A SMOOTH SPACE OF TETRAHEDRA

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ABSTRACT. We construct a symmetric compactification of the space of all labeled n-simplices in $\mathbb{P}^n$, and show that our space is smooth in the case of tetrahedra in $\mathbb{P}^3$.

1. INTRODUCTION

1.1. Let $\mathbf{k}$ be an infinite algebraically closed field of characteristic zero, and for $n \geq 3$ let $\mathbb{P}^{n-1}$ be the $(n-1)$-dimensional projective space over $\mathbf{k}$. Let $P \subset \mathbb{P}^{n-1}$ be a set of n labeled points in general position. Then by taking all possible linear spans of subsets of P, one determines a configuration of flats in $\mathbb{P}^{n-1}$ arranged to form a simplex.

Now consider the set $\mathscr{Z}^\circ$ of all such configurations, as P ranges over all sets of n general points. Then $\mathscr{Z}^\circ$ is naturally a smooth quasi-projective variety, and has a canonical singular compactification $\mathscr{Z}$. One is interested in this configuration space for many reasons:

- For $n = 3$, the space $\mathscr{Z}$ is the space of triangles in the plane. In [15], Schubert described a desingularization of $\mathscr{Z}$, and used it to study enumerative problems involving triangles [16, 14, 3].
- The space $\mathscr{Z}$ is a *configuration variety* in the sense of Magyar [11, 19]. Such spaces arise naturally in the study of *generalized Schur modules*. These are (reducible) GL_n-modules that generalize the classical Schur modules, and have been studied in various guises by many authors [1, 10, 20, 9, 12, 13, 17]. One hopes that configuration varieties will play a role in a Borel-Weil theory for these modules.
- Suppose $\mathbf{k} = \mathbb{C}$. Let $\mathscr{B}$ be the building for $SL_n(\mathbf{k})$, and let $\mathscr{C}$ be the associated Coxeter complex [18]. Then $\mathscr{Z}$ can be interpreted as the space of maps of $\mathscr{C}$ into $\mathscr{B}$. By considering other algebraic groups, one obtains a collection of configuration spaces related to the *Bott-Samuelson varieties* of Demazure [4]. In particular, $\mathscr{Z}$ can be regarded as a *canonical* Bott-Samuelson variety associated to all reduced expressions of the longest word of the Weyl group of SL_n.

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- The space $\mathcal{Z}$ is a natural generalization of the Fulton-MacPherson space [6] $X := \mathbb{P}^{n-1}[n]$. This variety adds data to an open set of the product $\prod_{i=1}^{n} \mathbb{P}^{n-1}$ that records how points approach the (large) diagonals. In fact, X is canonically a desingularization of the space of all 1-*skeleta* of n-simplices in $\mathbb{P}^{n-1}$.

1.2. In this paper we construct a symmetric compactification $\widetilde{\mathcal{Z}}$ of $\mathcal{Z}^\circ$ that we call the space of *complete simplices*. Our main theorem is that $\widetilde{\mathcal{Z}}$ is nonsingular for $n = 4$, the case of labeled tetrahedra in $\mathbb{P}^3$. Although this and $n = 3$ are the only cases for which we know our space to be nonsingular, our construction and many of our results are valid for all n. In the case of tetrahedra, the complement of $\mathcal{Z}^\circ$ in the compactification $\widetilde{\mathcal{Z}}$ is a divisor with normal crossings, but we defer the proof of this and other facts about the intersection theory of $\widetilde{\mathcal{Z}}$ to a future paper [2].

1.3. To give a feel for $\widetilde{\mathcal{Z}}$, we describe it for $n = 3$. Let $p_1, p_2, p_3 \in \mathbb{P}^2$ be three distinct points, and let $l_{12}, l_{13}, l_{23} \subset \mathbb{P}^2$ be three distinct lines. Then $p_1, \ldots, l_{23}$ determine a point in $\mathcal{Z}^\circ$ if the line l_{ij} passes through the points p_i, p_j. This gives a map $\mathcal{Z}^\circ \to (\mathbb{P}^2)^3 \times (\hat{\mathbb{P}^2})^3$, where $\hat{\mathbb{P}^2}$ is the dual projective space, and the compactification $\mathcal{Z}$ is the closure of the image. It is easy to see that $\mathcal{Z}$ is 6-dimensional, and has a 3-dimensional singular locus given by $p_1 = p_2 = p_3$ and $l_{12} = l_{13} = l_{23}$.

Let $\mathbb{F} \subset \mathbb{P}^2 \times \hat{\mathbb{P}^2}$ be the variety of pairs (p, l) with $p \in l$. Then we have a map $\mathcal{Z} \to \mathbb{F}$ given by $z \mapsto (p_1, l_{12})$. This map is a fibration, and one can show that the fiber Z has an isolated singularity. In Section 8, we construct a local model for this singularity: it is the affine variety U with coordinate ring $\mathbf{k}[a, b, c, A, B, C]/I$, where

$$I = \langle a - b + c, A - B + C, aB - bA, aC - cA, bC - cB\rangle.$$

The coordinates a, b, c (respectively A, B, C) are related to the Plücker coordinates for $\mathbb{P}^2$ (resp. $\hat{\mathbb{P}^2}$), and the singular point u_0 corresponds to the maximal ideal $(a, \ldots, C)$.

1.4. To set the stage for our construction, we recall the geometry of Schubert's compactification, as described by Semple [16]. Let $u \in U$ be a regular point. Then one can show that the triangle $(p_1, \ldots, l_{23})$ corresponding to u determines a net of plane conics, which can be thought of as a point in $\mathbb{G}_3(\mathbf{k}^6)$, the Grassmannian of 3-planes in $\mathbf{k}^6$. Hence we have a map

$$U \smallsetminus \{u_0\} \to \mathbb{G}_3(\mathbf{k}^6), \tag{1}$$

and Schubert's compactification is defined to be the closure of the graph. It is not hard to see that this closure is nonsingular, and that the fiber over u_0 is a $\mathbb{P}^1$. Moreover, this construction is easily globalized—the map (1) becomes a map $\mathcal{Z}^\circ \to \mathbb{G}_3(\mathbf{k}^6)$, and again to compactify we take the closure of the graph.

1.5. Now we describe our construction. The key is a combinatorial interpretation of the ideal I. Let t and T be two triangles, and assign the coordinates a, b, c (respectively A, B, C) as "values" to the edges of the triangle t (resp. T). A valued pair t, T determines a point in U if and only if

1. the values (with appropriate signs) sum to zero around both t and T; and
2. the triple of values on t is proportional to the triple of values on T.

Thus the points of U can be encoded in the triangles t, T (Figure 1). We emphasize that these triangles are *combinatorial* objects and, in particular, should not be confused with the *geometric* triangle $p_1, \ldots, l_{23}$. Nevertheless, the valued triangles t, T completely capture the local geometry of U and, in our opinion, precisely quantify the notion of the "shape" of a geometric triangle in U.

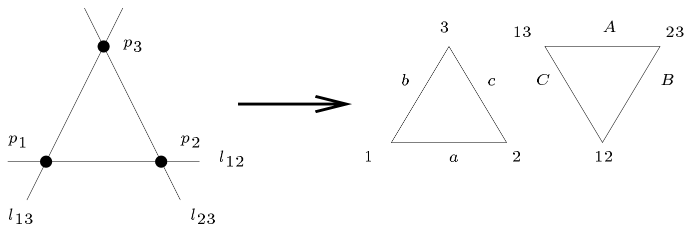

FIGURE 1.

1.6. In the spirit of Schubert, we desingularize U by adding data recording the infinitesimal shape of a geometric triangle as all points and lines come together. Motivated by the above description of I, we define a map

$$U \smallsetminus \{u_0\} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \tag{2}$$

by

$$(a, b, c, A, B, C) \longmapsto [a : b : c] \times [A : B : C]. \tag{3}$$

We can view a point in the image of (2) as a valued pair of triangles $\widetilde{t}$ and $\widetilde{T}$, but now the triple of values on each triangle is defined only up to a scalar multiple. Hence we have replaced Schubert's net of conics with the pair $\widetilde{t}$, $\widetilde{T}$.

Let $\widetilde{U}$ be the closure of the graph of (2), and let $\pi\colon \widetilde{U} \to U$ be the projection. Figure 2 shows a general point in $\widetilde{U}$. It is easy to show that $\widetilde{U}$ is nonsingular, and that again the fiber over u_0 is a projective line[1]. For an example of the data recorded by this fiber, assume $\mathbf{k} = \mathbb{C}$, and let $\varphi_1, \varphi_2\colon \mathbb{C} \to U$ be families of triangles given by

$$\varphi_1(s) = (s, s + s^2, s^2, s, s + s^2, s^2),$$
$$\varphi_2(s) = (s, s^2, s + s^2, s, s^2, s + s^2).$$

[1]However $\widetilde{U}$ is not the same as Schubert's compactification. See Remark 9.7.

As $s \to 0$, both curves approach u_0, but φ_1 brings the points p_2 and p_3 together faster than the other pairs, whereas φ_2 brings the points p_1 and p_3 together faster than the other pairs. In the lifts of these families to $\widetilde{U}$ the limits of $\pi^{-1} \circ \varphi_1$ and $\pi^{-1} \circ \varphi_2$ are different points in $\pi^{-1}(u_0)$ (Figure 3).

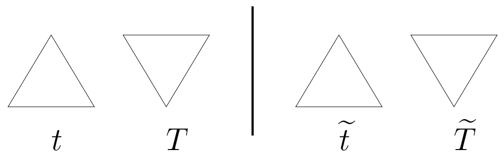

FIGURE 2.

FIGURE 3. The limits of $\pi^{-1} \circ \varphi_1$ and $\pi^{-1} \circ \varphi_2$.

1.7. We now outline the progression of the paper and describe our general construction. In Section 2 we define the singular space $\mathcal{Z}$. Section 3 contains the global picture of the material in 1.5–1.6, as well as the definition of our compactification $\widetilde{\mathcal{Z}}$. Our main idea is that the construction of $\widetilde{\mathcal{Z}}$ depends on the combinatorics of *hypersimplices*, in particular, the $(n-1)$-dimensional hypersimplices Δ^n_k, $k = 1, \dots, n-1$. These polytopes, discussed in [7], are intimately related to the geometry of the Grassmannians $\mathbb{G}^n_k$ of $(k-1)$-flats in $\mathbb{P}^{n-1}$, $k = 1, \dots, n-1$.

Let $\mathcal{E}$ be the set of all edges of all the hypersimplices Δ^n_k. The vertices of the hypersimplex Δ^n_k correspond to k-element subsets of $\{1, \dots, n\}$, hence any edge $\alpha \in \mathcal{E}$ corresponds to a pair $\{I, J\}$ of subsets with the same cardinality. For each $\alpha \in \mathcal{E}$, we associate a $\mathbb{P}^1$-bundle $\mathbf{P}_\alpha \to \mathcal{Z}$ with two canonical sections u^I_α and u^J_α. These sections correspond to the pair of flats V_I, V_J in a geometric simplex in $\mathbb{P}^{n-1}$ which are labeled by the subsets I and J, respectively. If F_I and F_J denote the corresponding linear subspaces of $\mathbf{k}^n$, then the sections u^I_α and u^J_α track the movement of F_I and F_J in the quotient $\mathbb{P}\big((F_I+F_J)/(F_I \cap F_J)\big) \cong \mathbb{P}^1$. We keep track of both sections simultaneously by defining $\mathbf{E}_\alpha$ to be the product bundle $\mathbf{P}_\alpha \times \mathbf{P}_\alpha$ and the section u_α to be the product of u^I_α and u^J_α. Letting $\mathbf{D}_\alpha$ denote the diagonal bundle in $\mathbf{E}_\alpha$, we see that the two flats V_I and V_J coincide precisely when the section u_α intersects the diagonal.

Let $\mathcal{H}$ be the set of faces of dimension at least 2 of all the Δ^n_k. For each $\beta \in \mathcal{H}$, let $\mathbf{E}_\beta$ be the product bundle

$$\mathbf{E}_\beta := \prod_{\alpha \subset \beta} \mathbf{E}_\alpha,$$

and let $\mathbf{D}_\beta$ be the product of the diagonals $\mathbf{D}_\alpha$. By taking products of the u_α we obtain a section $\mathbf{u}_\beta : \mathcal{Z} \to \mathbf{E}_\beta$, and this section intersects $\mathbf{D}_\beta$ precisely when all of the flats labeled by vertices of β coincide.

Now to record the infinitesimal shape of a simplex when flats come together, we blow-up the various bundles $\mathbf{E}_\beta$. Let $(\mathbf{E}_\beta)_\# \to \mathcal{Z}$ be the blow-up of $\mathbf{E}_\beta$ along $\mathbf{D}_\beta$, and define a product bundle

$$\mathbf{E}_\# := \prod_{\beta \in \mathcal{H}} (\mathbf{E}_\beta)_\#.$$

Via the sections $\mathbf{u}_\beta$ we have a map

$$\widetilde{\mathbf{u}} : \mathcal{Z}^\circ \longrightarrow \mathbf{E}_\#,$$

and we define $\widetilde{\mathcal{Z}}$, the space of *complete simplices*, to be the closure of the image of $\widetilde{\mathbf{u}}$.

