Counting faces of cubical spheres modulo two

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Abstract

Several recent papers have addressed the problem of characterizing the f-vectors of cubical polytopes. This is largely motivated by the complete characterization of the f-vectors of simplicial polytopes given by Stanley, Billera, and Lee in 1980 [19] [4]. Along these lines Blind and Blind [5] have shown that unlike in the simplicial case, there are parity restrictions on the f-vectors of cubical polytopes. In particular, except for polygons, all even dimensional cubical polytopes must have an even number of vertices. Here this result is extended to a class of zonotopal complexes which includes simply connected odd dimensional manifolds. This paper then shows that the only modular equations which hold for the f-vectors of all d-dimensional cubical polytopes (and hence spheres) are modulo two. Finally, the question of which mod two equations hold for the f-vectors of PL cubical spheres is reduced to a question about the Euler characteristics of multiple point loci from codimension one PL immersions into the d-sphere. Some results about this topological question are known [7] [10] [14] and Herbert's result we translate into the cubical setting, thereby removing the PL requirement. A central definition in this paper is that of the derivative complex, which captures the correspondence between cubical spheres and codimension one immersions.

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1 Introduction

Several recent papers have addressed the problem of characterizing the f-vectors of cubical polytopes [1] [2] [3] [5] [11] [13]. A *cubical d-polytope* is a d-dimensional convex polytope all of whose boundary faces are combinatorially equivalent to cubes. The f-vector of a cubical complex K is the vector $f(K) = (f_0, f_1, f_2, \ldots, f_{d-1})$ or $\sum f_i t^i$, where f_i denotes the number of i-dimensional faces in K. The motivation for characterizing the f-vectors of cubical polytopes stems from the success of the characterization effort for f-vectors of simplicial polytopes [19] [4].

One of the first cubical f-vector results was that of Blind and Blind [5], who showed that unlike in the simplicial case, there are parity restrictions on the f-vectors of cubical polytopes. In particular, except for polygons, all even dimensional cubical polytopes must have an even number of vertices. In Section 2 we extend this result to a class of complexes including all simply connected odd dimensional zonotopal manifolds. The case of particular interest to us is cubical spheres.

In Section 3 we show that the only modular equations which hold for the f-vectors of all d-dimensional cubical polytopes are modulo two. Since the linear equations which hold for spheres and polytopes are the same, this result also holds for cubical spheres.

In the remainder of the paper we focus on piecewise linear (PL) cubical spheres, which are cubical complexes whose realizations are PL-homeomorphic to the boundary of a simplex [16]. This is a more general class of complexes than boundaries of cubical polytopes.

In Section 4, we define the derivative complex of any cubical complex. For a PL cubical d-sphere K the derivative complex is a codimension one PL cubical manifold with an immersion into $|K|$ such that the image is the $(d-1)$-skeleton of the dual to the cubical structure on K. This construction captures the correspondence between cubical spheres and normal crossing codimension one immersions which is crucial to the remainder of the paper. It is called the derivative complex because it acts as a derivation with respect to products and disjoint union.

In Section 5, we use the concept of derivative complexes to prove a new modulo two equation for f-vectors of cubical spheres analogous to a topological result about the corresponding immersions [7] [10].

In Section 6, we construct a non-canonical PL cubical sphere from any codimension one normal crossing PL immersion into the sphere. These are related by the fact that modulo two the number of k-cubes in the complex is the Euler characteristic of the k-fold self-intersection set of the immersion.

Combining this with the derivative complex of Section 4 shows that the set of f-vectors of PL cubical d-spheres is the same modulo two as the set of Euler characteristics of multiple point loci from normal crossing codimension one PL immersions into the d-sphere.

In Section 7 we use this correspondence to translate other results about immersions to results about piecewise linear cubical spheres. For these results we do not know if the PL requirement can be removed.

Throughout the paper we will use standard poset terminology [18] chapter 3, which we review here.

For a poset P and $x \in P$, we denote by $\bigwedge x$ the *order ideal* $\{z \in P : z \leq x\}$, and by $\bigvee x$ the *filter* $\{z \in P : z \geq x\}$. We define the *link* of x in P to be the poset $\bigvee x \backslash \{x\}$. P^{op} denotes the dual poset to P, *i.e.,* the underlying set of P with the reverse order. We denote by $|P|$ the (simplicial) complex of chains in P. A map between posets is called a poset map if it preserves order.

A poset is *cubical* if each order ideal $\bigwedge x$ is a product of copies of I, the three element face poset of an interval, excluding the empty set (so I has a maximum but no minimum element).

Cubical posets are ranked, the rank of an element being the number of links in a maximal chain ending at this element. Thus in a cubical complex, rank is the same as dimension. A cubical complex is a cell complex whose face poset is a cubical poset P such that $\widehat{P}$ (*i.e.,* P with a minimum and maximum element adjoined) is a lattice. Throughout this paper when we consider the face poset of any cubical complex we will always exclude the empty set.

