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ABSTRACT

Let K be a convex polyhedron in $\mathbf{R}^3$. Let A(x) denote the area of the intersection of K with a plane orthogonal to the x-axis. We show that A(x) need not be a *convex* function of x. Nevertheless we show that A(x) is *unimodal*. Let K be tangent to the xy plane and let R(w) denote the area of the intersection of K with a plane that is rotated by an angle w about the y axis. We show that there exist convex polyhedra such that R(w) is a *multimodal* function. In fact, for a convex polyhedron of n vertices, R(w) may have O(n) local maxima. As an application of the unimodality of A(x) we show that given a convex polyhedron K in $\mathbf{R}^3$ and a directional vector v, the section of K orthogonal to v with maximum area may be computed in O(n log n) time. On the other hand the multimodality of R(w) implies $\Omega(n)$ lower bounds on the complexity of certain geometric problems for convex polyhedra.

1. Introduction

Let Q be a convex polygon of n vertices in the plane. It was shown in [DJ73] that if $l(x)$ is the length of the intersection of a line orthogonal to the x-axis with Q, then $l(x)$ is a *unimodal* function of x, i.e., it has one local maximum. It is shown in [Ch80], [CD87] that given a line L and a convex polygon Q then the perpendicular distance function of the vertices of Q from L is bimodal from which a unimodal function is constructed. Such unimodality properties of functions are important for the design of efficient searching algorithms because they permit the application of *prune-and-search* strategies such as binary search or Fibonacci search [Ki53]. For example, the latter property implies that given a convex polygon Q and a line L, the point in Q furthest from L can be computed in O(log n) time [Ch80]. This in turn implies that given a diagonal of Q, the maximum-area triangle determined by this diagonal and a third vertex of Q may be found in O(log n) time. Such unimodality properties also simplify proofs of geometric properties. For example, Pach [Pa78] gave a 9-case combinatorial argument to prove that the minimal-area triangle determined by three vertices of a convex polygon has two sides which are edges of the polygon. An immediate simple proof of this result follows from the second of the above unimodality properties. Note that this observation allows the minimal-area triangle to be found in O(n) time, a marked improvement over the O(n^3) naive approach. As an application of the first unimodality property mentioned above we point out in passing that it provides an alternate proof to that of Kirkpatrick and Snoeyink [KS93] that the longest vertical cut of a convex polygon can be found in O(log n) time.

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On the other hand, let q be a vertex of Q and consider the distance function $r(z)=d(q,z)$ as z travels around the boundary of Q starting at q and ending at q, where $d(q,z)$ denotes the Euclidean distance between q and z. Then $r(z)$ need not be unimodal. In fact it was shown in [ATB82] that for a convex polygon with n vertices $r(z)$ may have $O(n)$ local maxima. These results have implications for computing several properties of polygons such as, for example, the diameter of a convex polygon [Ab90], [To84] and the minimum vertex distance between two convex polygons [MT85].

In this note we extend these results from distances in two dimensions to areas in three dimensions. Let K be a convex polyhedron in $\mathbf{R}^3$. Let $A(x)$ denote the area of the intersection of K with a plane orthogonal to x. We show that $A(x)$ need not be a convex function of x but is *unimodal*. Let K be tangent to the xy plane and let $R(w)$ denote the area of the intersection of K with a plane that is rotated about the y axis. We show that there exist convex polyhedra such that $R(w)$ is a multimodal function of the angle of rotation w. In fact, for a convex polyhedron of n vertices, $R(w)$ may have $O(n)$ local maxima. As an application of the unimodality of $A(x)$ we show that given a convex polyhedron K in $\mathbf{R}^3$ and a directional vector v, the section of K orthogonal to v with maximum area may be computed in $O(n \log n)$ time. On the other hand the multimodality of $R(w)$ implies $\Omega(n)$ lower bounds on the complexity of several geometric problems for convex polyhedra.