1.8. Next, in Sections 5–8 we relate these objects to the pictures in 1.5–1.6. By fixing a "flag at infinity" (Section 5) we trivialize the various bundles locally, and express the blow-ups as graphs of rational maps to projective spaces (Section 6). Using these trivializations, we obtain local models U and $\widetilde{U}$ for $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$, respectively (Section 7). In these models, the pair of triangles t, T of Figure 2 becomes the set of $(n-1)$-dimensional hypersimplices, and the pair $\widetilde{t}$, $\widetilde{T}$ becomes the set of all faces of dimension ≥ 2. We attach coordinates to the edges in these figures that can be expressed in terms of Plücker coordinates for certain Grassmannians (Remark 8.3). In Section 8, we describe the equations among the edge coordinates that define (generically) the variety $\widetilde{U}$.

1.9. A crucial part of our proof of the main theorem is that the local model $\widetilde{U}$ has the structure of a vector bundle, with fiber $\mathbf{k}^{n-1}$, over a projective variety Ξ (Section 9). The group $GL_n(\mathbf{k})$ acts on $\widetilde{\mathcal{Z}}$ and the subgroup stabilizing our local model $\widetilde{U}$ is a maximal torus in G. This torus acts with dense orbit on the fibers of the vector bundle and the variety Ξ is precisely the set of torus fixed points in $\widetilde{U}$. We call Ξ the *core* of $\widetilde{U}$. For example, in the triangle space, Ξ is the $\mathbb{P}^1$ lying over $u_0 \in U$.

The importance of the core is that it captures most of the geometry of $\widetilde{U}$, and hence of $\widetilde{\mathcal{Z}}$. In particular, we show that $\widetilde{\mathcal{Z}}$ is nonsingular if and only if Ξ is nonsingular. Moreover, we show that to prove the latter, it suffices to check smoothness at certain *special points* in the core (Section 10).

1.10. The verification of nonsingularity in the case of tetrahedra (Section 11) is, unfortunately, a direct calculation using the combinatorics of the special points in the core. Though all of our results up until Section 11 hold for all n, there are two obstacles to verifying smoothness for $n > 4$. First, the combinatorics required to find all of the types of special points becomes unwieldy for $n > 4$. Second, there are families of special points at which the verification of smoothness requires finding some fairly subtle equations among the edge variables (cf. Lemma 11.8).

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2. The Basic Variety

2.1. Notation. Throughout we let $\mathbf{k}$ be an infinite, algebraically closed field of characteristic zero for convenience, although much of what is say is valid in arbitrary characteristic. Let V be the n-dimensional $\mathbf{k}$-vector space $\mathbf{k}^n$ with standard basis $e_1, \dots, e_n$. Let $\mathbb{P}^{n-1} = \mathbb{P}(V)$ be the projective space of lines in V. Let G be the algebraic group $GL_n(\mathbf{k})$. We denote the finite set $\{1, \dots, n\}$ by $[\![n]\!]$ and the permutation group of $[\![n]\!]$ by S_n.

For any set A and any finite set S, we let A^S denote the set of functions from S to A. We let $\|S\|$ denote the cardinality of the set S; thus, any ordering of S determines a bijection from A^S to the product $A^{\|S\|}$.

2.2. Let $\mathbb{G}_i = \mathbb{G}_i(V)$ be the Grassmannian of i-dimensional subspaces of V. For any subset $I \subset [\![n]\!]$, we let $\mathbb{G}_I = \mathbb{G}_i$, where $i = \|I\|$, and we let $E_I \in \mathbb{G}_I$ be the subspace spanned by $\{e_i \mid i \in I\}$. For ease of notation, we include the cases $\mathbb{G}_0$ and $\mathbb{G}_n$, though each is just a single point.

Let $\mathcal{X}$ be the product of Grassmannians

$$\mathcal{X} = \prod_{I \subset [\![n]\!]} \mathbb{G}_I,$$

and let $\pi_I \colon \mathcal{X} \to \mathbb{G}_I$ be the projection to the Ith factor. The group G acts on each $\mathbb{G}_I$ and on $\mathcal{X}$ by left multiplication, and each projection is G-equivariant.

Definition 2.3. Let $p_0 \in \mathcal{X}$ be the point defined by $\pi_I(p_0) = E_I$ for all $I \subset [\![n]\!]$. Let $\mathcal{Z}^\circ \subset \mathcal{X}$ be the G-orbit of p_0, and let $\mathcal{Z} \subset \mathcal{X}$ be the closure of $\mathcal{Z}^\circ$. The space $\mathcal{Z}$ (respectively $\mathcal{Z}^\circ$) is called the *space of simplices (resp. nondegenerate simplices) in* $\mathbb{P}^{n-1}$.

Remark 2.4. Note that since G preserves incidence of subspaces of V, for any $p \in \mathcal{Z}$ we have $\pi_I(p) \subset \pi_J(p)$ if $I \subset J$.

2.5. The symmetric group S_n acts on the subsets of $[\![n]\!]$ preserving cardinality and, therefore, on $\mathcal{X}$ by permuting the factors. On the other hand, the symmetric group acts on $\mathcal{X}$ as the permutation subgroup in $G = GL_n(k)$. To avoid confusion between these actions, we represent the first action as a right action, and the second action (as with the G-action) as a left action. Thus, for $\sigma \in S_n$ and $p \in \mathcal{X}$, $p \cdot \sigma$ is defined by $\pi_I(p \cdot \sigma) = \pi_{\sigma \cdot I}(p)$ for all $I \subset [\![n]\!]$. Unless otherwise stated, when refering to the symmetric group action on $\mathcal{X}$ we shall always mean the right action.

Proposition 2.6. *The symmetric group action on $\mathcal{X}$ restricts to an action on $\mathcal{Z}$.*

Proof. It suffices to show that the symmetric group action restricts to an action on $\mathcal{Z}^\circ$. But this is obvious since $(g \cdot p_0) \cdot \sigma = (g\sigma^{-1}) \cdot p_0$ for all $\sigma \in S_n$ and $g \in G$. $\square$

Remark 2.7. For any point $p \in \mathcal{Z}$, the collection of subspaces $\{\pi_I(p) \mid I \subset [\![n]\!]\}$ is invariant under the (right) S_n-action. Thus, we can think of this action as "changing the labels" on a geometric simplex. Note, however, that the way we have defined the action is such that if a subspace of p is labeled by $I \subset [\![n]\!]$, then the corresponding subspace of $p \cdot \sigma$ is labeled by the subset $\sigma^{-1} \cdot I$.

3. The Global Resolution

3.1. We will construct a resolution of $\mathcal{Z}$ based on the combinatorics of hypersimplices, and so we recall basic facts about them. For more details, see [7].

Let $\{e_i \mid i \in [\![n]\!]\}$ be the standard basis of $\mathbb{R}^n$. If $I \subset [\![n]\!]$, then e_I denotes the vector $\sum_{i \in I} e_i$. Then the *simplex* Δ^n is the closed convex hull of $\{e_i \mid i \in [\![n]\!]\}$. More generally, if $1 \le k \le n-1$, then the *hypersimplex* Δ^n_k is the closed convex hull of

$$\{e_I \mid I \subset [\![n]\!] \quad \text{and} \quad \|I\| = k\}.$$

For each $k \in [\![n]\!]$, let $\mathcal{P}_k$ be the face poset of Δ^n_k. The combinatorial structure of the hypersimplices can be described as follows. Let β be a collection of subsets of $[\![n]\!]$, let $I^-(\beta)$ be the intersection $\bigcap \beta$, and let $I^+(\beta)$ be the union $\bigcup \beta$. The collection β is called *saturated* if there exists some $k \in [\![n]\!]$ such that $\|I\| = k$ for all $I \in \beta$ and β contains every k-element subset J satisfying $I^-(\beta) \subset J \subset I^+(\beta)$. The number k is called the *rank of* β, and the cardinality of the set $I^+(\beta) \smallsetminus I^-(\beta)$ is called the *dimension of* β. We denote the set of saturated collections (of rank k) by $\mathcal{F}$ (respectively $\mathcal{F}_k$), noting that $\mathcal{F}$ is partially ordered by inclusion of sets.

Proposition 3.2. *The map $\mathcal{F}_k \to \mathcal{P}_k$ given by mapping β to the convex hull of $\{e_I \mid I \in \beta\}$ is an isomorphism of posets. The dimension of β is equal to the dimension of the corresponding face.*

This proposition allows us to refer to elements of $\mathcal{F}$ as *faces* of the hypersimplices.

Example 3.3. Let $\mathcal{E}$ (respectively $\mathcal{E}_k$) be the subset of $\mathcal{F}$ (resp. $\mathcal{F}_k$) consisting of one-dimensional faces, or *edges*. Let $\alpha \in \mathcal{E}$, and let $I^+ = I^+(\alpha)$ and let $I^- = I^-(\alpha)$. Since $I^+ \smallsetminus I^-$ contains 2 elements, say i and j, there are four possibilities for an element of α. They are

$$I^-,$$
$$\{i\} \cup I^-, \{j\} \cup I^-,$$
$$\{i, j\} \cup I^-.$$

The only subcollection of these four sets that is saturated is $\{I, J\}$ where $I = \{i\} \cup I^-$, and $J = \{j\} \cup I^-$. Thus any edge in $\mathcal{E}_k$ is of the form $\{I, J\}$ where $\|I\| = \|J\| = k$ and $\|I \cap J\| = k - 1$ (or equivalently, $\|I\| = \|J\| = k$ and $\|I \cup J\| = k + 1$).

Example 3.4. Let $\mathcal{T}$ (respectively $\mathcal{T}_k$) be the subset of $\mathcal{F}$ (respectively $\mathcal{F}_k$) consisting of two-dimensional faces. Let $\beta \in \mathcal{E}$, and let $I^+ = I^+(\beta)$ and let $I^- = I^-(\beta)$. Since $I^+ \smallsetminus I^-$ contains 3 elements, say i_1, i_2, and i_3, there are eight possibilities for an element of β. They are

$$I^-,$$
$$\{i_1\} \cup I^-, \{i_2\} \cup I^-, \{i_3\} \cup I^-,$$
$$\{i_2, i_3\} \cup I^-, \{i_1, i_3\} \cup I^-, \{i_1, i_2\} \cup I^-,$$
$$\{i_1, i_2, i_3\} \cup I^-.$$

The only saturated subcollections of these (of dimension 2) are the second and third lines. They are $\beta = \{I_1, I_2, I_3\}$ where $I_1 = \{i_1\} \cup I^-$, $I_2 = \{i_2\} \cup I^-$, $I_3 = \{i_3\} \cup I^-$, and $\beta^* = \{J_1, J_2, J_3\}$ where $J_1 = \{i_2, i_3\} \cup I^-$, $J_2 = \{i_1, i_3\} \cup I^-$, and $J_3 = \{i_1, i_2\} \cup I^-$. Thus, 2-dimensional faces are all triangles, and we call two faces of the form β and β^* *related triangles*. Note that β is in $\mathcal{T}_k$ and β^* is in $\mathcal{T}_{k+1}$ where $k = \|I^-\| + 1$.

Example 3.5. Each $\mathcal{F}_k$ has a unique maximal element γ_k corresponding to the full hypersimplex Δ_k^n. By definition

$$\gamma_k = \{I \subset [\![n]\!] \mid \|I\| = k\}.$$

Example 3.6. For $n = 4$, Figure 4 shows the three maximal hypersimplices. The rank-1 hypersimplex Δ_1^4 and the rank-3 hypersimplex Δ_3^4 are both tetrahedra, while the rank-2 hypersimplex Δ_2^4 is an octahedron. As drawn, the figure encodes two important relationships among subfaces. First, suppose $\alpha_k = \{I_k, J_k\}$, $k = 1, 2$, are two edges in $\mathcal{E}$, and suppose i_k (resp. j_k) is the unique element of I_k (resp. J_k) not in $I_k \cap J_k$. Then the two edges are parallel in the figure if and only if $\{i_1, j_1\} = \{i_2, j_2\}$. Similarly, two triangles β and β^* are related triangles if and only if in the figure they are in adjacent maximal faces and one differs from the other by a 180° rotation (and possibly a dilation).

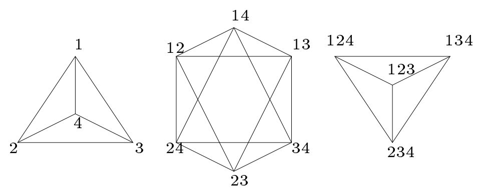

FIGURE 4. Hypersimplices for $n = 4$.

3.7. We now establish a correspondence between edges of the hypersimplices and certain $\mathbb{P}^1$-bundles over the space of simplices. For any $I \subset [\![n]\!]$, we let $\mathbf{F}_I \to \mathcal{X}$ be the pull-back of the canonical $\|I\|$-plane bundle on $\mathbb{G}_I$ under the projection π_I. Restricted to $\mathcal{Z}$, the incidence conditions imply that if $I \subset J$, then $\mathbf{F}_I \to \mathcal{Z}$ is a subbundle of $\mathbf{F}_J \to \mathcal{Z}$.

Let $\alpha = \{I, J\}$ be an edge in $\mathcal{E}$. Let $\mathbf{Q}_\alpha \to \mathcal{Z}$ be the quotient bundle $\mathbf{F}_{I\cup J}/\mathbf{F}_{I\cap J}$, and let $\mathbf{P}_\alpha$ be the projectivized bundle $\mathbb{P}(\mathbf{Q}_\alpha) \to \mathcal{Z}$. Since $I \cup J$ has 2 more elements than $I \cap J$, the bundle $\mathbf{Q}_\alpha$ is a 2-plane bundle and $\mathbf{P}_\alpha$ is a $\mathbb{P}^1$-bundle.