To keep notation to a minimum, we will usually denote a cell complex and its face poset by the same symbol. An element of a face poset is called an i-face, if it has rank i. If two faces F and G are related by $F < G$ then F is called a face of G, and if also $rank(F) = rank(G) - 1$ then F is called a facet of G. A flag of faces is a set of faces which are totally ordered, *i.e.,* $F_1 < F_2 < \ldots < F_k$. The Euler characteristic $\chi(P)$ of a face poset P is the alternating sum of the number of faces of each rank.

2 Bicolorings and Even Vertices

We begin by generalizing the result of Blind and Blind which says that apart from polygons, every even dimensional cubical polytope must have an even number of vertices.

The proof entails first showing that apart from circles, cubical spheres are bicolorable and then that every bicolorable odd dimensional sphere has the same number of vertices of each color. Here a bicoloring of a complex is a choice from two colors (e.g. black and white) for each vertex so that each edge has one vertex of each color, (*i.e.* the 1-skeleton is bipartite).

Proposition 2.1 *If K is a cubical complex with $H^1(K;\mathbb{Z}/2\mathbb{Z}) = 0$, then K has a bicoloring.*

Proof: Note that if $K^1 \in C^1(K;\mathbb{Z}/2\mathbb{Z})$ is the cellular cochain which assigns one to every edge, then a bicoloring of K is a cochain $c \in C^0(K;\mathbb{Z}/2\mathbb{Z})$ such that $\delta c = K^1$. Since every 2-face of K has an even number of edges (four), K^1 is a cocycle. Thus K has a bicoloring if and only if $[K^1]$ is 0 in $H^1(K;\mathbb{Z}/2\mathbb{Z})$. In particular, if $H^1(K;\mathbb{Z}/2\mathbb{Z})$ is trivial, then K must have a bicoloring. □

A complex is n-Eulerian if the link of every vertex has Euler characteristic n.

Theorem 2.2 *If K is a bicolored n-Eulerian cubical complex with n nonzero, then the same number of vertices are assigned each color, and hence K has an even number of vertices.*

Proof: Let f_b and f_w denote the number of vertices of K colored black and white, respectively. Let $f_{b,i}$ and $f_{w,i}$ denote the number of pairs (v, k) where v is a vertex of the i-face $k \in K$ and v is colored black or white, respectively. Since K is n-Eulerian, the sum over the links of all black vertices gives

$$n \cdot f_b = f_{b,1} - f_{b,2} + \ldots + (-1)^{d-1} f_{b,d}.$$

Similarly for the white vertices. If a cube has dimension $i \geq 1$, the bicoloring of its vertices has the same number of each color, so we get $f_{b,i} = f_{w,i}$ for every $i \geq 1$. Thus the two equations above give $n \cdot f_b = n \cdot f_w$ and, since n is nonzero, this gives $f_b = f_w$. □

Corollary 2.3 *If K is a cubical d-sphere with $3 \leq d$ odd then K has an even number of vertices.*

Note that the proofs of the above proposition and theorem hold equally well if cubical complexes are replaced with zonotopal complexes. Thus every n-Eulerian zonotopal complex with $n \neq 0$ and trivial $\mathbb{Z}/2\mathbb{Z}$ first cohomology has an even number of vertices.

3 Modulo two

This section shows that the parity requirement on f_0 for odd-dimensional cubical polytopes is not unusual, in fact every modular equation which holds for all f-vectors of cubical $(d+1)$-polytopes (or all cubical d-spheres) is modulo two. This restriction on the modulus follows from the main result of this section, that the $\mathbb{Z}$-affine span L^d of all f-vectors of boundaries of cubical $(d+1)$-polytopes contains a full rank affine sublattice R^d consisting of all vectors in the $\mathbb{Q}$-affine span of L^d with only even entries. We show that R^d is in fact generated by the f-vectors of the boundaries of $(d+1)$-dimensional cubical zonotopes.

A *zonotope* is a polytope which can be generated by taking the Minkowski sum of a finite set of line segments [21] chapter 7. If a minimal generating set for a zonotope contains n line segments then the zonotope is said to have n *zones.* A *cubical zonotope* is a zonotope all of whose boundary faces are combinatorially equivalent to cubes. Let F_n^d denote the f-vector of the boundary of any cubical $(d+1)$-zonotope with $n+d+1$ zones, and $F_n^0 = 2$. Note that there are many such cubical zonotopes but all have the same f-vector. (This follows easily from the correspondence between cubical zonotopes and generic hyperplane arrangements [20].) In particular $F_0^d = (2+t)^{d+1} - t^{d+1}$ is the f-vector of the boundary of the $(d+1)$-cube.

