrams', John Wiley & Sons, Chichester, England (1992)

[57] T.Ono, Y.Kyoda, T.Masade, K.Hayase, T.Shibuja, M.Nakade, M.Inabe, H.Imai, K.Imai, D.Avis: 'A package for triangulations', *Proc. 12th. Ann. ACM Symp. on Computational Geometry*, (1996) pp. V17-V18

[58] D.A.Plaisted, J.Hong: 'A heuristic triangulation algorithm', *J. Algorithms* **8** (1987) pp. 405-437

[59] V.T.Rajan: 'Optimality of the Delaunay triangulation in R^d', *Discrete&Computational Geometry* **12** (1994) pp. 189-202

[60] S.Rippa: 'Minimal roughness property of the Delaunay triangulation', *Computer Aided Geometric Design* **7** (1990) pp. 489-497

[61] M.I.Shamos, D.Hoey: 'Closest point problems', *Proc. 16th IEEE Symp. on Foundations of Computer Science*, (1975) pp. 151-162

[62] J.Snoeyink: (personal communication.)

[63] R.Sibson: 'Locally equiangular triangulations', *The Computer Journal* **21** (1978) pp. 243-245

[64] P.Su, R.L.S.Drysdale: 'A comparison of sequential Delaunay triangulation algorithms', *Proc. 11th Ann. ACM Symp. on Computational Geometry*, (1995) pp. 61-70**21**

[65] C.A.Wang, F.Chin, Y.F.Xu: 'A new subgraph of minimum weight triangulations', *Journal of Combinatorial Optimization* **1** (1997) pp. 115-127

[66] Y.-F.Xu: 'Minimum weight triangulation problem of a planar point set', (Ph.D. thesis, Institute of Applied Mathematics, Academia Sinica, Beijing, 1992.)

[67] Y.-F.Xu: 'On stable line segments in all triangulations', *Appl.Math.-JCU* **11B** (1996) pp. 235-238

[68] B.-T.Yang, Y.-F.Xu, Z.-Y.You: 'A chain decomposition algorithm for the proof of a property on minimum weight triangulations', *Proc. 5th Int. Symp. on Algorithms and Computation (ISAAC), Lecture Notes in Computer Science 834*, Springer-Verlag (1994) pp. 423-427

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