For any $p \in \mathcal{Z}$, the fiber $(\mathbf{Q}_\alpha)_p$ (respectively $(\mathbf{P}_\alpha)_p$) is canonically identified with $\pi_{I\cup J}(p)/\pi_{I\cap J}(p)$ (resp. $\mathbb{P}(\pi_{I\cup J}(p)/\pi_{I\cap J}(p))$). The $\mathbf{k}$-span of any $a(p) \in \pi_{I\cup J}(p) \smallsetminus \pi_{I\cap J}(p)$ determines an element of $(\mathbf{P}_\alpha)_p$ that we denote by $[a(p)]$.

3.8. Each $\mathbb{P}^1$-bundle $\mathbf{P}_\alpha \to \mathcal{Z}$ has two canonical sections corresponding to the two vertices I and J of the edge α. These sections are defined as follows. Let $p \in \mathcal{Z}$. Since the planes $\pi_I(p)$ and $\pi_J(p)$ are both contained in $\pi_{I\cup J}(p)$ and both contain $\pi_{I\cap J}(p)$, they each define elements of $(\mathbf{P}_\alpha)_p$, the fiber over p of the bundle $\mathbf{P}_\alpha$. Allowing p to vary, we obtain sections $u^I_\alpha : \mathcal{Z} \to \mathbf{P}_\alpha$ and $u^J_\alpha : \mathcal{Z} \to \mathbf{P}_\alpha$ corresponding to I and J, respectively.

To keep track of both sections, we define the *edge bundle* $\mathbf{E}_\alpha \to \mathcal{Z}$ to be the bundle $(\mathbf{P}_\alpha)^\alpha \to \mathcal{Z}$. (Note that $\mathbf{E}_\alpha$ is isomorphic (over $\mathcal{Z}$) to the product $\mathbf{P}_\alpha \times \mathbf{P}_\alpha$, but this isomorphism depends on a choice of ordering for $\alpha = \{I, J\}$.) The product of the two sections u^I_α and u^J_α determines a section $u_\alpha : \mathcal{Z} \to \mathbf{E}_\alpha$.

3.9. More generally, we let $\mathcal{H}$ (respectively $\mathcal{H}_k$) be the subset of $\mathcal{F}$ (resp. $\mathcal{F}_k$) consisting of faces whose dimension is at least 2. For each $\beta \in \mathcal{H}$, we let $\mathcal{E}(\beta)$ be the set $\{\alpha \in \mathcal{E} \mid \alpha \subset \beta\}$ of edges in β and let $\mathbf{E}_\beta \to \mathcal{Z}$ be the product bundle

$$\mathbf{E}_\beta = \prod_{\alpha\in\mathcal{E}(\beta)} \mathbf{E}_\alpha.$$

We let $\mathbf{u}_\beta : \mathcal{Z} \to \mathbf{E}_\beta$ be the product of the sections u_α where α ranges over all edges in $\mathcal{E}(\beta)$.

We let $\mathbf{E} \to \mathcal{Z}$ be the product bundle

$$\mathbf{E} = \prod_{\beta\in\mathcal{H}} \mathbf{E}_\beta$$

and let $\mathbf{u}$ be the product of the sections $\mathbf{u}_\beta$ where β ranges over all elements of $\mathcal{H}$. The bundle $\mathbf{E}$ will be called the *ambient bundle*.

3.10. In order to get a smooth compactification, we need to keep track of additional data whenever certain collections of the subspaces $\{\pi_I(p)\}$ coincide. The simplest of these "degenerations" correspond to those points in $\mathcal{Z}$ where the sections u^I_α and u^J_α coincide or, equivalently, to places where the section u_α intersects the diagonal in $\mathbf{E}_\alpha \cong \mathbf{P}_\alpha \times \mathbf{P}_\alpha$. To record infinitesimal changes at these degenerations, we blow-up our bundles along products of these diagonals. Since the diagonal bundle in $\mathbf{E}_\alpha$ is codimension-one for any edge $\alpha \in \mathcal{E}$, blowing up $\mathbf{E}_\alpha$ along the diagonal has no effect. We, therefore, restrict our attention to the bundles $\mathbf{E}_\beta$ where $\beta \in \mathcal{H}$.

For each $\alpha \in \mathcal{E}$, we let $\mathbf{D}_\alpha \subset \mathbf{E}_\alpha$ denote the diagonal bundle, and for each $\beta \in \mathcal{H}$ we let $\mathbf{D}_\beta \subset \mathbf{E}_\beta$ denote the product bundle

$$\mathbf{D}_\beta = \prod_{\alpha \in \mathcal{E}(\beta)} \mathbf{D}_\alpha.$$

Let $b_\beta : (\mathbf{E}_\beta)_\# \to \mathbf{E}_\beta$ be the blow-up of $\mathbf{E}_\beta$ along $\mathbf{D}_\beta$. Then $(\mathbf{E}_\beta)_\#$ is a locally trivial bundle over $\mathcal{Z}$ with fiber isomorphic to the blow-up of $(\mathbb{P}^1)^{\mathcal{E}(\beta)} \times (\mathbb{P}^1)^{\mathcal{E}(\beta)}$ along the diagonal.

Let $\mathbf{E}_\# \to \mathcal{Z}$ be the product bundle

$$\mathbf{E}_\# = \prod_{\beta \in \mathcal{H}} (\mathbf{E}_\beta)_\#.$$

We call this bundle the *complete ambient bundle*. Taking a product of the compositions of projections with blow-up maps, we obtain a proper birational morphism of bundles $\mathbf{E}_\# \to \mathbf{E}$, which we denote by b.

3.11. Restricted to the nondegenerate simplices $\mathcal{Z}^\circ$, each section $\mathbf{u}_\beta : \mathcal{Z} \to \mathbf{E}_\beta$ avoids the locus $\mathbf{D}_\beta$. Hence, the composition $b_\beta^{-1} \circ \mathbf{u}_\beta$ defines a rational map $\mathcal{Z} \dashrightarrow (\mathbf{E}_\beta)_\#$ which is regular on $\mathcal{Z}^\circ$. We denote this map by $\widetilde{\mathbf{u}}_\beta$. Taking products of these $\widetilde{\mathbf{u}}_\beta$ as β runs over elements of $\mathcal{H}$, we obtain a rational section $\mathcal{Z} \dashrightarrow \mathbf{E}_\#$, which we denote by $\widetilde{\mathbf{u}}$. Note that $\widetilde{\mathbf{u}}$ is regular on the open set $\mathcal{Z}^\circ$, and restricted to this set, the composition $b \circ \widetilde{\mathbf{u}}$ coincides with the section $\mathbf{u}$.

Definition 3.12. The closure in $\mathbf{E}_\#$ of the image $\widetilde{\mathbf{u}}(\mathcal{Z}^\circ)$ will be called the *complete space of simplices* and denoted by $\widetilde{\mathcal{Z}}$.

The bundle projection $\mathbf{E}_\# \to \mathcal{Z}$ restricts to a surjective birational morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z}$.

3.13. Since all of the bundles $\mathbf{E}_\beta$ are constructed from canonical bundles, they admit a natural G-action which lifts the action on $\mathcal{Z}$. Moreover, since the diagonals are preserved by this action, the blown-up bundles $(\mathbf{E}_\beta)_\#$ also admit natural G-actions which lift the action on $\mathcal{Z}$. It is clear that the sections $\widetilde{\mathbf{u}}_\beta$ and, therefore, the section $\widetilde{\mathbf{u}} : \mathcal{Z}^\circ \to \mathbf{E}_\#$ commute with this action, giving the following.

Proposition 3.14. *The action of G on $\mathbf{E}_\#$ restricts to an action on $\widetilde{\mathcal{Z}}$, and the map $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is equivariant.*

3.15. The S_n-action on $\mathcal{Z}$ also lifts to an action on $\mathbf{E}$ and to an action on $\mathbf{E}_\#$ as follows. Let $\sigma \in S_n$. Since $\pi_I(p \cdot \sigma) = \pi_{\sigma \cdot I}(p)$ for all $I \subset [\![n]\!]$ and $p \in \mathcal{Z}$, there are canonical isomorphisms of fibers

$$(\mathbf{F}_I)_p \to (\mathbf{F}_{\sigma^{-1} \cdot I})_{p \cdot \sigma}.$$

More generally, the S_n-action on subsets of $[\![n]\!]$ extends in the obvious way to an action on the poset $\mathcal{F}$, and there are canonical isomorphisms of fibers

$$(\mathbf{E}_\beta)_p \to (\mathbf{E}_{\sigma^{-1}.\beta})_{p\cdot\sigma}$$

for all $\beta \in \mathcal{H}$. These isomorphisms define bundle isomorphisms

$$\mathbf{E}_\beta \to \mathbf{E}_{\sigma^{-1}.\beta}$$

and

$$(\mathbf{E}_\beta)_\# \to (\mathbf{E}_{\sigma^{-1}.\beta})_\#$$

which lift the action of σ on $\mathcal{Z}$. Taking a product of the compositions of projections $\mathbf{E} \to \mathbf{E}_\beta$ with the isomorphisms $\mathbf{E}_\beta \to \mathbf{E}_{\sigma^{-1}.\beta}$ as β runs over all elements of $\mathcal{H}$, we obtain a bundle automorphism $\rho_\sigma : \mathbf{E} \to \mathbf{E}$ which lifts the action of σ on $\mathcal{Z}$. Likewise, taking a product of the compositions of projections $\mathbf{E}_\# \to (\mathbf{E}_\beta)_\#$ with the isomorphisms $(\mathbf{E}_\beta)_\# \to (\mathbf{E}_{\sigma^{-1}.\beta})_\#$, we obtain an automorphism $\widetilde{\rho}_\sigma : \mathbf{E}_\# \to \mathbf{E}_\#$ which lifts the action of σ on $\mathcal{Z}$. It is clear from the definition of ρ_σ (resp., $\widetilde{\rho}_\sigma$) that for any $\sigma_1, \sigma_2 \in S_n$, we have $\rho_{\sigma_2}\rho_{\sigma_1} = \rho_{\sigma_1\sigma_2}$ (resp., $\widetilde{\rho}_{\sigma_2}\widetilde{\rho}_{\sigma_1} = \widetilde{\rho}_{\sigma_1\sigma_2}$); hence, $y \cdot \sigma := \rho_\sigma(y)$ (resp., $y \cdot \sigma := \widetilde{\rho}_\sigma(y)$) defines a (right) S_n-action on $\mathbf{E}$ (resp., on $\mathbf{E}_\#$) which lifts the action on $\widetilde{\mathcal{Z}}$.

Proposition 3.16. *The action of S_n on $\mathbf{E}_\#$ restricts to an action on $\widetilde{\mathcal{Z}}$, and the map $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is equivariant.*

Proof. It suffices to show that the S_n-action commutes with the section $\widetilde{\mathbf{u}}$ when restricted to $\mathcal{Z}^\circ$, i.e., that $\widetilde{\mathbf{u}}(p) \cdot \sigma = \widetilde{\mathbf{u}}(p \cdot \sigma)$ for all $p \in \mathcal{Z}^\circ$. We leave the calculation to the reader. □

3.17. The proof of the following theorem is the main point of this article.

Theorem 3.18. *If $n = 4$, $\widetilde{\mathcal{Z}}$ is a smooth variety.*

For completeness we state the following conjecture.

Conjecture 3.19. *If $n > 4$, $\widetilde{\mathcal{Z}}$ is a smooth variety.*

4. The Fibration over the Flag Variety

4.1. Let $\mathbb{F} = \mathbb{F}(V)$ be the variety of full flags F_* of linear subspaces in V:

$$F_* := \{F_0 := \{0\} \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n := V\}.$$

Alternatively, we may think of $\mathbb{F}$ as the variety of full flags of flats in $\mathbb{P}^{n-1}$.

The group G acts transitively on $\mathbb{F}$ and the stabilizer of any flag F_* is a Borel subgroup $B(F_*)$. For any p in $\mathcal{Z}$, the incidence conditions imply that the subspaces $\{0\}$, $\pi_{\{1\}}(p)$, $\pi_{\{1,2\}}(p)$, …, $\pi_{\{1,2,\dots,n-1\}}(p)$, V form a flag. We denote this flag by $\pi(p)$.

Definition 4.2. For any flag $F_* \in \mathbb{F}$, let $Z(F_*)$ be the set of points p in $\mathcal{Z}$ such that $\pi(p) = F_*$. The variety $Z(F_*)$ is called the *space of simplices based at* F_*. The subset $Z^\circ(F_*) = \mathcal{Z}^\circ \cap Z(F_*)$ is called the *space of nondegenerate simplices based at* F_*.

If $F_* = \pi(p_0)$, then p_0 lies in $Z(F_*)$, as does the entire $B(F_*)$-orbit of p_0. The variety $Z^\circ(F_*)$ is precisely this $B(F_*)$-orbit, and $Z(F_*)$ is the closure of $Z^\circ(F_*)$ in $\mathcal{X}$.