Since $F_0^d \in R^d \cap L^d$, we can form the lattices $R^d - F_0^d = \{y | y + F_0^d \in R^d\}$ and $L^d - F_0^d = \{y | y + F_0^d \in L^d\}$.

Theorem 3.1 *For every d, the lattice $R^d - F_0^d$ is a full rank sublattice of $L^d - F_0^d$.*

Proof: Let E_n^d denote the $\mathbb{Z}$-affine span of $\{F_i^d\}_{i\geq n}$.

Note that $F_n^d = F_{n-1}^d + (1+t)F_n^{d-1}$ for each n. Hence $E_n^d = \langle (1+t)E_{n+1}^{d-1}\rangle_{\mathbb{Z}} + F_n^d$, where $\langle\ \rangle_{\mathbb{Z}}$ denotes $\mathbb{Z}$-linear span, and $\langle\ \rangle_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear span. Thus, by induction on d, E_n^d is independent of n, so write $E^d = E_n^d$ and we have $E^d = \langle (1+t)E^{d-1}\rangle_{\mathbb{Z}} + F_0^d$. To begin the induction note that $E_n^0 = \{2\}$ for all n.

We will show that rank $E^d =$ rank L^d, so $E^d - F_0^d$ is a full rank sublattice of $L^d - F_0^d$. Then we will show that $R^d = E^d$, completing the proof.

Let $v : L^d \to \mathbb{Z}$ denote evaluation at -1. Then $v(F_0^d) = 1 + (-1)^d$ while $v(p) = 0$ for any $p \in \langle (1+t)E^{d-1}\rangle_{\mathbb{Z}}$. Thus if d is even, the rank of E^{d+1} equals the rank of $\langle E^d\rangle_{\mathbb{Z}}$ which is one more than the rank of E^d. Since rank $E^0 = 0$ and rank $E^d \leq$ rank $L^d = \lfloor \frac{d+1}{2} \rfloor$ for all $d \geq 1$ [9], we have rank $E^d =$ rank L^d for all $d \geq 1$.

Now we show that $E^d = R^d$. Since zonotopes are centrally symmetric, $E^d \subseteq R^d$, so we just need to show that $R^d \subseteq E^d$. It is clear that $R^1 \subseteq E^1$, so we proceed to prove that $R^d \subseteq E^d$ by induction on d. Fix $w \in R^d - F_0^d$. Since $E^d - F_0^d$ is a full rank sublattice of $L^d - F_0^d$, we have $R^d - F_0^d \subseteq \langle L^d - F_0^d \rangle_{\mathbb{Q}} = \langle E^d - F_0^d \rangle_{\mathbb{Q}} = \langle (1+t)\langle E^{d-1} \rangle_{\mathbb{Z}} \rangle_{\mathbb{Q}} = (1+t)\langle E^{d-1} \rangle_{\mathbb{Q}}$. Thus we can write $w \in R^d - F_0^d$ as $w = (1+t)\sum_i \lambda_i w_i$, for some $w_i \in E^{d-1}$ and $\lambda_i \in \mathbb{Q}$. We will say that a vector is even if it has all even entries. Since $w = (1+t)\sum_i \lambda_i w_i$ is even, $u = \sum_i \lambda_i w_i$ is also even.

If d is even, then $\langle E^{d-1} \rangle_{\mathbb{Z}} = E^{d-1}$, so u is an even vector in the $\mathbb{Q}$-affine span of E^{d-1}. In particular, this means that u is in R^{d-1}. But $R^{d-1} \subseteq E^{d-1}$ (by induction on d), so we have $u \in E^{d-1}$, which implies $w = (1+t)u \in E^d - F_0^d$.

If d is odd, we use that $v(F_0^{d-1}) = 2$ and $\langle E^{d-2} \rangle_{\mathbb{Z}} = E^{d-2}$ to show that $w \in E^d - F_0^d = \langle (1+t)(\langle (1+t)E^{d-2} \rangle_{\mathbb{Z}} + F_0^{d-1}) \rangle_{\mathbb{Z}} = (1+t)\langle (1+t)E^{d-2} + F_0^{d-1} \rangle_{\mathbb{Z}}$. We can write $w = (1+t)u$ as above and $u = \mu_0 F_0^{d-1} + (1+t)\sum_{i \geq 1} \mu_i v_i$ for some $v_i \in E^{d-2}$ and $\mu_i \in \mathbb{Q}$ with $\mu_0 = \sum_{i \geq 1} \mu_i$. Since u is even, $v(u)$ must be even, so $(1/2)v(u) = \mu_0 = \sum_{i \geq 1} \mu_i = n$ for some integer n. Now set $y = \sum_{i \geq 1} \mu_i v_i$ is an even vector in $\langle E^{d-2} \rangle_{\mathbb{Q}}$ and $\langle E^{d-2} \rangle_{\mathbb{Z}} = E^{d-2}$, so $y \in R^{d-2}$. But $R^{d-2} \subseteq E^{d-2}$ (by induction on d), so we have $y \in E^{d-2}$, which implies $w = (1+t)(nF_0^{d-1} + y) \in E^d - F_0^d$.