2. Translational Sections

Let K be a convex polyhedron in $\mathbf{R}^3$. Let $A(x)$ denote the area of the intersection of K with a plane orthogonal to x. In this section we show that $A(x)$ is not necessarily a *convex* function but must be a *unimodal* function of x. The proof rests on the Brunn-Minkowski Theorem, a powerful inequality between areas of polygons in the plane [S-Y93]. Suppose that we are given three convex polygons in $\mathbf{R}^2$, Q_1, Q_2 and Q_3, such that Q_2 is a linear combination of Q_1, and Q_3, i.e., $Q_2 = \lambda Q_1 + (1-\lambda)Q_3$, where $0 \le \lambda \le 1$. In other words, for any fixed λ, Q_2 is the set of points $\lambda p + (1-\lambda)q$ for all points p and q such that $p \in Q_1$ and $q \in Q_2$. Let $A(Q_i)$ denote the area of Q_i for i=1,2,3. Then the Brunn-Minkowski theorem states that:

$$\sqrt{A(Q_2)} \ge \lambda \sqrt{A(Q_1)} + (1-\lambda)\sqrt{A(Q_3)}$$

in which equality holds if, and only, if Q_1, and Q_3 are *homothetic*. Recall that a homothet of a convex set Q has the form $x + \lambda Q$ where $\lambda > 0$ and $x \in \mathbf{R}^2$.

Now let us return to our polyhedron K in $\mathbf{R}^3$. We first show that $A(x)$ is unimodal. Assume that $A(x)$ is not unimodal, i.e., that there exists at least one local minimum. Let x^* denote the value of x for which $A(x)$ assumes this local minimum. Let S_2 denote the intersection of K with a plane orthogonal to x at x^*, and let $A(S_2)$ denote the corresponding minimal area. Since x^* is the point of minimal area there must exist a sufficiently small ε greater than zero such that at $x^*- \varepsilon$ and at $x^*+ \varepsilon$ the intersections S_1 and S_3 of K with parallel planes intersecting x at $x^*- \varepsilon$ and $x^*+ \varepsilon$ must have areas, $A(S_1)$ and $A(S_3)$ respectively, such that $A(S_1) > A(S_2)$ and $A(S_3) > A(S_2)$. Without loss of generality, assume $A(S_1) \ge A(S_3)$, and consider the solid $\overline{S}$ defined as the convex hull of the union of S_1 and S_3. Furthermore, denote by $A(\overline{S}_2)$ the area of the intersection of solid $\overline{S}$ with a plane orthogonal to x at x^*. Lyusternik [Ly66] has shown that all the steps in the proof of the Brunn-Minkowski theorem in the plane hold for polygons which are parallel sections of the solid $\bar{S}$. Therefore we have that:

$$\sqrt{A(\bar{S}_2)} \geq \lambda\sqrt{A(S_1)} + (1-\lambda)\sqrt{A(S_3)}$$

Furthermore, since solid $\bar{S}$ is the convex hull of subsets contained in K it follows that $\bar{S}$ must be contained in K and therefore $A(S_2) \geq A(\bar{S}_2)$. Therefore we have:

$$\sqrt{A(S_2)} \geq \lambda\sqrt{A(S_1)} + (1-\lambda)\sqrt{A(S_3)}$$

In the above construction $A(S_1) \geq A(S_3)$ by assumption and therefore $A(S_2) \geq A(S_3)$ which contradicts the fact that a local minimum implies $A(S_3) > A(S_2)$. We have therefore proved the following.

Theorem 2.1: Given a convex polyhedron K in $\mathbf{R}^3$, the intersection area of K with a plane orthogonal to x is a unimodal function of x.

Theorem 2.1 can be strengthened by showing that $A(x)$ need not be concave. Recall that a function $f(x)$ is concave if $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$. Now consider the following example in which two homothets Q_1 and Q_3 are such that $A(Q_1) = 1$ and $A(Q_3) = 4$. Furthermore, let $Q_2 = (1/2)Q_1 + (1/2)Q_3$. The Brunn-Minkowski theorem implies that

$$\sqrt{A(Q_2)} = \left(\frac{1}{2}\right)\cdot 1 + \left(\frac{1}{2}\right)\cdot 2 = \left(\frac{3}{2}\right)$$

so that $A(Q_2) = 9/4$. However,

$$\left(\frac{1}{2}\right)A(Q_1) + \left(\frac{1}{2}\right)A(Q_3) = \left(\frac{1}{2}\right)\cdot 1 + \left(\frac{1}{2}\right)\cdot 4 = \left(\frac{10}{4}\right) > A(Q_2)$$

contradicting the definition of concavity.

An example of a 12-vertex convex polyhedron is described in the appendix for which the function $A(x)$ is neither convex nor concave. A plot of $A(x)$ is also illustrated.