4.3. The following proposition will allow us to focus attention on $Z(F_*)$ instead of $\mathcal{Z}$. It follows from the fact that $\mathcal{Z} \to \mathbb{F}$ can be identified with the bundle

$$G \times_{B(F_*)} Z(F_*) \to G/B(F_*).$$

Proposition 4.4. *The projection $\pi : \mathcal{Z} \to \mathbb{F}$ is a locally trivial G-equivariant fibration whose fiber over a flag $F_* \in \mathbb{F}$ is $Z(F_*)$.*

4.5. If F_* is any flag in $\mathbb{F}$, then the section $\widetilde{u} : \mathcal{Z}^\circ \to \mathbf{E}_\#$ is also well defined when restricted to $Z^\circ(F_*)$, the nondegenerate simplices based at F_*. This allows us to restrict the global resolution to the fiber of $\pi : \mathcal{Z} \to \mathbb{F}$.'

Definition 4.6. The closure in $\mathbf{E}_\#$ of the image $\widetilde{u}(Z^\circ(F_*))$ is called the *complete space of simplices based at* F_* and denoted by $\widetilde{Z}(F_*)$.

Since the restricted bundle $\mathbf{E}_\#|_{Z(F_*)} \to Z(F_*)$ is closed in $\mathbf{E}_\#$, $\widetilde{Z}(F_*)$ is a subvariety of $\mathbf{E}_\#|_{Z(F_*)}$, so the bundle projection defines a surjective birational morphism $\widetilde{Z}(F_*) \to Z(F_*)$.

Let $\widetilde{\pi} : \widetilde{\mathcal{Z}} \to \mathbb{F}$ be the composition of $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ with the fibration $\pi : \mathcal{Z} \to \mathbb{F}$. The following lemma reduces the proof of the main theorem to proving that $\widetilde{Z}(F_*) \to Z(F_*)$ is a smooth $B(F_*)$-equivariant resolution for any given $F_* \in \mathbb{F}$. As in the case of the fibration $\mathcal{Z} \to \mathbb{F}$, it follows from the fact that $\widetilde{\mathcal{Z}} \to \mathbb{F}$ is the associated bundle for the $B(F_*)$-action on $\widetilde{Z}(F_*)$.

Lemma 4.7. *The projection $\widetilde{\pi} : \widetilde{\mathcal{Z}} \to \mathbb{F}$ is a locally trivial G-equivariant fibration whose fiber over the flag $F_* \in \mathbb{F}$ is $\widetilde{Z}(F_*)$.*

5. Fixing the Flag at Infinity

5.1. Given a flag $A_* \in \mathbb{F}$, let $\mathcal{Z}(A_*)$ be the set of all $p \in \mathcal{Z}$ in general position to A_*. That is, $\mathcal{Z}(A_*)$ consists of all $p \in \mathcal{Z}$ such that the $\|I\|$-plane $\pi_I(p)$ is transverse to A_i for all $i \in [\![n]\!]$ and $I \subset [\![n]\!]$. Let $\mathbf{E}_\#(A_*) \to \mathcal{Z}(A_*)$ be the restriction of the ambient bundle $\mathbf{E}_\#$ to $\mathcal{Z}(A_*)$. Let $\mathcal{Z}^\circ(A_*) = \mathcal{Z}(A_*) \cap \mathcal{Z}^\circ$, and let $\widetilde{\mathcal{Z}}(A_*)$ be the closure of $\widetilde{u}(\mathcal{Z}^\circ(A_*))$ in the restricted bundle $\mathbf{E}_\#(A_*)$. Bundle projection defines a map $\widetilde{\mathcal{Z}}(A_*) \to \mathcal{Z}(A_*)$ which is an isomorphism on the preimage of $\mathcal{Z}^\circ(A_*)$. The Borel subgroup $B(A_*)$ of G stabilizing the flag A_* acts on $\mathcal{Z}(A_*)$, $\mathcal{Z}^\circ(A_*)$, $\mathbf{E}_\#(A_*)$, and $\widetilde{\mathcal{Z}}(A_*)$. Both the bundle projection $\mathbf{E}_\#(A_*) \to \mathcal{Z}(A_*)$ and its restriction map $\widetilde{\mathcal{Z}}(A_*) \to \mathcal{Z}(A_*)$ are $B(A_*)$-equivariant.

The subvariety $\mathcal{Z}(A_*)$ is the intersection of $\mathcal{Z}$ with an affine open subset of the product $\mathcal{X}$ of Grassmannians; hence, it is an affine open subvariety of $\mathcal{Z}$. It follows that the restricted bundle $\mathbf{E}_\#(A_*)$ is an open subvariety of the complete ambient bundle $\mathbf{E}_\#$, hence $\widetilde{Z}(A_*)$ is an open subvariety of $\widetilde{Z}$. Moreover, since any $p \in \mathcal{Z}$ is contained in some $\mathcal{Z}(A_*)$, any $\widetilde{p} \in \widetilde{\mathcal{Z}}$ is contained in some $\widetilde{\mathcal{Z}}(A_*)$. Summarizing, we have the following lemma.

Lemma 5.2. *The collection $\{\mathcal{Z}(A_*)\colon A_* \in \mathbb{F}\}$ is an affine open cover for $\mathcal{Z}$ and the collection $\{\widetilde{\mathcal{Z}}(A_*)\colon A_* \in \mathbb{F}\}$ is an open cover for $\widetilde{\mathcal{Z}}$. The group G acts transitively on these covers.*

5.3. Let $\mathbb{F}(A_*) \subset \mathbb{F}$ be the affine open subset consisting of all flags in general position to A_*. Let $\pi : \mathcal{Z}(A_*) \to \mathbb{F}(A_*)$ be the restriction of the the fibration $\mathcal{Z} \to \mathbb{F}$. Note that $B(A_*)$ acts transitively on $\mathbb{F}(A_*)$ and that both π and the composition $\widetilde{\pi}$ of $\widetilde{\mathcal{Z}}(A_*) \to \mathcal{Z}(A_*)$ with π are $B(A_*)$-equivariant. For any $F_* \in \mathbb{F}(A_*)$, we let $Z(F_*, A_*) = \mathcal{Z}(A_*) \cap Z(F_*)$, $Z^\circ(F_*, A_*) = \mathcal{Z}^\circ(A_*) \cap Z(F_*)$, and $\widetilde{Z}(F_*, A_*) = \widetilde{\mathcal{Z}}(A_*) \cap \widetilde{Z}(F_*)$. We note that $\widetilde{Z}(F_*, A_*)$ is the closure of $\widetilde{u}(Z^\circ(F_*, A_*))$ in the restricted bundle $\mathbf{E}_\#(A_*)$. Equivalently, if $\mathbf{E}_\#(F_*, A_*)$ denotes the restriction of the complete ambient bundle to $Z(F_*, A_*)$, then $\widetilde{Z}(F_*, A_*)$ is the closure of $\widetilde{u}(Z^\circ(F_*, A_*))$ in $\mathbf{E}_\#(F_*, A_*)$. Proposition 4.4 and Lemma 4.7 have the following local version.

Lemma 5.4. *The map $\pi : \mathcal{Z}(A_*) \to \mathbb{F}(A_*)$ (respectively $\widetilde{\pi} : \widetilde{\mathcal{Z}}(A_*) \to \mathbb{F}(A_*)$) is a locally trivial $B(A_*)$-equivariant fibration whose fiber over $F_* \in \mathbb{F}(A_*)$ is $Z(F_*, A_*)$ (respectively $\widetilde{Z}(F_*, A_*)$).*

5.5. Let $F_* \in \mathbb{F}(A_*)$. Restricting the collections of Lemma 5.2 to the fibers $Z(F_*)$ and $\widetilde{Z}(F_*)$, we have the following lemma.

Lemma 5.6. *The collection $\{Z(F_*, A_*)\colon A_* \in \mathbb{F}(F_*)\}$ is an affine open cover for $Z(F_*)$ and the collection $\{\widetilde{Z}(F_*, A_*)\colon A_* \in \mathbb{F}(F_*)\}$ is an open cover for $\widetilde{Z}(F_*)$. The group $B(F_*)$ acts transitively on these covers.*

Definition 5.7. The subvariety $\widetilde{Z}(F_*, A_*)$ is called a *chart for $\widetilde{\mathcal{Z}}$.*

Note that a chart $\widetilde{Z}(F_*, A_*)$ is open in the fiber $\widetilde{Z}(F_*)$ but is not open in $\widetilde{\mathcal{Z}}$. The group G acts transitively on the collection of charts, and the stabilizer of any given chart is a maximal torus. Namely, the stabilizer of the chart $\widetilde{Z}(F_*, A_*)$ is the unique maximal torus $T(F_*, A_*)$ contained in both Borel subgroups $B(F_*)$ and $B(A_*)$. The varieties $Z(F_*, A_*)$, $Z^\circ(F_*, A_*)$ and $\widetilde{Z}(F_*, A_*)$ are all $T(F_*, A_*)$-stable and the restriction $\widetilde{Z}(F_*, A_*) \to Z(F_*, A_*)$ of the bundle map is $T(F_*, A_*)$-equivariant.

6. Bundle Trivializations

6.1. Restricting to the open subvariety $\mathcal{Z}(A_*)$ of $\mathcal{Z}$, we can find open subbundles $\mathbf{E}_\beta(A_*) \to \mathcal{Z}(A_*)$ of the bundles $\mathbf{E}_\beta$ that have affine space fibers and that contain the images of the section $\mathbf{u}_\beta$. The open subvariety $\widetilde{\mathcal{Z}}(A_*)$ of $\mathcal{Z}$ can then be described by applying the construction of Section 3 to these affine bundles and their blow-ups. We first define trivializations of the affine bundles. These trivializations depend on two choices: a total ordering of the subsets of $[\![n]\!]$ and a basis for V compatible with the flag A_*. For the former, we use the following.

Definition 6.2. The *standard total ordering* of the power set of $[\![n]\!]$ is the ordering defined by $I < J$ if and only if

1. I has fewer elements than J, or
2. I and J have the same cardinality and the smallest element of $I \smallsetminus (I \cap J)$ is less than the smallest element of $J \smallsetminus (I \cap J)$.

6.3. Let A_* be a flag in $\mathbb{F}$, and choose an element $a \in G$ such that $a(e_i) \in A_{n+1-i} \smallsetminus A_{n-i}$ for $i = 1, \dots, n$. We let $a_i = a(e_i)$ for all i, giving a basis $a_1, \dots, a_n$ for V. Recall that for any $I \subset [\![n]\!]$, $\mathbf{F}_I \to \mathcal{Z}$ is the $\|I\|$-plane bundle whose fiber over p is canonically identified with $\pi_I(p) \subset V$. Let $\mathbf{F}_I(A_*)$ be the restriction of this bundle to $\mathcal{Z}(A_*)$. The choice a determines sections $s_I : \mathcal{Z}(A_*) \to \mathbf{F}_I(A_*)$ for all $I \subset [\![n]\!]$ as follows. Let I be any subset of $[\![n]\!]$ and let $i = \|I\|$. For each $p \in \mathcal{Z}(A_*)$, the intersection $\pi_I(p) \cap A_{n+1-i}$ is a one-dimensional subspace of V, and the intersection $\pi_I(p) \cap A_{n-i}$ is zero-dimensional. It follows that there is a unique vector $s_I(p) \in \pi_I(p) \cap A_{n+1-i}$ of the form

$$s_I(p) = a_i + \sigma_I^{i+1}(p)a_{i+1} + \cdots + \sigma_I^n(p)a_n.$$

We let $s_I : \mathcal{Z}(A_*) \to \mathbf{F}_I(A_*)$ be the section defined by $p \mapsto s_I(p)$. Note that if I is a proper subset of J, then $s_J(p) \in \pi_J(p) \smallsetminus \pi_I(p)$ for all $p \in \mathcal{Z}(A_*)$. Figure 5 shows the $n = 3$ case. The solid triangle represents the point p, while the hollow point, dotted line, and dotted square represent the fixed flag A_*.

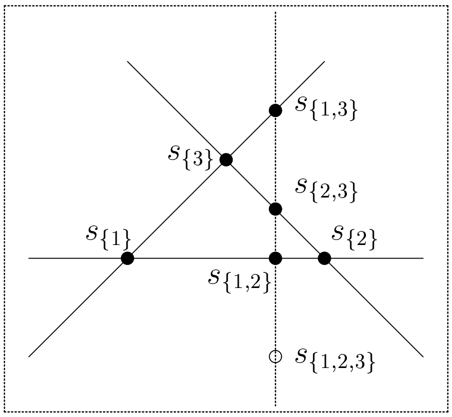

FIGURE 5.

6.4. Let $\alpha = \{I, J\}$ be an edge in $\mathcal{E}$. Since $s_{I\cup J}(p) \in \pi_{I\cup J}(p) \smallsetminus \pi_{I\cap J}(p)$ for all $p \in \mathcal{Z}(A_*)$, the section $s_{I\cup J}$ defines a nonzero section of the restricted bundle $\mathbf{Q}_\alpha|_{\mathcal{Z}(A_*)}$ and, hence, a section $[s_{I\cup J}]$ of the restricted bundle $\mathbf{P}_\alpha|_{\mathcal{Z}(A_*)}$. We let $\mathbf{P}_\alpha(A_*)$ denote the complement of this section. Having removed a point from each fiber of the $\mathbb{P}^1$-bundle $\mathbf{P}_\alpha$, we have that $\mathbf{P}_\alpha(A_*)$ is an affine line bundle over $\mathcal{Z}(A_*)$ which is open in the restricted bundle $\mathbf{P}_\alpha|_{\mathcal{Z}(A_*)}$.