Thus we have that for any $w \in R^d - F_0^d$, w must be in $E^d - F_0^d$; so $R^d \subseteq E^d$, as desired. $\square$

Note that the affine equations satisfied by all f-vectors of cubical polytopes are the same as those satisfied by all f-vectors of cubical spheres, since these equations can be derived from the Dehn-Sommerville equations for simplicial spheres [17] by considering the links of faces in cubical polytopes and spheres. Thus the $\mathbb{Z}$-affine span of all f-vectors of cubical d-spheres contains L^d as a full rank affine sublattice, and so the theorem holds also for cubical d-spheres in place of the boundaries of cubical $(d+1)$-polytopes.

Corollary 3.2 *The only modular equations which hold among the components of the f-vectors of all cubical d-polytopes (and hence spheres) are modulo two.*

Proof: Let $V = \frac{1}{2}(R^d - F_0^d)$ denote the lattice of integral points in $\langle R^d - F_0^d \rangle_{\mathbb{Q}}$. Then $L^d - F_0^d \subset V$, since $R^d - F_0^d$ is a full-rank sublattice of $L^d - F_0^d$. A modulo m equation on L^d is a $\mathbb{Z}$-linear map $S : V \to \mathbb{Z}/(m\mathbb{Z})$ which restricts to 0 on $L^d - F_0^d$. We may assume that S is surjective, otherwise we can get an equivalent surjective modular equation with modulus the order of the image of S. Thus $S(v) \equiv_m 1$ for some integral v in $\langle \frac{1}{2}(R^d - F_0^d) \rangle_{\mathbb{Q}}$. Then $2v$ has all even entries and is in the same $\mathbb{Q}$-linear span, so $2v \in R^d - F_0^d \subset L^d - F_0^d$. But then $2 \equiv_m S(2v) \equiv_m 0$, so $m = 2$. $\square$

Note that a slight extension of the above argument shows there are no modular restrictions on the f-vectors of arbitrary cubical d-manifolds – since zonotopes are centrally symmetric, $1/2$ the f-vector of the boundary of any cubical zonotope is the f-vector of a cubical subdivision of real projective space.

4 Derivative Complexes

In this section we give the more straightforward direction for the equivalence between PL (piecewise linear) cubical d-spheres and codimension one PL normal crossing immersions into the d-sphere. The topological objects involved are described in ([13] Section 6), where they are attributed to MacPherson and Stanley.

If K is a cubical poset, define a new cubical poset NK with elements the ordered pairs $(b, c) \in K \times K$ such that the join of b and c covers both, while b and c have no meet. Thus b and c are opposite facets of their join. The partial order on NK is the partial order on K taken component-wise. Let $\epsilon : NK \to NK$ denote the involution $\epsilon(b, c) = (c, b)$. Then the *derivative complex* of K is the quotient poset $DK = NK/\epsilon$. Note that NK and DK are both cubical posets and NK is simply a double cover of DK. An element $\{b, c\} = (b, c)/\epsilon \in DK$ corresponds to a slice through the interior of the join of b and c, parallel to b and c. An element $(b, c) \in NK$ corresponds to the side of $\{b, c\} \in DK$ which faces b in K. See Figure 1 for geometric realizations of NK and DK.

Note that

$$f(DK, t) = \frac{d}{dt} f(K, t)$$

and D and N act as derivations with respect to product and disjoint union:

$$D(K_1 \times K_2) = (DK_1 \times K_2) \cup (K_1 \times DK_2).$$

Finally we have the map $j : NK \to K$ taking (b, c) to the join of b and c, and similarly, $j : DK \to K$. Both maps induce isomorphisms on the links of faces, so if $|K|$ is a piecewise linear manifold then so are $|DK|$ and $|NK|$. Furthermore $j^* : |DK| \to |K|$ is a codimension one normal crossing immersion. If $\dim(|K|) = d$, then the image of j^* is the $(d-1)$-skeleton of the dual to the cubical structure of K. See Figure 2 for an illustration on the boundaries of cubical 3-polytopes.

In Section 6 we will show that any codimension one normal crossing immersion into S^d is enumeratively equivalent, modulo two, to the immersion of a derivative complex. Note that $f_i(K) = (-1)^{d-i}\chi(\{s \in |K|||j^{*-1}(s)| = i\})$ since the set of points $\{s \in |K|||j^{*-1}(s)| = i\}$ is simply a disjoint union of open $(d-i)$-balls, one for each i-face of K. (Here χ denotes Euler characteristic with closed supports, so $\chi(R^n) = (-1)^n$.) This relation will be used in Section 6.