3. Rotational Sections

In this section we show through an example that if instead of translating a plane across a convex polyhedron, we rotate the plane about some axis then unimodality of the sectional area no longer holds. In fact we construct a convex n-vertex polyhedron whose rotational sectional area function contains $O(n)$ local maxima as the plane in question is rotated. The construction is inspired by the convex polygon construction of [ATB82] that yields a multimodal function of the distances from one vertex to all the others (refer to Fig. 3.1).

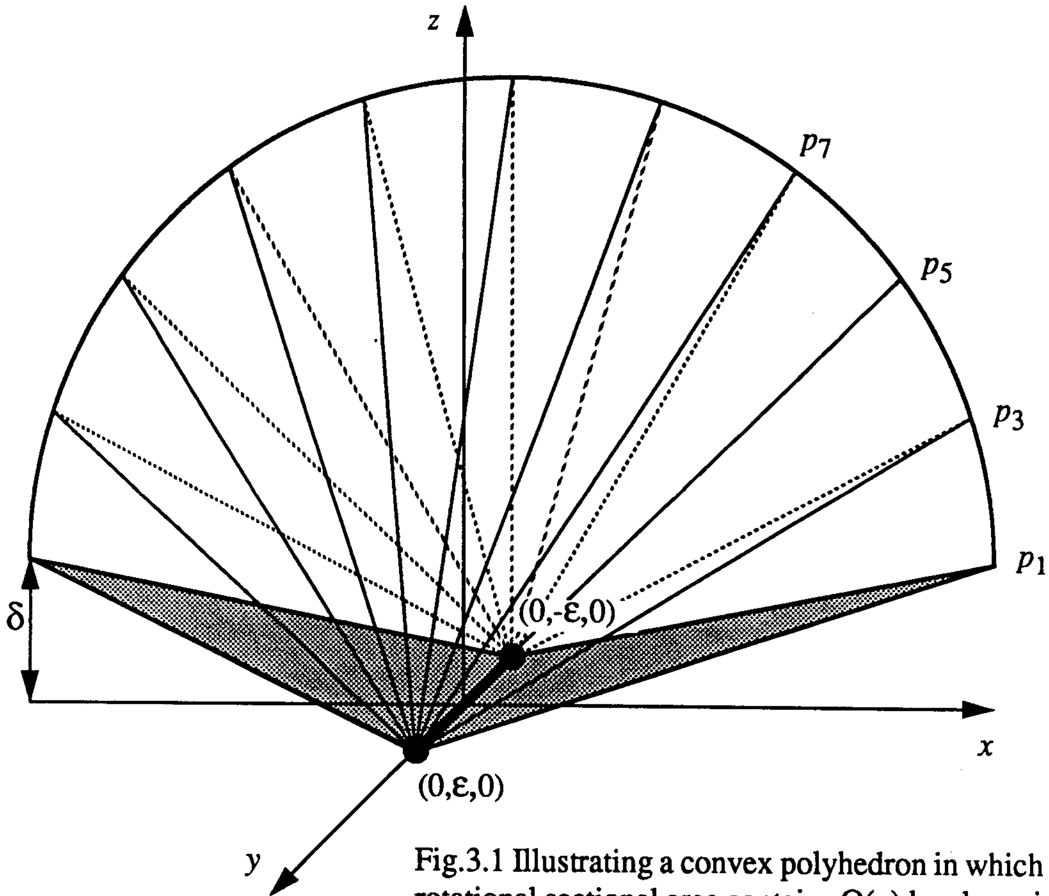

Fig.3.1 Illustrating a convex polyhedron in which the rotational sectional area contains O(n) local maxima.