Since $s_{I\cup J}(p) \in \pi_{I\cup J}(p) \smallsetminus \pi_I(p)$ and $s_I(p) \in \pi_I(p) \smallsetminus \pi_{I\cap J}(p)$, the images of $s_I(p)$ and $s_{I\cup J}(p)$ span the fiber $(\mathbf{Q}_\alpha)_p$ for all $p \in \mathcal{Z}(A_*)$. It follows that any element $y = y_p$ in $\mathbf{P}(A_*)_p$ can be written in the form

$$[s_I(p) + f^I_\alpha(y) \cdot s_{I\cup J}(p)]$$

for a unique element $f^I_\alpha(y) \in \mathbf{k}$. We let $f^I_\alpha : \mathbf{P}_\alpha(A_*) \to \mathbf{k}$ be the function defined fiberwise by $y_p \mapsto f^I_\alpha(p)$. Likewise, by interchanging I and J, we obtain a function $f^J_\alpha : \mathbf{P}_\alpha(A_*) \to \mathbf{k}$. The product of bundle projection with either f^I_α or f^J_α defines a trivialization

$$\mathbf{P}_\alpha(A_*) \to \mathcal{Z}(A_*) \times \mathbf{k}.$$

Let $\mathbf{E}_\alpha(A_*) = (\mathbf{P}_\alpha(A_*))^\alpha$. Then $\mathbf{E}_\alpha(A_*)$ is an affine $(\mathbf{k}\times\mathbf{k})$-bundle over $\mathcal{Z}(A_*)$ and is open in the restricted bundle $\mathbf{E}_\alpha|_{\mathcal{Z}(A_*)}$. Assume $I < J$ in the standard ordering, and let f_α be the product $f^I_\alpha \times f^J_\alpha$. Then the product of bundle projection and f_α defines a trivialization

$$\mathbf{E}_\alpha(A_*) \to \mathcal{Z}(A_*) \times (\mathbf{k}\times\mathbf{k}).$$

6.5. More generally, let $\beta \in \mathcal{H}$ and defin $\mathbf{E}_\beta(A_*)$ to be the product

$$\mathbf{E}_\beta(A_*) = \prod_{\alpha\in\mathcal{E}(\beta)} \mathbf{E}_\alpha(A_*).$$

Then $\mathbf{E}_\beta(A_*) \to \mathcal{Z}(A_*)$ is an affine bundle of dimension $2\|\mathcal{E}(\beta)\|$ and is open in the restricted bundle $\mathbf{E}_\beta|_{\mathcal{Z}(A_*)}$.

Let $\mathrm{pr}^\beta_\alpha : \mathbf{E}_\beta(A_*) \to \mathbf{E}_\alpha(A_*)$ be projection to the indicated factor, and let

$$\mathbf{f}_\beta : \mathbf{E}_\beta(A_*) \to (\mathbf{k}\times\mathbf{k})^{\mathcal{E}(\beta)}$$

be the product of the compositions $f_\alpha \circ \mathrm{pr}^\beta_\alpha$ as α runs over the elements of $\mathcal{E}(\beta)$. Then the product of the bundle projection with $\mathbf{f}_\beta$ defines a trivialization

$$\mathbf{E}_\beta(A_*) \to \mathcal{Z}(A_*) \times (\mathbf{k}\times\mathbf{k})^{\mathcal{E}(\beta)}.$$

Let $\mathbf{E}(A_*) \to \mathcal{Z}(A_*)$ be the product of the bundles $\mathbf{E}_\beta(A_*)$ as β ranges over elements of $\mathcal{H}$. It is an affine bundle and is an open subset of the restricted bundle $\mathbf{E}|_{\mathcal{Z}(A_*)}$. Let $\mathrm{pr}_\beta : \mathbf{E} \to \mathbf{E}_\beta$ be the projection to the indicated factor, and let

$$\mathbf{f} : \mathbf{E}(A_*) \to \prod_{\beta\in\mathcal{H}} (\mathbf{k}\times\mathbf{k})^{\mathcal{E}(\beta)}$$

be the product of the maps $\mathbf{f}_\beta \circ \mathrm{pr}_\beta$. Then the product of bundle projection with $\mathbf{f}$ defines a trivialization

$$\mathbf{E}(A_*) \to \mathcal{Z}(A_*) \times \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}.$$

6.6. For each $\alpha \in \mathcal{E}$, let $\mathbf{D}_\alpha(A_*)$ be the diagonal in $\mathbf{E}_\alpha(A_*) = (\mathbf{P}_\alpha(A_*))^\alpha$. It is a 1-dimensional affine subbundle of $\mathbf{E}_\alpha(A_*)$. Similarly, for any $\beta \in \mathcal{H}$, the product bundle

$$\mathbf{D}_\beta(A_*) = \prod_{\alpha \in \mathcal{E}(\beta)} \mathbf{D}_\alpha(A_*)$$

is an $\|\mathcal{E}(\beta)\|$-dimensional affine subbundle of $\mathbf{E}_\beta(A_*)$. We let $\mathbf{E}_\beta(A_*)_\#$ denote the blow-up of $\mathbf{E}_\beta(A_*)$ along $\mathbf{D}_\beta(A_*)$. In order to work with the bundles $\mathbf{E}_\beta(A_*)_\#$, in addition to giving trivializations, we define closed embeddings

$$\mathbf{E}_\beta(A_*)_\# \to \mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

First, we recall some standard facts about embedding blow-ups of affine space.

Let $(\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}_\# \to (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$ be the blow-up of $(\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$ along $D^{\mathcal{E}(\beta)}$ where $D \subset \mathbf{k} \times \mathbf{k}$ is the diagonal. Let $\mathrm{pr}_1 : (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to \mathbf{k}^{\mathcal{E}(\beta)}$ be the projection which picks out the first coordinate from each factor of $\mathbf{k} \times \mathbf{k}$ and let $\mathrm{pr}_2 : (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to \mathbf{k}^{\mathcal{E}(\beta)}$ be the projection which picks out the second. Then $\mathbf{x} \mapsto [\mathrm{pr}_2(\mathbf{x}) - \mathrm{pr}_1(\mathbf{x})]$ defines a rational map

$$(\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \dashrightarrow \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

which is regular off of $D^{\mathcal{E}(\beta)}$, and the graph of this rational map defines a closed embedding

$$(\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}_\# \to (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

We use a parametrized version of this embedding for our setting. We let $\widetilde{\mathbf{f}}_\beta : \mathbf{E}_\beta(A_*) \dashrightarrow \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$ be the rational map defined by $y \mapsto [\mathrm{pr}_2(\mathbf{f}_\beta(y)) - \mathrm{pr}_1(\mathbf{f}_\beta(y))]$. The following lemma implies that the locus in $\mathbf{E}_\beta(A_*)$ where this map fails to be regular is precisely the blow-up locus $\mathbf{D}_\beta(A_*)$. The proof is just an unraveling of definitions of the various maps.

Lemma 6.7. *Let $\beta \in \mathcal{H}$. Then $y \in \mathbf{E}_\beta(A_*)$ is in $\mathbf{D}_\beta(A_*)$ if and only if $\mathbf{f}_\beta(y) \in D^{\mathcal{E}(\beta)}$ (i.e., if and only if $\mathrm{pr}_1(\mathbf{f}_\beta(y)) = \mathrm{pr}_2(\mathbf{f}_\beta(y))$).*

Taking the graph of $\widetilde{\mathbf{f}}_\beta$, we then obtain a closed embedding

$$\mathbf{E}_\beta(A_*)_\# \to \mathbf{E}_\beta(A_*) \times \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

with the blow-up map being projection to the first factor. Combining this with the trivialization in 6.5, we obtain a closed embedding

$$\mathbf{E}_\beta(A_*)_\# \to \mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

6.8. Finally, we let

$$\mathbf{E}(A_*)_\# = \prod_{\beta\in\mathcal{H}} \mathbf{E}_\beta(A_*)_\#.$$

It is a bundle over $\mathcal{Z}(A_*)$ and an open subvariety of the complete ambient bundle $\mathbf{E}_\#$. Let $\widetilde{\mathbf{f}}$ be the rational map

$$\mathbf{E}(A_*) \dashrightarrow \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

obtained by taking the product over all $\beta \in \mathcal{H}$ of the compositions $\widetilde{\mathbf{f}}_\beta \circ \mathrm{pr}_\beta$. From 6.6 we then have a closed embedding

$$\mathbf{E}(A_*)_\# \to \mathbf{E}(A_*) \times \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

whose image is the graph of the rational map $\widetilde{\mathbf{f}}$.

Combining this with the trivializations in 6.5, we have a closed embedding

$$\mathbf{E}(A_*)_\# \to \mathcal{Z}(A_*) \times \prod_{\beta\in\mathcal{H}} (\mathbf{k}\times\mathbf{k})^{\mathcal{E}(\beta)} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

The image is the closure of the image of the rational map $g \times \mathbf{f} \times \widetilde{\mathbf{f}}$ where $g : \mathbf{E}(A_*) \to \mathcal{Z}(A_*)$ is the bundle projection.

7. Restriction to the Local Model

7.1. In order to use the trivializations and embeddings of Section 6, we need to show that $\widetilde{\mathcal{Z}}(A_*)$ is a closed subvariety of the bundle $\mathbf{E}(A_*)_\#$.

Let $b : \mathbf{E}_\#|\mathcal{Z}(A_*) \to \mathbf{E}|_{\mathcal{Z}(A_*)}$ be the product of blow-up maps from 3.10. Then b is invertible on the image $\mathbf{u}(\mathcal{Z}^\circ(A_*))$ and $\widetilde{\mathbf{u}}(p) = b^{-1}(\mathbf{u}(p))$ for all $p \in \mathcal{Z}^\circ(A_*)$. It follows that $\widetilde{\mathcal{Z}}(A_*)$ is the closure of $b^{-1}(\mathbf{u}(\mathcal{Z}^\circ(A_*)))$ in $\mathbf{E}_\#|_{\mathcal{Z}(A_*)}$.

Recall that the bundle $\mathbf{E}(A_*)$ is an open subvariety of the bundle $\mathbf{E}|_{\mathcal{Z}(A_*)}$. Because the blow-up loci in the bundles $\mathbf{E}_\beta(A_*)$ are precisely the intersections of these bundles with the blow-up loci in the respective bundles $\mathbf{E}_\beta|_{\mathcal{Z}(A_*)}$, the bundle $\mathbf{E}(A_*)_\#$ is the complete inverse image of $\mathbf{E}(A_*)$ under the map $b : \mathbf{E}_\#|_{\mathcal{Z}(A_*)} \to \mathbf{E}|_{\mathcal{Z}(A_*)}$.

The definitions in 6.3 imply that for any edge $\alpha = \{I, J\}$, the restriction of the section u^I_α to $\mathcal{Z}(A_*)$ coincides with the section $[s_I] : \mathcal{Z}(A_*) \to \mathbf{P}_\alpha|_{\mathcal{Z}(A_*)}$, and moreover, this section does not intersect the section $[s_{I\cup J}]$. It follows that the image $u^I_\alpha(\mathcal{Z}(A_*))$ is contained in the affine line bundle $\mathbf{P}_\alpha(A_*)$. In fact, since the restrictions of the sections u_α, $\mathbf{u}_\beta$, and $\mathbf{u}$ are products of such sections, their images are contained in the affine bundles $\mathbf{E}_\alpha(A_*)$, $\mathbf{E}_\beta(A_*)$, and $\mathbf{E}(A_*)$, respectively. In particular, $\mathbf{u}(\mathcal{Z}(A_*)) \subset \mathbf{E}(A_*)$.

Proposition 7.2. *The variety $\widetilde{\mathcal{Z}}(A_*)$ coincides with the closure of $b^{-1}(\mathbf{u}(\mathcal{Z}^\circ(A_*)))$ in $\mathbf{E}(A_*)_\#$.*

Proof. Let Y be the closure of $b^{-1}(\mathbf{u}(\mathcal{Z}^\circ(A_*)))$ in $\mathbf{E}(A_*)_\#$. Obviously, $Y \subset \widetilde{\mathcal{Z}}(A_*)$. Let $\widetilde{p} \in \widetilde{\mathcal{Z}}(A_*)$. Then $b(\widetilde{p})$ is in the closure of $\mathbf{u}(\mathcal{Z}^\circ(A_*))$ in $\mathbf{E}|_{\mathcal{Z}(A_*)}$. Since $\mathbf{u}$ is a regular section of the bundle $\mathbf{E}|_{\mathcal{Z}(A_*)} \to \mathcal{Z}(A_*)$, its image is closed, hence $b(\widetilde{p}) \in \mathbf{u}(\mathcal{Z}(A_*))$. By the previous paragraph, $\mathbf{u}(\mathcal{Z}(A_*))$ is contained in $\mathbf{E}(A_*)$, and since the blow-up $\mathbf{E}(A_*)_\#$ is the complete inverse image $b^{-1}(\mathbf{E}(A_*))$ in $\mathbf{E}|_{\mathcal{Z}(A_*)}$, we must have $\widetilde{p} \in \mathbf{E}(A_*)_\#$. This completes the proof. □

7.3. Let $\alpha = \{I, J\}$ be an edge in $\mathcal{E}$. With respect to the trivialization $\mathbf{E}_\alpha(A_*) \to \mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})$ of 6.4, the section $u_\alpha : \mathcal{Z}(A_*) \to \mathbf{E}_\alpha(A_*)$ corresponds to the section of the trivial bundle $\mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})$ given by $p \mapsto (p, f_\alpha(u_\alpha(p)))$. Let $\mathrm{pr}_i : \mathbf{k} \times \mathbf{k} \to \mathbf{k}$, $i = 1, 2$, be the projections to the first and second factors. We can now define a function that measure the difference between the two sections u^I_α and u^J_α. (Note that the definition depends on our choice of ordering for subsets of $[\![n]\!]$ and our choice of basis a.)