5 A Chain Lemma

In this section we translate a known result about codimension one immersions [10] into a result about cubical complexes. For even dimensional cubical polytopes of dimension at least 6, this result gives a modulo two condition different from the Blind-Blind [5] condition that f_0 must be even.

If K is a PL cubical d-sphere, then $j^* : |DK| \to |K|$ is a codimension one normal crossing PL immersion into the d-sphere, so a result of [7], [10] tells us that if d is odd, then the number of degree d intersection points is congruent modulo two to the Euler characteristic $\chi(|DK|)$. In particular, $f_d(K) \equiv_2 f_0(DK) + f_1(DK) + f_2(DK) + \dots f_{d-1}(DK)$. Since $f_i(DK) = (i+1)f_{i+1}(K)$, this implies that $f_d(K) \equiv_2 f_1(K) + f_3(K) + \dots + f_d(K)$, or $f_1(K) + f_3(K) + \dots f_{d-2}(K) \equiv_2 0$, for all PL cubical d-spheres K with d odd. Here we prove this result directly for a class of cubical complexes including all odd-dimensional cubical spheres, thus removing the PL requirement.

We first prove a lemma about simplicial flags of K, DK and NK. Some notation will be useful. If K is a poset, let SK denote the simplicial poset of flags of K, and $C_*(K) = C_*(|K|; \mathbb{Z}/2\mathbb{Z})$ the simplicial chain complex with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We identify elements of $C_*(K)$ with subsets of the set of elements of SK so that both set and chain complex notations make sense (*e.g.*, $\cap, +, \partial$).

Let $\epsilon_* : C_*(NK) \to C_*(NK)$ denote the extension of the involution $\epsilon : NK \to NK$, and $\sigma = 1 + \epsilon_* : C_*(NK) \to C_*(NK)$. One can check that $\partial\sigma = \sigma\partial$ and $\sigma(C_*(NK))$ is a chain subcomplex isomorphic to $C_*(DK)$.

Define a degree 0 map $\gamma = j^{-1} : C_*(K) \to C_*(NK)$ by $\gamma(a_0 < \dots < a_r) = \sum_I (e_0 < \dots < e_r)$ where $(e_0 < \dots < e_r) \in I$ if and only if $j(e_i) = a_i$ for all i. The image of γ is contained in $\sigma(C_*(NK))$.

Define a degree -1 map $\tau : C_*(K) \to C_*(NK)$ by $\tau(a_0 < \dots < a_r) = \sum_H (e_1 < \dots < e_r)$ where $(e_1 < \dots < e_r) \in H$ if and only if for every i,

$e_i = (b_i, c_i)$ such that $b_i \geq a_0$ and a_i is the join of b_i and c_i.

Now suppose $e_1 < \ldots < e_{k-1} < e_{k+1} < \ldots < e_r$ and for each i, $e_i = (b_i, c_i)$ such that $b_i \geq a_0$ and a_i is the join of b_i and c_i. Then if $k > 1$ there is a unique e_k such that $(e_1 < \ldots < e_{k-1} < e_k < e_{k+1} < \ldots < e_r) \in H$, but if $k = 1$ then there exists such an e_k if and only if $a_1 \not\leq b_2$. Based on this observation one can verify that $\partial\tau = \tau\partial$, and also that $\sigma\tau = \partial\gamma + \gamma\partial$.

To prove the following lemma we often need to fix a side for each face of SDK. Such a choice corresponds to a chain $W \in C_*(NK)$ with $\sigma W = SNK$. (Recall, since we identify elements of $C_*(NK)$ with subsets of the set of elements in SNK, here we mean that σW is the chain of faces of SNK with coefficient one for every face.) Fix such a chain W. For every $a = (a_1 < \ldots < a_r) \in SK$, define $p_a \in C_*(K)$ as follows. If a_1 is minimal in K, let $p_a = 0$ (*i.e.,*, the empty chain, by our identification of subsets of SK with elements in $C_*(K)$). If a_1 is not minimal, let $p_a = \{(p_0 < a_1 < \ldots < a_r)\}$ with p_0 minimal and $W \cap \gamma a = \tau p_a$. Note that since p_0 is minimal, $\gamma p_a = 0$.

Figure 3 shows P, τP, and γP, where K is the solid 3-cube and P is the boundary of a glued pair of triangles in the subdivided 3-cube SK. For this example, let W choose the side of each face in SDK which faces the facet with smaller ternary labeling. (*i.e.,* if $(b, c) \in NK$ occurs in a flag in W then b is less than c with respect to the ternary labeling of faces shown.) Then $W \cap \tau P = \{(001, 201)\} + \{(002, 202)\}$ and $\sigma(W \cap \tau P) = \{(001, 201)\} + \{(201, 001)\} + \{(002, 202)\} + \{(202, 002)\}$.