First construct the circle C determined by the equation $x^2 + z^2 = 1$. We refer to z as the vertical direction. We will use C only above the xy-plane. Next, we create three types of points in space that will become the vertices of our desired polyhedron. There are two type-I points located at $(0,\varepsilon,0)$ and $(0,-\varepsilon,0)$ where ε is a fixed positive number. Let C' denote the portion of C with z-coordinate greater than a sufficiently small but positive δ. We now place $(n-2)/2$ type-II points equally spaced on C'. C' is now partitioned into $[(n-2)/2]-1$ arcs. Consider each arc determined by a pair of adjacent points $arc(p_i,p_j)$ on C'. Let $e[p_i,p_j]$ denote the straight line segment connecting p_i and p_j. The remaining $[(n-2)/2]-1$ type-III points are placed "in-between" the points already on C'. More precisely, corresponding to each arc $arc(p_i,p_j)$ we place a type-III point on the xz-plane strictly below $arc(p_i,p_j)$ and strictly above $e[p_i,p_j]$. Finally we take the convex hull of all the n points as our polyhedron K. To see that all n points appear in K and that K is a convex polyhedron it suffices to establish that every vertex of K is convex. A vertex v of a polyhedron K is convex provided that there exists a plane H through v such that all vertices of K adjacent to v lie in one open half-space determined by H (call it H*) and such that if H is translated by a sufficiently small amount in a direction towards any point in H*, then the intersection of H with K in a sufficiently small neighborhood of v is a convex polygon. The type-I vertices are convex because all other vertices lie above the xy-plane in which they lie, because all other vertices lie on the xz-plane, and because each type-III vertex such as p_k has z-coordinate smaller than its vertical projection on $arc(p_{k-1},p_{k+1})$ and greater than its vertical projection on $e[p_{k-1},p_{k+1}]$. The type-III vertices are convex for the same reasons. The type-II vertices are convex for the additional reason that each such vertex p_j is such that its adjacent vertices p_{j-1} and p_{j+1} both lie strictly below the line contained in the *xz*-plane and tangent to C' at p_j.

It remains to show that there exists a plane such that when it is rotated the sectional area it makes with K is multimodal. Let R(*w*) denote the area of intersection of a plane H(*w*) that starts out as the *xy*-plane and is rotated about the *y*-axis by *w* degrees, $0 \leq w \leq 180$. When H(*w*) intersects K at points other than the edge e'= [(0,ε,0),(0,-ε,0)] the intersection forms a triangle. Every such intersecting triangle contains e' as its base. Now whenever H(*w*) intersects a type-II vertex the height of the triangle is equal to the radius *r* of circle C and thus the area equals 0.5e'*r*. On the other hand, whenever H(*w*) intersects a type-III vertex the height of the triangle is equal to a value *r*' less than the radius *r* and therefore the area equals 0.5e'*r*' < 0.5e'*r*. Therefore as H(*w*) is rotated each type-III vertex yields a local minimum for R(*w*). We have thus established the following.

Theorem 3.1: There exist convex *n*-vertex polyhedra such that if a plane is rotated about some axis in space, the area of intersection of the rotating plane with such a polyhedron yields a function of angle of rotation which may contain O(*n*) local maxima.

This theorem has implications for the problem of computing the maximum-area triangle determined by three vertices of P, two of which determine a pre-specified diagonal of P. Recall that this problem in two dimensions can be solved in O(log *n*) time. On the other hand, theorem 3.1 implies the following.

Theorem 3.2: Given a convex polyhedron K of *n* vertices and a specified diagonal *d* of P, computing the maximum-area triangle determined by *d* and a third vertex of K takes time Θ(*n*).

4. Applications

4.1 Computing the maximum area section of a convex polyhedron

In this section we show how the unimodality property of the sectional area allows us to find, given an *n*-vertex convex polyhedron K and a direction ϕ, the maximum-area intersection of K with a plane orthogonal to ϕ. Without loss of generality we assume ϕ to be the *x*-axis.

First we show how to compute in linear time the maximum area cross-section of a special type of polyhedron called a *drum*. Let P = $[p_1, p_2, \ldots, p_m]$ be a convex polygon in a plane H_1 in $\mathbf{R}^3$. Let Q = $[q_1, q_2, \ldots, q_n]$ be a convex polygon in a plane H_2 which is parallel to, but not identical to, H_1 and parallel to the *yz*-plane. Assume both polygons are labelled in counterclockwise order when viewed from $x = +\infty$. The convex hull of P ∪ Q is called the *drum* defined by P and Q. Let us denote it D. We show how to compute the maximum area cross-section of D in the direction orthogonal to the *x*-axis in O(*m*+*n*) time.