Definition 7.4. The *edge coordinate associated to* α is the function $\chi_\alpha : \mathcal{Z}(A_*) \to \mathbf{k}$ defined by

$$\chi_\alpha = \frac{1}{2}\left(\mathrm{pr}_2 \circ f_\alpha \circ u_\alpha - \mathrm{pr}_1 \circ f_\alpha \circ u_\alpha\right).$$

Our choice of the sections $\{s_I\}$ is compatible with the edge coordinates in the following sense.

Proposition 7.5. *Let $\alpha = \{I, J\}$ be an edge in $\mathcal{E}$ (with $I < J$). Then*

1. $s_J(p) = s_I(p) + \chi_\alpha(p) s_{I\cup J}(p)$ *for any* $p \in \mathcal{Z}(A_*)$, *and*
2. $\chi_\alpha = f^J_\alpha \circ u^J_\alpha = -f^I_\alpha \circ u^I_\alpha$.

Proof. By definition of f_α and u_α, we know

$$\chi_\alpha(p) = \frac{1}{2}\left(f^J_\alpha(u^J_\alpha(p) - f^I_\alpha(u^I_\alpha(p))\right). \tag{4}$$

Let $i = \|I\| = \|J\|$. From 7.1 we know $u^J_\alpha(p) = [s_J(p)] = [s_I(p) + f^J_\alpha(u^J_\alpha(p)) s_{I\cup J}(p)$ in $(\mathbf{P}_\alpha)_p = \mathbb{P}(\pi_{I\cup J}(p)/\pi_{I\cap J}(p))$. This means that the vector $s_J(p)$ is a multiple of $s_I(p) + f^J_\alpha(u^J_\alpha(p)) s_{I\cup J}(p)$ modulo the subspace $\pi_{I\cap J}(p)$. But $\pi_{I\cap J}(p)$ is in general position to the flag A_*, hence the intersection $\pi_{I\cap J}(p) \cap A_{n+1-i}$ is the zero subspace. Since both $s_J(p)$ and $s_I(p) + f^J_\alpha(u^J_\alpha(p)) s_{I\cup J}(p)$ are contained in $A_{n+1-i} = \mathrm{Span}\{a_i, \dots, a_n\}$, and since the a_i coefficient of both is 1, this means the two vectors must, in fact, be equal. That is, $s_J(p) = s_I(p) + f^J_\alpha(u^J_\alpha(p)) s_{I\cup J}(p)$. By symmetry, $s_I(p) = s_J(p) + f^I_\alpha(u^I_\alpha(p)) s_{I\cup J}(p)$. Subtracting the two equations gives

$$2s_J(p) = 2s_I(p)) + (f^J_\alpha(u^J_\alpha(p)) - f^I_\alpha(u^I_\alpha(p))) s_{I\cup J}(p),$$

and substituting (4) into this gives the first part of the proposition. Adding the two equations gives

$$f^J_\alpha(u^J_\alpha(p)) + f^I_\alpha(u^I_\alpha(p)) = 0,$$

and substituting this into (4) gives the second part of the proposition. □

7.6. An important feature of the edge coordinates is that together with the projection $\pi : \mathcal{Z}(A_*) \to \mathbb{F}(A_*)$, they completely determine the variety $\mathcal{Z}(A_*)$. To prove this, we describe a total ordering on the subsets of $[\![n]\!]$ that is more compatible with our choice of coordinates, and reflects the choice of the projection π.

The new ordering is defined inductively as follows. The first n subsets, in the new order, are the standard subsets

$$I_1 = \{1\}, I_2 = \{1, 2\}, \dots , I_n = \{1, \dots , n\}.$$

We let $\mathcal{I}_n = \{I_1, \dots I_n\}$, and assume that we have added elements (in the new order) to obtain $\mathcal{I}_{k-1}$ (so $k > n$). Thus, $\mathcal{I}_{k-1}$ contains the first $k - 1$ elements in the new ordering. Let J be the minimal subset (in the standard ordering) such that

1. $J \notin \mathcal{I}_{k-1}$, and
2. there exists an edge $\alpha = \{I, J\} \in \mathcal{E}$ such that I, $I \cap J$, and $I \cup J$ are all elements of $\mathcal{I}_{k-1}$.

We let J be the next element of the new ordering and set $\mathcal{I}_k = \mathcal{I}_{k-1} \cup \{J\}$. Continuing, we obtain a total ordering of the subsets of $[\![n]\!]$.

Example 7.7. The $n = 4$ case is shown in the table below with the top row indicating the new order

J	1	12	123	1234	2	13	3	23	124	14	4	24	134	34	234
$I \cap J$					∅	1	∅	2	12	1	∅	2	13	3	23
I					1	12	1	12	123	12	1	12	123	13	123
$I \cup J$					12	123	13	123	1234	124	14	124	1234	134	1234

7.8. For $p \in \mathcal{Z}(A_*)$, let $\chi(p)$ be the point in $\mathbf{k}^{\mathcal{E}}$ whose α-coordinate is $\chi_\alpha(p)$. We then have the following lemma.

Lemma 7.9. *The map $p \mapsto (\pi(p), \chi(p))$ defines a closed embedding*

$$\mathcal{Z}(A_*) \to \mathbb{F}(A_*) \times \mathbf{k}^{\mathcal{E}}.$$

Moreover, if $F_* \in \mathbb{F}(A_*)$, then composing with projection to the second factor and restricting to $Z(F_*, A_*)$ gives a closed embedding

$$Z(F_*, A_*) \to \mathbf{k}^{\mathcal{E}}.$$

Proof. First we show that the map is injective. For this it suffices to show that if p_1 and p_2 are two distinct points of $\mathcal{Z}(A_*)$ such that $\pi(p_1) = \pi(p_2)$, then there is some $\alpha \in \mathcal{E}$ such that $\chi_\alpha(p_1) \neq \chi_\alpha(p_2)$.

Now if $p_1 \neq p_2$, then there is some $J \subset [\![n]\!]$ such that $\pi_J(p_1) \neq \pi_J(p_2)$. Choose the minimal such J in the total ordering of 7.6. Then there exists an edge $\alpha = \{I, J\} \in \mathcal{E}$ such that $\pi_{I\cap J}(p_1) = \pi_{I\cap J}(p_2)$, $\pi_I(p_1) = \pi_I(p_2)$, and $\pi_{I\cup J}(p_1) = \pi_{I\cup J}(p_2)$. We claim $\chi_\alpha(p_1) \neq \chi_\alpha(p_2)$. Since the sections s_I are uniquely determined once $a \in G$ has been chosen, we also have $s_{I\cap J}(p_1) = s_{I\cap J}(p_2)$, $s_I(p_1) = s_I(p_2)$, and $s_{I\cup J}(p_1) = s_{I\cup J}(p_2)$. If $\chi_\alpha(p_1) = \chi_\alpha(p_2)$, then from Proposition 7.5 we would have

$$s_J(p_1) = s_I(p_1) + \chi_\alpha(p_1)s_{I\cup J}(p_1) = s_I(p_2) + \chi_\alpha(p_2)s_{I\cup J}(p_2) = s_J(p_2).$$

But since $\pi_J(p_1)$ (respectively $\pi_J(p_2)$) is the span of $\pi_{I\cap J}(p_1) = \pi_{I\cap J}(p_2)$ and $s_J(p_1)$ (resp, $s_J(p_2)$), we would then have $\pi_J(p_1) = \pi_J(p_2)$ contradicting our choice of J. It follows that $\chi_\alpha(p_1) \neq \chi_\alpha(p_2)$.

Finally, we show that the differential of the map is injective on tangent spaces. The tangent space $\mathbf{T}\mathcal{Z}$ is obviously a subvariety of the tangent bundle $\mathbf{T}\mathcal{X}$, and the latter can be naturally identified with the bundle

$$\mathbf{T}\mathcal{X} = \bigoplus_{I\subset [\![n]\!]} \mathbf{Hom}(\mathbf{F}_I, \mathbf{V}/\mathbf{F}_I),$$

whose fiber over $p \in \mathcal{X}$ is the vector space

$$\mathbf{T}_p\mathcal{X} = \bigoplus_{I\subset [\![n]\!]} \mathrm{Hom}(\pi_I(p), V/\pi_I(p))$$

(see, e.g., [8, chapter 16]). The incidence conditions on $\mathcal{Z}$ imply that for $p \in \mathcal{Z}$, any point $(\phi_J) \in \mathbf{T}_p\mathcal{Z}$ satisfies:

$$\phi_J|_{\pi_I(p)} \equiv \phi_I \pmod{\pi_J(p)} \quad \text{whenever } I \subset J. \tag{5}$$

This implies that for any $p \in \mathcal{Z}(A_*)$, an element $(\phi_J) \in \mathbf{T}_p\mathcal{Z}$ is completely determined by the images $\{\phi_J(s_J(p)) \mid J \subset [\![n]\!]\}$ (since the vectors $\{s_I(p) \mid I \subset J\}$ contain a basis for the subspace $\pi_J(p)$).

Now since $\pi : \mathcal{Z}(A_*) \to \mathbb{F}(A_*)$ is a locally trivial fibration with fiber $Z(F_*, A_*)$, it suffices to show that $d\chi_p$ is injective for all $p \in Z(F_*, A_*)$. Let $\rho(t)$ be any smooth (in $\mathcal{X}$) curve in $Z(F_*, A_*)$ such that $\rho(0) = p$, and assume that $(\chi_\alpha \circ \rho)'(0) = 0$ for all $\alpha \in \mathcal{E}$. Then we have to show that the tangent vector (ϕ_J) to the curve $\rho(t)$ is zero at p. We can determine this tangent vector inductively as follows. For each subset J, we have, by definition,

$$\phi_J(s_J(p)) = (s_J \circ \rho)'(0) \bmod \pi_J(p).$$

Since $\pi(\rho(t))$ is constant, it follows that $(s_{I_j} \circ \rho)'(0) = 0$ for the standard subsets $I_j \in \{I_1, \dots I_n\}$. Given any other $J \subset [\![n]\!]$, we can then assume by induction that $(s_I \circ \rho)'(0) = 0$ for all $I < J$ (in the ordering of 7.6). Let $\alpha = \{I, J\}$ be an edge in $\mathcal{E}$ such that $I \cap J$, I, and $I \cup J$ are all less than J in the ordering of 7.6. Then differentiating the equation

$$s_J(\rho(t)) = s_I(\rho(t)) + \chi_\alpha(\rho(t))s_{I\cup J}(\rho(t))$$

of Proposition 7.5 gives

$$(s_J \circ \rho)'(0) = (s_I \circ \rho)'(0) + ((\chi_\alpha \circ \rho)'(0))\, s_{I\cup J}(p) + \chi_\alpha(p)(s_{I\cup J} \circ \rho)'(0),$$

which, by induction, is zero. Hence $\phi_J(s_J(p)) = 0$ for all J. The conditions (5) then imply that $\phi_J = 0$ for all J, and this completes the proof. □

Example 7.10. When $n = 4$, this lemma provides an embedding of $Z(F_*, A_*)$ into $\mathbf{k}^{\mathcal{E}} = \mathbf{k}^{\mathcal{E}_1} \times \mathbf{k}^{\mathcal{E}_2} \times \mathbf{k}^{\mathcal{E}_3} \cong \mathbf{k}^6 \times \mathbf{k}^{12} \times \mathbf{k}^6$. This allows us to view a point p in $Z(F_*, A_*)$ as a "valued figure" by assigning to each edge α in Figure 4 the value of the function $\chi_\alpha(p)$.

7.11. With respect to the trivializations

$$\mathbf{E}_\beta(A_*) \to \mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$$

of 6.5, the sections $\mathbf{u}_\beta$ correspond to sections of the trivial bundle $\mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$ of the form $p \mapsto (p, \mathbf{f}_\beta(\mathbf{u}_\beta(p)))$.

With respect to the closed embedding

$$\mathbf{E}_\beta(A_*)_\# \to \mathcal{Z}(A_*) \times (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

of 6.6, the rational section $\widetilde{\mathbf{u}}_\beta$ corresponds to the rational map $p \mapsto (p, \mathbf{f}_\beta(\mathbf{u}_\beta(p)), \widetilde{\mathbf{f}}_\beta(\mathbf{u}_\beta(p))$.

And finally, with respect to the closed embedding

$$\mathbf{E}(A_*)_\# \to \mathcal{Z}(A_*) \times \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

of 6.8, the rational section $\widetilde{\mathbf{u}}$ corresponds to the rational map $p \mapsto (p, \mathbf{f}(\mathbf{u}(p)), \widetilde{\mathbf{f}}(\mathbf{u}(p)))$.

Since $\widetilde{\mathcal{Z}}(A_*)$ is defined as the closure of the image $\widetilde{\mathbf{u}}(\mathcal{Z}^\circ(A_*))$, we have the following embedding.