Lemma 5.1 *If K is a cubical complex and $P = \partial Q$ for some $Q \in C_*(K)$ and $\gamma P = \partial\sigma T$ for some $T \in C_*(NK)$ then $\tau P = \partial\sigma U$ for some $U \in C_*(NK)$.*

Proof: Let $R = Q + \sum_{a \in Q} \partial p_a$, so $\partial R = \partial Q = P$. We also have

$$\gamma R = \gamma Q + \sum_{a \in Q} \gamma\partial p_a = \gamma Q + \gamma Q = 0,$$

since

$$\gamma\partial p_a = \partial\gamma p_a + \sigma\tau p_a = \sigma\tau p_a = \sigma(W \cap \gamma a) = \gamma a$$

and the coefficients are in $\mathbb{Z}/2\mathbb{Z}$. Thus $\sigma\tau R = (\gamma\partial + \partial\gamma)R = \gamma P$.

Now note that for any $U \in C_*(NK)$, we can write $U = \sigma A + B \cap \sigma U$ for some $A, B \in C_*(NK)$ with $\sigma B = SNK$. So with $U = \tau R$, we have for some $A, B \in C_*(NK)$ with $\sigma B = SNK$, and

$$\tau R = \sigma A + B \cap \sigma\tau R = \sigma A + B \cap \gamma P.$$

Thus

$$\tau P = \tau\partial R = \partial\tau R = \partial(\sigma A + B\cap\gamma P) = \partial(\sigma A + B\cap\partial\sigma T) = \partial(\sigma A + \sum_{t\in T}(B\cap\partial\sigma t)).$$

Now for each $t \in T$ there exist $A_t, B_t \in C_*(NK)$ with $\sigma B_t = SNK$ and $\partial t \subseteq B_t$ and $B = B_t + \sigma A_t$. So

$$B \cap \partial\sigma t = (B_t + \sigma A_t) \cap \partial\sigma t = (B_t \cap \partial\sigma t) + \sigma(A_t \cap \partial t) = \partial t + \sigma(A_t \cap \partial t).$$

Thus

$$\tau P = \partial(\sigma A + \sum_{t\in T}(\partial t + \sigma(A_t \cap \partial t))) = \partial\sigma U,$$

where $U = A + \sum_{t\in T}(A_t \cap \partial t) \in C_*(NK)$. □

Now some careful choices together with the above lemma give us the desired result.

Theorem 5.2 *If K is an odd dimensional cubical d-sphere then $f_1(K) + f_3(K) + \ldots + f_{d-2}(K)$ is even.*

Proof: We will show that there exists a set of edges in SDK with $f_1(K) + f_3(K) + \ldots + f_{d-2}(K)$ boundary vertices (modulo 2), hence $f_1(K) + f_3(K) + \ldots + f_{d-2}(K)$ must be even.

We will define a set of simplicial chains $P_i \in C_i(K)$ which satisfy the conditions of Lemma 5.1 by induction on i, so that finally we may conclude that γP_0 is a symmetric boundary in $C_*(NK)$, and from this we get the desired zero boundary in $C_*(DK)$.

Let $[d, d-i] = \{d, d-1, \ldots, d-i\}$ and let $K_{[d,d-i]}$ denote the $[d, d-i]$ rank-selected subposet of K. Then define $P_i = (SK + S(K_{[d,d-i]})) \cap C_i(K)$.

One can check that $\gamma P_i = (SNK + S(NK)_{[d-1,d-i-1]}) \cap C_i(NK)$ by noting that every $(i+1)$-chain $e_0 < e_1 < \ldots < e_i$ in $C_i(NK)$ has a unique preimage $\gamma^{-1}(e_0 < e_1 < \ldots < e_i) = (j(e_0) < j(e_1) < \ldots < j(e_i))$ in $C_i(K)$. It is also not hard to check that $\tau P_i = \gamma P_{i-1}$ for any cubical complex, by noting that for every chain $e_1 < e_2 < \ldots < e_i$ in $C_i(NK)$ there is an odd number of $(i-1)$-chains in $\tau^{-1}(e_1 < e_2 < \ldots < e_i)$, in particular all chains of the form $a_0 < j(e_1) < j(e_2) < \ldots < j(e_i)$ where $a_0 \le b_1$ and $e_1 = (b_1, c_1)$. (Note that there is an odd number of such a_0 since the number of faces in a cube, including the cube itself, is 3^t where t is the dimension of the cube.) Since all coefficients are mod two, any odd number is equivalent to one. This shows that $\tau P_i = (SNK + S(NK)_{[d-1,d-i]}) \cap C_{i-1}(NK)$, so $\tau P_i = \gamma P_{i-1}$, as desired.