The edges of D include the edges of P, the edges of Q and additional edges of the form $p_i q_j$. Suppose there are *k* of these additional edges and denote them by $p_{i_1} q_{j_1}, \ldots, p_{i_k} q_{j'}$

Since D is a convex polyhedron in $\mathbf{R}^3$ we have $k = O(m+n)$. Note that the intersection of D and any plane H which lies between and is parallel to H_1 and H_2 is a polygon in the plane H with k vertices. We denote this polygon D(H) and label its vertices $r_1, r_2, \ldots, r_k$. Each of these vertices is the intersection of H with one of the k edges of D defined above. Assume the edges of D and the vertices of D(H) are labelled such that for $t = 1,\ldots,k$,

$$r_t = H \cap p_{i_t} q_{j_t}$$

and that the vertices r_t appear in counterclockwise order when viewed from $x = +\infty$. We assume that D is given in such a way that the vertices of D(H) can be recovered in this order in $O(m+n)$ time. Let r_t have coordinates (y_t, z_t).

The area of D(H) is given by the standard formula:

$$Area\{D(H)\} = (\frac{1}{2})\left((y_1 z_2 + y_2 z_3 + \ldots + y_k z_1) - (z_1 y_2 + z_2 y_3 + \ldots + z_k y_1)\right)$$

By the definition of H, there exists some λ between zero and one such that

$$H = \lambda H_1 + (1-\lambda) H_2$$

It follows that for $t = 1,\ldots, k$

$$r_t = \lambda p_{i_t} + (1-\lambda) q_{i_t}$$

The coordinates of the vertices of P and Q are known. We may therefore substitute for y_t and z_t in the area equation for D(H) to obtain a formula that is quadratic in λ in $O(m+n)$ time. This formula may be maximized by elementary calculus over the range $0 \le \lambda \le 1$ in constant time. Summarizing, we have the following result.

Theorem 4.1: The maximum-area cross-section of a drum of k vertices, in the direction orthogonal to its defining polygons, can be computed in $O(k)$ time.

We are now ready to solve the problem for an arbitrary convex polyhedron K. If we create a list L stored as an array that contains all the vertices of K sorted by x-coordinates, theorem 2.1 allows us to perform prune-and-search on this list. Accordingly, as a pre-processing step we first sort all the vertices of K in $O(n \log n)$ time and remove any duplicate vertices in $O(n)$ time. We may discard duplicates because they yield the same cutting planes and therefore the same sectional areas. Let p_{med} be the median vertex of L and p_{med-1} and p_{med+1} its two adjacent vertices. We construct three planes orthogonal to the x-axis, H_{med}, H_{med-1} and H_{med+1} through the three corresponding vertices in $O(1)$ time. We intersect these three planes with K in $O(n)$ time with the algorithm of Chazelle and Dobkin [CD87] to obtain three polygons P_{med}, P_{med-1} and P_{med+1}, respectively that in turn determine two drums D_{med-} defined by P_{med}, and P_{med-1} and D_{med+}, defined by P_{med} and P_{med+1}. Next we compute the areas of P_{med}, P_{med-1} and P_{med+1} denoted by A_{med}, A_{med-1} and A_{med+1}, respectively, in $O(n)$ time. To decide which half of L to discard at each pruning step we compare the values of the three areas. If $A_{med-1} < A_{med} < A_{med+1}$ we discard all vertices of L below p_{med} and recurse on the sub-list remaining. If $A_{med-1} > A_{med} > A_{med+1}$ we discard all vertices of L above p_{med} and recurse. If $A_{med} > max[A_{med-1}, A_{med+1}]$ the optimal solution must lie between p_{med-1} and p_{med+1}. Then we use theorem 4.1 twice to solve the optimization problem in $O(n)$ time for drums D_{med-} and D_{med+} and select the higher of the two values encountered. Since at each pruning step we discard half of the vertices of L from further consideration and the test to determine which half is thrown away takes $O(n)$ time we have proved the following:

Theorem 4.2: The maximum-area cross-section of a convex polyhedron of n vertices, orthogonal to a pre-specified direction, can be computed in $O(n \log n)$ time.

4.2 Computing the longest vertical cut of a convex polygon

Consider the two-dimensional version of our problem and let Q be a convex polygon of n vertices in $\mathbf{R}^2$ that supports binary search. What is asked for in this problem is the longest vertical chord of Q. This problem finds application in computing the densest double-lattice packing of a convex polygon [Mo91]. Kirkpatrick and Snoeyink [KS93] described an elegant prune-and-search algorithm for computing the longest vertical cut (or chord) of Q in $O(\log n)$ time. Their key lemma, that allows them to discard one fourth of the vertices at each pruning step, is that the longest cut is determined by points on the boundary of Q that must admit parallel lines of support of Q. As a preprocessing step they split the boundary of Q into two chains Q_{up} and Q_{down}. This is accomplished by finding the vertices of Q with minimum and maximum x-coordinates in $O(\log n)$ time with the algorithm described in [Ch80].