Lemma 7.12. *$\widetilde{\mathcal{Z}}(A_*)$ is isomorphic to the closure of the image of the map*

$$\mathcal{Z}^\circ(A_*) \to \mathcal{Z}(A_*) \times \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

given by $p \mapsto (p, \mathbf{f}(\mathbf{u}(p)), \widetilde{\mathbf{f}}(\mathbf{u}(p)))$. Moreover, if F_* is any flag in $\mathbb{F}(A_*)$, then $\widetilde{Z}(F_*, A_*)$ is isomorphic to the closure of the image of $Z^\circ(F_*, A_*)$ under this same map.

7.13. The ambient variety of the embedding of Lemma 7.12 can be reduced to a smaller ambient variety as follows. Recall that for each of the maximal elements $\gamma_k \in \mathcal{H}$ ($k = 1, \ldots, n-1$), we have $\mathcal{E}_k = \mathcal{E}(\gamma_k)$, and the vector space $\mathbf{k}^{\mathcal{E}}$ can be identified with the product

$$\mathbf{k}^{\mathcal{E}_1} \times \cdots \times \mathbf{k}^{\mathcal{E}_{n-1}}.$$

Let h be the projection

$$\prod_{\beta \in \mathcal{H}} \mathbf{k}^{\mathcal{E}(\beta)} \to \mathbf{k}^{\mathcal{E}}$$

that picks out the factors corresponding to the maximal faces. Let pr_1 (resp., pr_2) be the projection

$$\prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to \prod_{\beta \in \mathcal{H}} \mathbf{k}^{\mathcal{E}(\beta)}$$

that picks out the first (resp., second) coordinate from each $\mathbf{k} \times \mathbf{k}$ factor, and let

$$d : \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to \prod_{\beta \in \mathcal{H}} \mathbf{k}^{\mathcal{E}(\beta)}$$

be the difference $d = \mathrm{pr}_2 - \mathrm{pr}_1$.

It follows from the definition of χ in 7.8, that the composition of

$$\mathbf{f} \circ \mathbf{u} : \mathcal{Z}(A_*) \to \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$$

with

$$h \circ d : \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to \mathbf{k}^{\mathcal{E}}$$

is precisely the map χ.

Lemma 7.14. *$\widetilde{\mathcal{Z}}(A_*)$ is isomorphic to the closure of the image of the map*

$$\mathcal{Z}^\circ(A_*) \to \mathcal{Z}(A_*) \times \mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

given by $p \mapsto (p, \chi(p), \widetilde{\mathbf{f}}(\mathbf{u}(p)))$. Moreover, if F_* is any flag in $\mathbb{F}(A_*)$, then $\widetilde{Z}(F_*, A_*)$ is isomorphic to the closure of the image of $Z^\circ(F_*, A_*)$ under this same map.

Proof. Let pr^β_α be the projection $\prod_\beta (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \to (\mathbf{k} \times \mathbf{k})$ corresponding to the edge α in the factor $(\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)}$, and let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbf{k} \times \mathbf{k} \to \mathbf{k}$ be the projections onto the first and second factors, respectively. Consider the closed subvariety W of

$$\mathcal{Z}(A_*) \times \prod_{\beta \in \mathcal{H}} (\mathbf{k} \times \mathbf{k})^{\mathcal{E}(\beta)} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

defined by the equations

$$\mathrm{pr}_1 \circ \mathrm{pr}^\beta_\alpha = -\,\mathrm{pr}_2 \circ \mathrm{pr}^\beta_\alpha = \mathrm{pr}_1 \circ \mathrm{pr}^{\gamma_k}_\alpha = -\,\mathrm{pr}_2 \circ \mathrm{pr}^{\gamma_k}_\alpha$$

for all $k \in \{1, \ldots, n-1\}$, $\beta \in \mathcal{H}_k$, and $\alpha \in \mathcal{E}(\beta)$. By the second part of Proposition 7.5, for any $p \in \mathcal{Z}^\circ(A_*)$, the point $(p, \mathbf{f}(\mathbf{u}(p)), \widetilde{\mathbf{f}}(\mathbf{u}(p)))$ lies in this subvariety, hence, so does the closure of the image of $\mathcal{Z}^\circ(A_*)$. But the map $\mathrm{Id} \times (h \circ d) \times \mathrm{Id}$ restricts to an isomorphism from W to the smaller variety

$$\mathcal{Z}(A_*) \times \mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

Since the map $p \mapsto (p, \chi(p), \widetilde{\mathbf{f}}(\mathbf{u}(p)))$ is the composition of the map $\mathcal{Z}^\circ(A_*) \to W$ of Lemma 7.12 with $(\mathrm{Id} \times (h \circ d) \times \mathrm{Id})|_W$, the closure of its image is isomorphic $\widetilde{Z}(A_*)$. $\square$

7.15. Combining Lemma 7.14 and Lemma 7.9, (and setting $\widetilde{\boldsymbol{\chi}} = \widetilde{\mathbf{f}} \circ \mathbf{u}$) we have the following.

Lemma 7.16. *$\widetilde{\mathcal{Z}}(A_*)$ is isomorphic to the closure of the image of the (regular) map*

$$\mathcal{Z}^\circ(A_*) \to \mathbb{F}(A_*) \times \mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

given by $p \mapsto (\pi(p), \chi(p), \widetilde{\boldsymbol{\chi}}(p))$. If F_* is any flag in $\mathbb{F}(A_*)$, then the fiber $\widetilde{Z}(F_*, A_*)$ is isomorphic to the closure of the image of $Z^\circ(F_*, A_*)$ under the map

$$Z^\circ(F_*, A_*) \to \mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

given by $p \mapsto (\chi(p), \widetilde{\boldsymbol{\chi}}(p))$.

Example 7.17. For $n = 4$, this lemma implies that $\widetilde{Z}(F_*, A_*)$ embeds in the variety

$$\mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}) \cong \mathbf{k}^6 \times \mathbf{k}^{12} \times \mathbf{k}^6 \times \mathbb{P}^5 \times \mathbb{P}^{11} \times \mathbb{P}^5 \times (\mathbb{P}^2)^4 \times (\mathbb{P}^2)^8 \times (\mathbb{P}^2)^4.$$

Thus, we can view a point $\widetilde{p} \in \widetilde{Z}(F_*, A_*)$ as a valued figure of the form Figure 6. Each edge α inside the box in the upper left corner is assigned the value $\chi_\alpha(p)$, where p is the image of $\widetilde{p}$ under the projection $\widetilde{Z}(F_*, A_*) \to Z(F_*, A_*)$ (cf. Example 7.10). Below the box are the three maximal faces, and to the right of the box (in their respective columns) are the triangular subfaces of these maximal faces. Each of the edges outside the box, therefore, is assigned the value of a projective coordinate x_α^β where β indicates the face in $\mathcal{H}$ and α indicates the edge of this face. Note, though, that since the x_α^β are homogeneous coordinates, the assignment of these values within a given face β is only well-defined up to multiplication by a nonzero scalar.

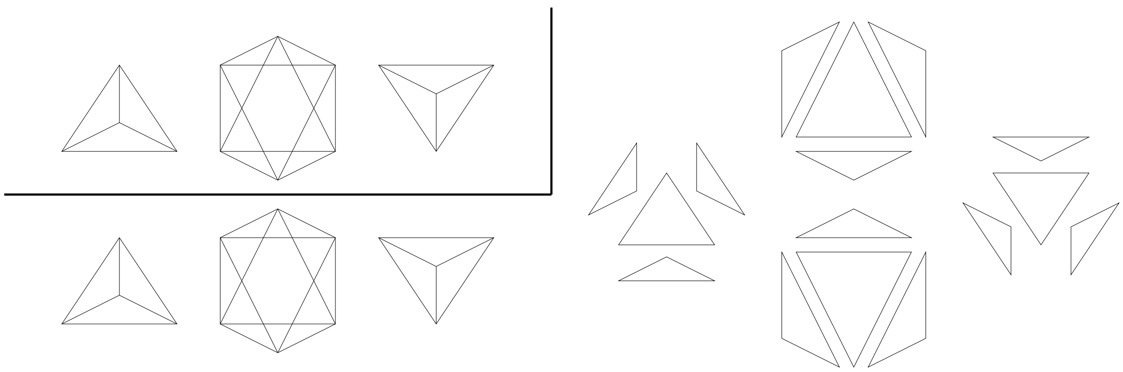

FIGURE 6. All proper faces when $n = 4$.

7.18. To check that $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is a G-equivariant resolution, it suffices by Lemma 5.2 to check that $\widetilde{\mathcal{Z}}(A_*) \to \mathcal{Z}(A_*)$ is a $B(A_*)$-equivariant resolution for any given A_*. Lemma 5.4 then reduces this proof to showing that the map of fibers $\widetilde{Z}(F_*, A_*) \to Z(F_*, A_*)$ is a $T(F_*, A_*)$-equivariant resolution for any given $F_* \in \mathbb{F}(A_*)$. With this in mind, we make the following choices to get our "local model".

Let $E_* \in \mathbb{F}$ denote the *fixed flag*

$$E_* := \{\{0\} \subset E_{\{1\}} \subset E_{\{1,2\}} \subset \cdots \subset E_{\{1,\dots,n\}} = V\},$$

and let E_*^{op} denote the *opposite fixed flag*

$$E_*^{op} := \{\{0\} \subset E_{\{n\}} \subset E_{\{n,n-1\}} \subset \cdots \subset E_{\{n,\dots,1\}} = V\}.$$

Let $B = B(E_*)$, $B^{op} = B(E_*^{op})$ and $T = T(E_*, E_*^{op})$. Then B is the subgroup of upper triangular matrices, B^{op} is the subgroup of lower triangular matrices, and T is the diagonal torus in G.

To get our bundle trivializations and local coordinates, we let $a \in G$ be the identity transformation $a_i = a(e_i) = e_i$ for $i = 1, \dots, n$. Let U be the image of $Z(E_*, E_*^{op})$ under the closed embedding of Lemma 7.9, and let U° be the image of $Z^\circ(E_*, E_*^{op})$ under the same embedding. Let $\widetilde{U}$ be the isomorphic image of $\widetilde{Z}(E_*, E_*^{op})$ under the embeddings of Lemma 7.16. U is, therefore, an affine subvariety of $\mathbf{k}^{\mathcal{E}}$, and $\widetilde{U}$ is a closed subvariety of $\mathbf{k}^{\mathcal{E}} \times \prod_\beta \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$.

Definition 7.19. The pair $(U, \widetilde{U})$ with the projection $\widetilde{U} \to U$ is called the *local model* for the chart $\widetilde{Z}(E_*, E_*^{op})$ and the projection $\widetilde{Z}(E_*, E_*^{op}) \to Z(E_*, E_*^{op})$.

Our main theorem has then been reduced to the following.

Theorem 7.20. *$\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is a smooth G-equivariant resolution if and only if $\widetilde{U} \to U$ is a smooth T-equivariant resolution.*

8. Equations for the Local Model

8.1. Let $\mathbf{k}[\{x_\alpha\}]$ be the coordinate ring for the affine space $\mathbf{k}^{\mathcal{E}}$. Most of the defining equations for the subvariety $U \subset \mathbf{k}^{\mathcal{E}}$ can be described in terms of the triangles $\mathcal{T}$ of Example 3.4. First we identify the coordinates x_α of a point x in the image of the embedding $Z(E_*, E_*^{op}) \to \mathbf{k}^{\mathcal{E}}$.

For $I \subset [\![n]\!]$, the sections $s_I : Z(E_*, E_*^{op}) \to \mathbf{F}_I(E_*^{op})$ (as in 6.3) are determined as follows. Since $a_i = e_i$ for $i = 1, \dots, n$, for any $p \in Z(E_*, E_*^{op})$, $s_I(p)$ will be the vector in $\pi_I(p) \cap E_{\{i,i+1,\dots,n\}}$ of the form

$$s_I(p) = e_i + \sigma_I^{i+1}(p) e_{i+1} + \cdots + \sigma_I^n(p) e_n$$

where $i = \|I\|$.

Proposition 8.2. *Let $\alpha = \{I, J\}$ be any edge in $\mathcal{E}$ (with $I < J$), and let $i = \|I\| = \|J\|$. If $x \in U$ is the image of $p \in Z(E_*, E_*^{op})$, then*

$$x_\alpha = \sigma_J^{i+1}(p) - \sigma_I^{i+1}(p),$$

and for any $k > i+1$ we have

$$x_\alpha \sigma_{I\cup J}^k(p) = \sigma_J^k(p) - \sigma_I^k(p).$$

Proof. The isomorphism $Z(E_*, E_*^{op}) \to U \subset \mathbf{k}^{\mathcal{E}}$ is given by $p \mapsto x$ where $x_\alpha = \chi_\alpha(p)$; hence, by Proposition 7.5, we have

$$s_J(p) = s_I(p) + x_\alpha s_{I\cup J}(p).$$

The equations of the proposition are just the resulting equations in the coordinates. □

Note, in particular, that if $x \in U^\circ$ is the image of a nondegenerate $p \in Z^\circ(E_*, E_*^{op})$, then $x_\alpha \neq 0$.