Similarly, it is easy to check that $\partial P_i = 0$, since the preimage (under ∂) of any chain in $C_{i-1}(K)$ contains an even number of i-chains. More explicitly, the first term of P_i has trivial boundary because the Euler characteristic of the link of any face in a cubical sphere of any dimension is even, and a cube of any dimension has an even number of non-maximal, non-empty faces, while to check that the second term has trivial boundary uses that every codimension one face is in two facets, and that every $d - i - 1$ cube contains an even number of facets. Thus, if $0 < i < d$, then $\partial P_i = 0$ so $H_i(|K|; \mathbb{Z}/2\mathbb{Z}) = 0$ implies that $P_i = \partial Q_i$ for some $Q_i \in C_{i+1}(K)$. Finally, simply plugging in $i = d - 1$ we get $\gamma P_{d-1} = 0 = \partial\sigma 0$, and so by induction and Lemma 5.1, if $0 \leq i < d$ then $\gamma P_i = \partial\sigma U_i$ for some $U_i \in C_{i+1}(NK)$. Thus $\gamma P_0/\epsilon_*$ is a boundary and hence $|\gamma P_0/\epsilon_*| = \frac{1}{2}(\sum_j 2j f_j(K) + 2d f_d(K))$ is even. Thus $f_1(K) + f_3(K) + \ldots + f_{d-2}(K) \equiv_2 \sum_j j f_j(K) + d f_d(K) \equiv_2 0$, as desired. □

Note that the proof holds more generally for any d and any cubical complex K with no faces of dimension more than d, the link of every face having even Euler characteristic, and $H_i(|K|, \mathbb{Z}/2\mathbb{Z}) = 0$ for $0 < i < d$. Thus, for all such K, we have $d f_d(K) \equiv_2 \sum_i i f_i(K)$.

6 Immersions to Cubations

In Section 2, we constructed a codimension one PL normal crossing immersion into the d-sphere which we called the derivative complex, from any PL cubical d-sphere. We now complete this correspondence. In particular, we prove the following:

Theorem 6.1 *Given a codimension one normal crossing PL immersion $y : M \to S^d$, there exists a PL cubical d-sphere K such that, modulo two, the Euler characteristics for the multiple point loci of y are the same as for the immersion of the derivative complex, $j : |DK| \to |K|$. In particular, $\chi(\{s \in S^d \| |y^{-1}(s)| = i\}) \equiv_2 f_i(K)$.*

As a result, the PL case of our Theorem 5.2 proves the result which motivated it [7] [10]:

Corollary 6.2 *If d is odd and $y : M \to S^d$ is a codimension one normal crossing PL immersion, then the number of degree d intersection points is congruent modulo two to the Euler characteristic $\chi(M)$.*

In the next section we will give other results about normal crossing immersions which correspond to counting facets of PL cubical spheres modulo two.

Our proof of Theorem 6.1 involves combinatorial constructions of posets with PL-equivalent realizations. These equivalences rely on regularity of the cell complexes involved, as in the following lemma.

Lemma 6.3 *If Q is the face lattice of a regular cell complex and $g : P \to Q$ is a poset map so that for every $q \in Q$ we have that $|g^{-1}(\bigwedge q)|$ is a PL ball with dimension the rank of q and boundary $|g^{-1}(\bigwedge q \backslash \{q\})|$ then $|P|$ is PL-equivalent to $|Q|$.*

Proof: Construct a PL homeomorphism between $|P|$ and $|Q|$ by building in one cell of Q at a time, and completing the k-skeleton before moving to $(k+1)$-cells. To extend the map over a k-cell $q \in Q$ note that there is already a chosen PL homeomorphism between $|\bigwedge q \backslash \{q\}|$ and $|g^{-1}(\bigwedge q \backslash \{q\})|$, which are homeomorphic to the $(k-1)$-sphere. We can extend this homeomorphism over $|g^{-1}(\bigwedge q)|$ to $|\bigwedge q|$ by coning, as in [16] Lemma 1.10. $\square$

Proof of Theorem 6.1: Given a PL codimension one normal crossing immersion $y : M \to S^d$, choose a triangulation T of S^d so that M has a triangulation making y simplicial. (T exists by the definition of PL.)

The construction of the cubical sphere will be done in two steps, each producing a PL sphere.

Let J be the poset with elements

$$\{(t, C_1, C) | t \in T, C_1 \subseteq C \in S(\bigwedge t)\}$$

and partial order given by $(t, C_1, C) \geq (t', C_1', C')$ if $t \geq t'$ and $C_1' \subseteq C_1 \subseteq C \subseteq C'$. Let $p : J \to T$ be the map $(t, C_1, C) \mapsto t$. Note that the dual poset, J^{op}, is simply the cubical barycover [2] of the barycentric subdivision of T with the subdivided faces of the $(d-1)$-skeleton of T thickened, as shown in Figure 4. Since T need not be dual cubical, J^{op} need not be cubical.