For their prune-and-search strategy they pick q_{up}, the median vertex of Q_{up} as well q_{down}, the median vertex of Q_{down}. By examining the adjacent vertices of q_{up} and q_{down} they construct in O(1) time non-parallel tangents to Q at q_{up} and q_{down} that meet at some point p outside Q. Their "parallel-line-of-support" lemma allows them to discard one fourth of the vertices of Q. In particular, if p is right of q_{down} which, in turn, is right of q_{up} then the vertices of Q_{down} and to the right of q_{down} may be discarded. The other three cases are analogous.

Recall that $l(x)$, the length of the intersecting segment of a line parallel to the y-axis with Q was shown in [JD73] to be a unimodal function of x. Here we merely wish to point out that the "parallel-line-of-support" lemma is not necessary and that the correctness of their algorithm follows from the unimodality property of $l(x)$.

5. Acknowledgement

We thank James Annan of Oxford University for communicating an idea for a proof of the unimodality theorem. Our proof which is based on the Brunn-Minkowski inequality was independently obtained by Bernard Chazelle and David Dobkin of Princeton University. The unfolding in Fig. A3 was produced by a *Mathematica* package developed by Makoto Namiki and Komei Fukuda. Finally, we thank Luc Devroye for his expertise in *Postscript* and for making figure A2.

6. References

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Appendix

Below we illustrate a convex polyhedron for which A(x) is neither convex nor concave. An unfolding of this polyhedron given in Fig. A3.

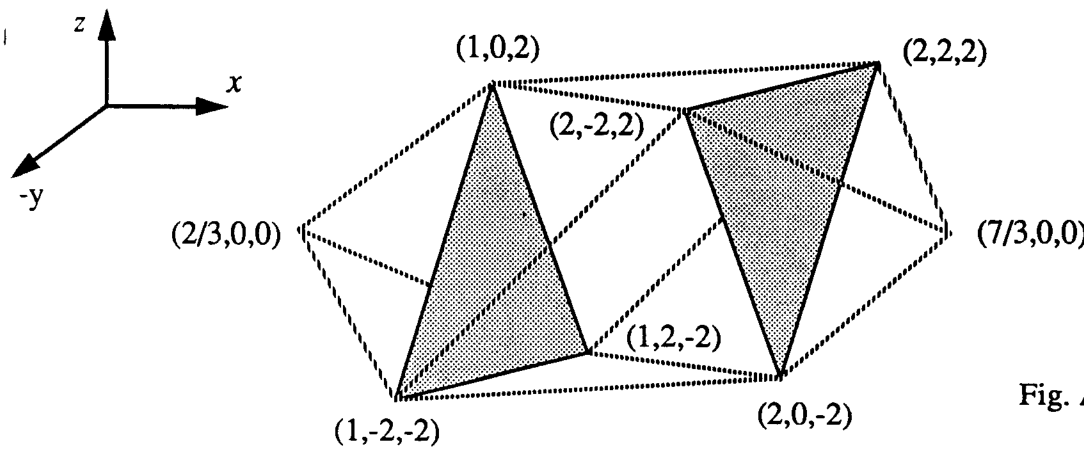


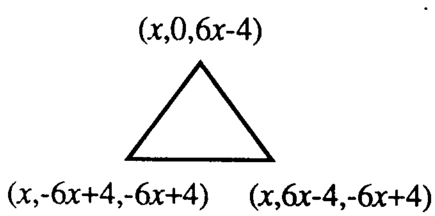
Fig. A1

Area function (a plot of A(x) is illustrated in Fig. A2)

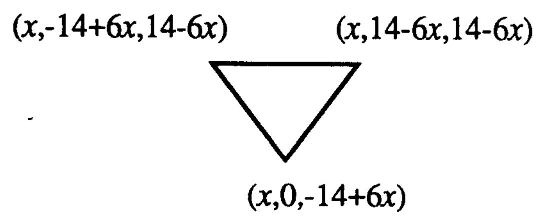
$$A(x) = \begin{cases} 8(3x-2)(3x-2) \ldots\ldots\ldots (2/3 \le x \le 1) \\ -24+48x+(-16)x^2 \ldots\ldots\ldots (1 \le x \le 2) \\ 8(7-3x)(7-3x) \ldots\ldots\ldots (2 \le x \le 7/3) \end{cases}$$