One can check that $p : J \to T$ satisfies the conditions of the lemma. Thus $|J|$ and $|T|$ are PL equivalent, so $|J|$ is a PL sphere. Let $y : M \to |J|$ denote the induced simplicial map of the associated subdivision of M into $|J|$. This again is a PL normal crossing immersion.

Locally, $y(M) \subset |J|$ is a collection of normal crossing hyperplanes subdividing the open star of $t \in J$ into sectors, and hence there is a map n_t taking each face in the open star of t to its ambient sector. Formally, for every $t \in J$

we define $n_t : \bigvee t \to (I^{op})^{|y^{-1}t|}$ where I^{op} denotes the three element poset with a minimum but no maximum element. Each copy of I^{op} in the range of n_t corresponds to one of the normal crossing hyperplanes subdividing the open star of t, in particular, the minimum element of I^{op} corresponds to faces in that hyperplane, and the other two elements of I^{op} correspond to the two sides of that hyperplane. The maps $\{n_t\}$ are consistent, in the sense that if $s \leq t$, then $n_s|_{\bigvee t}$ is constant and maximal on some copies of I^{op} and agrees with n_t on the other copies of I^{op}.

Let K be the quotient of J by the equivalence relation putting $(t, C_1, C) \sim (t', C_1, C)$ if $n_{MC}t = n_{MC}t'$, where MC is the maximum element of C. Then K^{op} is cubical, as K is the result of combining enough faces in J that the only faces of T which remain thickened are the normal crossing faces in $y(M)$. See Figure 5.

Note that the fact that the immersion is normal crossing implies that $B = |\bigwedge n_t^{-1} r|$ is always a ball with boundary the union of $B \cap (\bigwedge(\bigvee t)\backslash\{\bigvee t\})$ and $n_t^{-1}(\bigwedge r \backslash \{r\})$. Thus the quotient map from J to K satisfies the conditions of the lemma, so K is a PL d-sphere and K^{op} is a PL cubical d-sphere.

We now show that $f(K^{op})$ satisfies the desired modulo two equations. For any $t \in T$, let $U(t)$ be the set of elements $(t', C_1, C) \in K$ for which $t = MC$, the maximum element of C. See Figure 6.

Now note that there is an odd number of rank n faces in $U(t)$ if and only if $n = d - |y^{-1}t|$. To see this, define an involution i on $U(t)$ by $i(s, C_1, C) = (s', C_1', C')$ if the following hold:

1. C_1', C' are obtained from C_1, C respectively by replacing each face c strictly smaller than t by its complement $t \backslash c$; and

2. $n_t s$ agrees with $n_t s'$ on exactly those copies of I^{op} for which either takes the minimum value, *i.e.,* s and s' are on opposite sides of all the local hyperplanes.

Then i is rank preserving and fixes only the rank $d - |y^{-1}t|$ element $(t, \{t\}, \{t\})$. Thus

$$\sum_{k \in K} x^{\text{rank}(k)} = \sum_{t \in T} \sum_{k \in U(t)} x^{\text{rank}(k)} \equiv_2 \sum_{t \in T} x^{d - |y^{-1}t|},$$

and hence

$$f_i(K^{op}) \equiv_2 \chi(\{s \in S^d \,||y^{-1}(s)| = i\})$$

for all i, as desired. □

7 Facets and Questions

Based on Theorem 6.1 and the smoothability of codimension one PL immersions (see [12] theorem 7.4) we can deduce the following about the number of facets of PL cubical spheres. The first follows from [6], the second from [14], and the others from [7].

1. There exists a PL cubical 3-sphere with an odd number of facets. (Let $j : RP^2 \to S^3$ be Boy's immersion, having a single degree 3 intersection point [6] [7].) Thus the $\mathbb{Z}$-affine span of f-vectors of cubical 3-spheres is completely known, *i.e.*, $f_0 \equiv_2 f_1 \equiv_2 f_2 + f_3 \equiv_2 0$.

2. If d is a multiple of 4 then a PL cubical d-sphere can have an odd number of facets if and only if $d = 4$.

3. If d is odd, then a PL cubical d-sphere can have an odd number of facets if and only if $d = 1, 3$ or 7.

4. If $d \equiv_4 2$, then a PL cubical d-sphere can have an odd number of facets only if $d = 2^n - 2$ for some n. Furthermore, examples of cubical d-spheres with an odd number of facets are known for $d = 2, 6, 14, 30$, and 62. (Eccles [8] showed that the existence of such spheres is equivalent to the existence of a framed d-manifold with Kervaire invariant one).

5. Edge orientable (in the sense of [11]) PL cubical spheres can have an odd number of facets if and only if $d \in \{1, 2, 4\}$.

Some interesting open questions are:
Do the above hold for cubations of arbitrary topological spheres?
Can even dimensional cubical spheres have an odd number of vertices?
For any given $d \geq 4$, what are all the modular equations for f-vectors of cubical d-spheres?

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