Cross sections

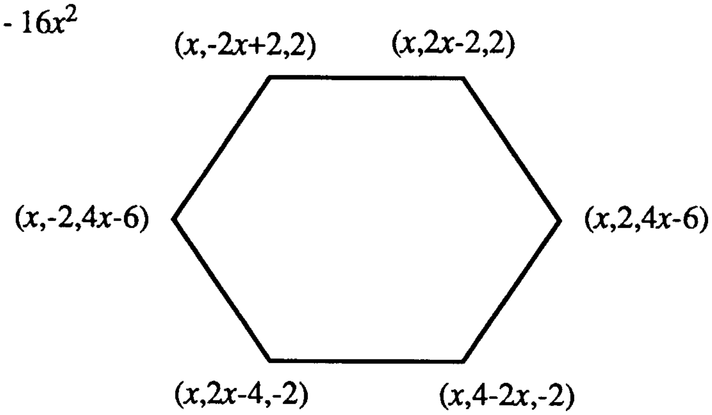
(a) $2/3 \le x \le 1$ Area = $8(3x - 2)(3x - 2)$



(c) $2 \le x \le 7/3$ Area = $8(7 - 3x)(7 - 3x)$



(b) $1 \le x \le 2$ Area = $-24 + 48x - 16x^2$



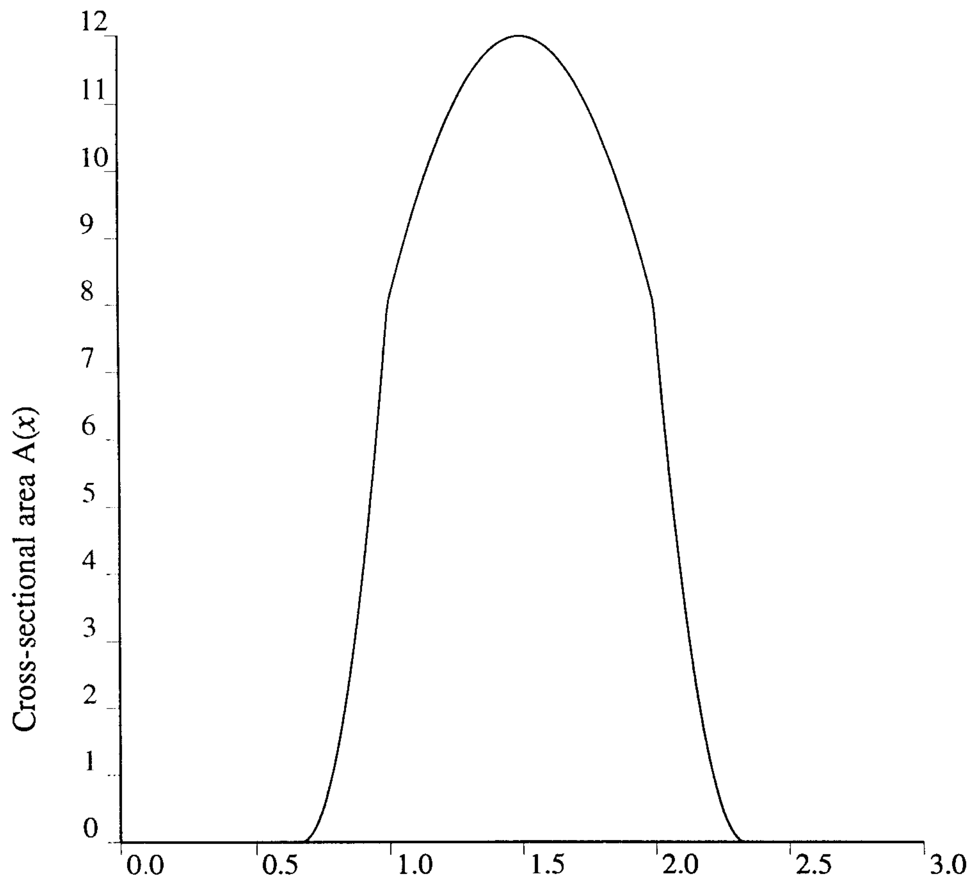


Fig. A2 The function $A(x)$ for the polyhedron of Fig. A1.

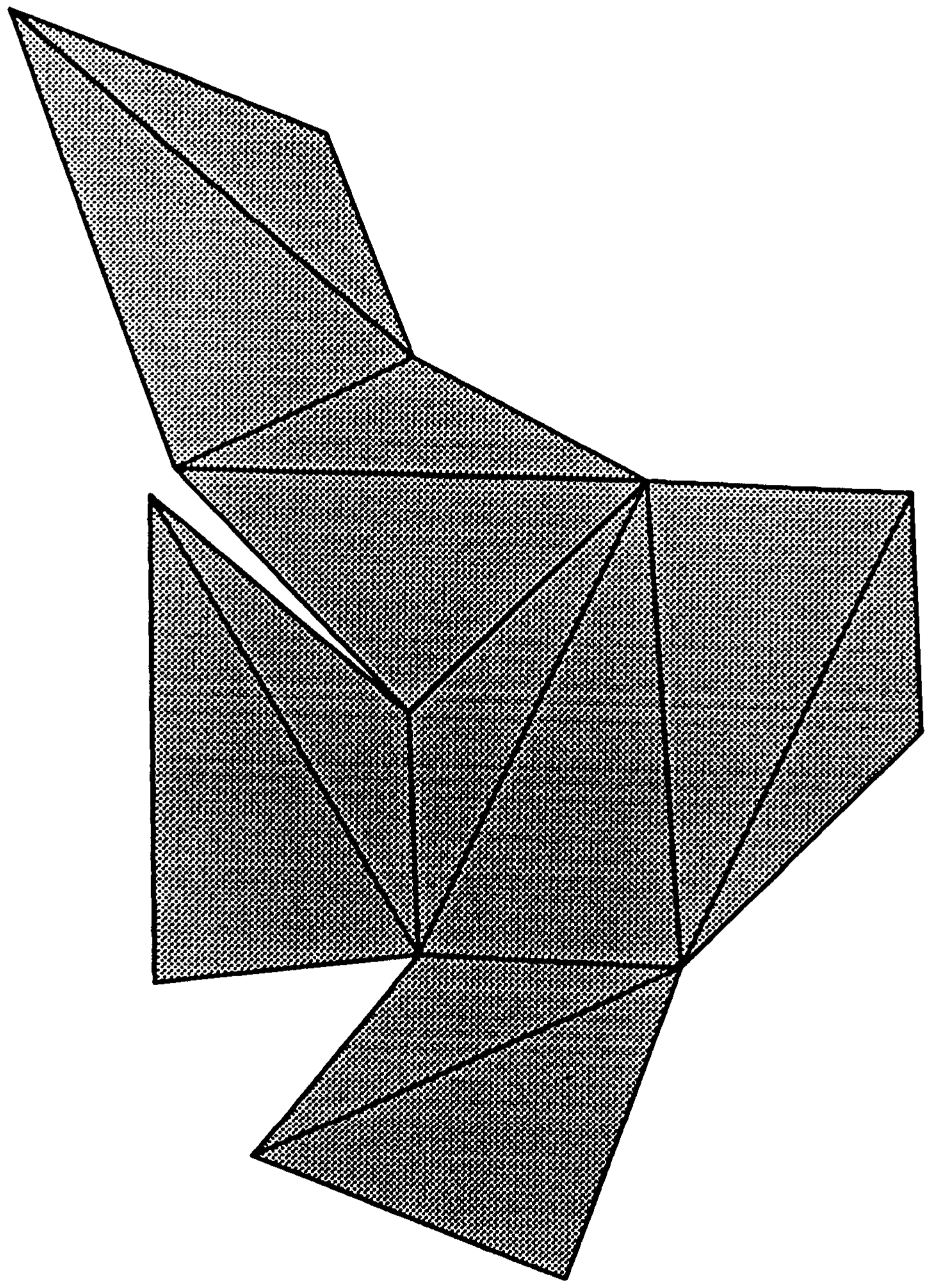


Fig. A3 The reader is encouraged to cut out the above unfolding of the polyhedron given in Fig. A1 in order to construct a physical model as a visual aid.