Remark 8.3. Consider the Plücker embedding $\mathbb{G}_i \to \mathbb{P}(\bigwedge^i V)$. Restricting p to be in the chart $Z(E_*, E_*^{op})$ implies that the Plücker coordinate dual to $e_1\wedge\cdots\wedge e_i$ is nonzero on $\pi_I(p)$. Denote this value by $f_I(p)$, and let $g_I(p)$ be the value of the coordinate dual to $e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1}$ evaluated on $\pi_I(p)$. Then the coefficient $\sigma_I^{i+1}(p)$ is the function $g_I(p)/f_I(p)$. It follows from Proposition 8.2, then, that the edge coordinate $\chi_\alpha(p)$ can be expressed in terms of these Plücker coordinates as

$$\chi_\alpha(p) = \frac{g_J(p)}{f_J(p)} - \frac{g_I(p)}{f_I(p)}.$$

8.4. Let $\beta \in \mathcal{T}_k$ and $\beta^* \in \mathcal{T}_{k+1}$ be any pair of related triangles. Let $I^- = I^-(\beta) = I^-(\beta^*)$, let $I^+ = I^+(\beta) = I^+(\beta^*)$, and let $I^+ \smallsetminus I^- = \{i_1, i_2, i_3\}$. We assume $i_1 < i_2 < i_3$, and let $I_1 = I^- \cup \{i_1\}$, $I_2 = I^- \cup \{i_2\}$, $I_3 = I^- \cup \{i_3\}$, $J_1 = I^- \cup \{i_2, i_3\}$, $J_2 = I^- \cup \{i_1, i_3\}$, and $J_3 = I^- \cup \{i_1, i_2\}$. Then $\beta = \{I_1, I_2, I_3\}$ and $\beta^* = \{J_1, J_2, J_3\}$. Note that $I_1 < I_2 < I_3$ and $J_1 > J_2 > J_3$. For these subsets, we have unique constants $b_j, c_j, d_j \in \mathbf{k}$, $j = 1, 2, 3$ such that that

$$\begin{array}{rcccccc}
s_{I_1}(p) & = & e_k & + & b_1 e_{k+1} & + & c_1 e_{k+2} \\
s_{I_2}(p) & = & e_k & + & b_2 e_{k+1} & + & c_2 e_{k+2} \\
s_{I_3}(p) & = & e_k & + & b_3 e_{k+1} & + & c_3 e_{k+2} \\
s_{J_1}(p) & = & & & e_{k+1} & + & d_1 e_{k+2} \\
s_{J_2}(p) & = & & & e_{k+1} & + & d_2 e_{k+2} \\
s_{J_3}(p) & = & & & e_{k+1} & + & d_3 e_{k+2} \\
s_{I^+}(p) & = & & & & & e_{k+2}
\end{array}$$

modulo the subspace $E_{\{k+3,\dots,n\}}$.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{E}_k$ be the edges $\{I_2, I_3\}$, $\{I_1, I_3\}$, and $\{I_1, I_2\}$, respectively, and let $\alpha_1^*, \alpha_2^*, \alpha_3^* \in \mathcal{E}_{k+1}$ be the edges $\{J_2, J_3\}$, $\{J_1, J_3\}$, and $\{J_1, J_2\}$, respectively. Applying

Proposition 8.2 to the edges of our related triangles gives

$$\begin{array}{lll} x_{\alpha_1} = b_3 - b_2 & x_{\alpha_1^*} = d_2 - d_3 & d_1 x_{\alpha_1} = c_3 - c_2 \\ x_{\alpha_2} = b_3 - b_1 & x_{\alpha_2^*} = d_1 - d_3 & d_2 x_{\alpha_2} = c_3 - c_1 \\ x_{\alpha_3} = b_2 - b_1 & x_{\alpha_3^*} = d_1 - d_2 & d_3 x_{\alpha_3} = c_2 - c_1. \end{array}$$

Eliminating constants, we then have the following.

Proposition 8.5. *For any pair of related triangles $\beta, \beta^* \in \mathcal{T}$, the (linear) polynomials*

$$x_{\alpha_1} - x_{\alpha_2} + x_{\alpha_3} \qquad \textit{and} \qquad x_{\alpha_1^*} - x_{\alpha_2^*} + x_{\alpha_3^*}$$

and all (quadratic) 2×2 minors of the matrix

$$\begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x_{\alpha_1^*} & x_{\alpha_2^*} & x_{\alpha_3^*} \end{pmatrix}$$

vanish if $x \in U$.

Example 8.6. With respect to the valued figures of Example 7.10 for the $n = 4$ case, the vanishing of the linear polynomials for $x \in U$ implies that the sum (with appropriate signs) of the values around any triangle in the figure is zero. The vanishing of the minors implies that the values on related triangles are proportional. We will sometimes refer to this fact by saying that related triangles *have the same shape (on x)*.

8.7. Let $Y \subset \mathbf{k}^{\mathcal{E}}$ be the affine variety defined by the vanishing of the polynomials in Proposition 8.5. Although U is obviously a subvariety of Y, in general (in fact for $n > 3$) the two varieties are not equal. We can show, however, that U is an irreducible component of Y. Let Y° be the open subvariety of Y defined by

$$Y^\circ = \{x \in Y \mid x_\alpha \neq 0 \text{ for all } \alpha \in \mathcal{E}\}.$$

Since $x_\alpha \neq 0$ for all $x \in U^\circ$, we know that $U^\circ \subset Y^\circ$.

Proposition 8.8. *The varieties U° and Y° coincide. In particular, U is an irreducible component of Y.*

Before proving the proposition, we first establish that Y° is an affine variety of dimension at most $n(n-1)/2$. Let $I_0 = \varnothing$, $I_1 = \{1\}$, $\dots$, $I_n = \{1, \dots, n\}$ be the standard flag of subsets of $[\![n]\!]$, and for $1 \leq k < m \leq n$, let $\alpha_{k,m}$ be the edge $\{I_k, I_{k-1} \cup \{m\}\}$ in $\mathcal{E}_k$.

Lemma 8.9. *For any $\alpha \in \mathcal{E}$, the coordinate function x_α can be expressed as a rational function of the variables $\{x_{\alpha_{k,m}} \mid 1 \leq k < m \leq n\}$ that is regular on Y°.*

Proof. Let $\alpha = \{\{i\}, \{j\}\}$ be an edge in $\mathcal{E}_1$ (and assume $i < j$). If $1 < i$, then applying the linear relation of Proposition 8.5 to the triangle $\beta = \{\{1\}, \{i\}, \{j\}\}$, we can express x_α linearly in terms of $x_{\alpha_{1,i}}$ and $x_{\alpha_{1,j}}$. Thus, the lemma holds for all $\alpha \in \mathcal{E}_1$. Next we assume that the lemma holds for all $\alpha = \{I, J\}$ where $\|I\| = \|J\| < k$.

We define a partial order on $\mathcal{E}$ by $\alpha = \{I, J\} < \alpha' = \{I', J'\}$ if and only if $I \leq I'$ (in the standard order), $J \leq J'$, and $\alpha \neq \alpha'$. With respect to this partial order, for each k, the edges $\alpha_{k,m}$ are precisely the minimal edges among the set

$$\{\alpha = \{I, J\} \in \mathcal{E}_k \mid I_k \leq I,\ I_{k-1} \leq I \cap J\}.$$

Hence, it will suffice by induction, to show that for any $\alpha = \{I, J\} \in \mathcal{E}_k$ such that $I_k < I$ or $I_{k-1} < I \cap J$, the variable x_α can be expressed as a rational function of the variables $\{x_{\alpha'} \mid \alpha' < \alpha\}$ that is regular on Y°.

Let $\alpha = \{I, J\}$ be an element of $\mathcal{E}_k$ (with $I < J$) such that either $I_k < I$ or $I_{k-1} < I \cap J$. Let i be the element of I that is not in $I \cap J$, let j be the element of J that is not in $I \cap J$ (so $i < j$), and let m be the maximal element of $I \cap J$. There are two cases.

First, suppose $i < m$. Letting $K = I \cap J \smallsetminus \{m\}$, we can then single out the pair of related triangles:

$$\beta = \{K \cup \{i\}, K \cup \{j\}, K \cup \{m\}\} \quad \text{and} \quad \beta^* = \{K \cup \{j, m\}, K \cup \{i, m\}, K \cup \{i, j\}\}.$$

Letting $\alpha_1 = \{K \cup \{i\}, K \cup \{j\}\}$, $\alpha_2 = \{K \cup \{j\}, K \cup \{m\}\}$, $\alpha_1^* = \{K \cup \{i, m\}, K \cup \{j, m\}\}$, and $\alpha_2^* = \{K \cup \{i, m\}, K \cup \{i, j\}\}$, one of the quadratic minors of Proposition 8.5 gives $x_{\alpha_1^*} x_{\alpha_2} = x_{\alpha_1} x_{\alpha_2^*}$. Solving for $x_\alpha = x_{\alpha_1^*}$, we have

$$x_\alpha = \frac{x_{\alpha_1}}{x_{\alpha_2}} x_{\alpha_2^*}$$

where α_1, α_2, and α_2^* are all strictly less than α in the partial order. Obviously, the right hand side is regular on Y°.

Second, suppose $m < i$. If $I_{k-1} < I \cap J$, then there exists an l such that $l < m$ and $l \notin I \cap J$. If $I_k < I$, then there exists an l such that $l < i$ and $l \notin I$. In either case, we have an l such that $l < i < j$ and $l \notin I \cap J$. Letting $K = (I \cap J) \cup \{l\}$, we then have a triangle $\beta = \{K, I, J\}$ with $K < I < J$. Applying the linear relations of Proposition 8.5 to this triangle we then have $x_\alpha = x_{\alpha_1} - x_{\alpha_2}$ where $\alpha_1 = \{K, J\}$ and $\alpha_2 = \{K, I\}$. But α_1 and α_2 are both strictly less that α in the partial order. □

Proof of the proposition. We define a map $\phi : Y^\circ \to Z^\circ(E_*, E_*^{op})$ as follows. Let $x \in Y^\circ$. For $k = 1, \ldots n$, we define $s_{k,k} \in V$ to be the standard basis vector e_k. For $1 \leq k < m \leq n$, we define $s_{k,m}$ (by induction on $m - k$) to be

$$s_{k,m} = s_{k,k} + x_{\alpha_{k,m}} s_{k+1,m}.$$

For $k = 1, \ldots n$, we let v_k be the vector $s_{1,k}$ and note that $\{v_1, \ldots, v_n\}$ is a basis for V. For each subset $I \subset [\![n]\!]$, we let $F_I \subset V$ be the subspace spanned by $\{v_i \mid i \in I\}$, and we define $\phi(x)$ to be the element $p \in Z^\circ(E_*, E_*^{op})$ satisfying $\pi_I(p) = F_I$ for all $I \subset [\![n]\!]$. Note that p is defined so that $s_{k,m} = s_I(p)$ where $I = I_{k-1} \cup \{m\}$. We leave it as an exercise for the reader to show that the composition $Y^\circ \to Z^\circ(E_*, E_*^{op}) \to Y^\circ$ is the identity (note that, by the lemma we just proved, it is enough to show that the coordinates $x_{\alpha_{k,m}}$ do not change under this composition). □

8.10. Let $\mathbf{k}[\{x_\alpha\};\{x^\beta_\alpha\}]$ be the multigraded ring corresponding to the quasiprojective variety

$$\mathbf{k}^{\mathcal{E}} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

Given a point $\widetilde{x}$ in this variety, we let x denote its image under the projection to $\mathbf{k}^{\mathcal{E}}$, $q(\widetilde{x})$ denote its image under projection to the full product of projective spaces, and for each $\beta\in\mathcal{H}$, we let $q^\beta(\widetilde{x})$ be its image under the projection to $\mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$. The variables $\{x^\beta_\alpha \mid \alpha\in\mathcal{E}(\beta)\}$ can be viewed as homogeneous coordinates for the image $q^\beta(\widetilde{x})\in\mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$.

For any $\beta\in\mathcal{H}$, let $\mathrm{pr}_\beta : \mathbf{k}^{\mathcal{E}} \to \mathbf{k}^{\mathcal{E}(\beta)}$ be the projection induced by the inclusion $\mathcal{E}(\beta)\subset\mathcal{E}$, and let $\widetilde{\mathrm{pr}}_\beta : \mathbf{k}^{\mathcal{E}} \dashrightarrow \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$ be the rational map defined by $x\mapsto[\mathrm{pr}_\beta(x)]$. Let

$$\widetilde{\mathrm{pr}} : \mathbf{k}^{\mathcal{E}} \dashrightarrow \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

be the rational map obtained by taking the product of the $\widetilde{\mathrm{pr}}_\beta$ as β runs over all elements of $\mathcal{H}$. Then a point $\widetilde{x}$ is in the image of $Z^\circ(E_*, E_*^{op})$ under the map of Lemma 7.16 if and only if $x\in U^\circ$ and $q(\widetilde{x}) = \widetilde{\mathrm{pr}}(x)$. Recall that $\widetilde{U}$ is the closure of this image.

Proposition 8.11. *If $\widetilde{x}\in\widetilde{U}$, then all of the following multi-homogeneous polynomials vanish.*

1. *(Linear polynomials) For any triangle $\beta\in\mathcal{T}$ and any face $\gamma\in\mathcal{F}$ such that $\beta\subset\gamma$,*
$$x_{\alpha_1}-x_{\alpha_2}+x_{\alpha_3} \qquad \textit{and} \qquad x^\gamma_{\alpha_1}-x^\gamma_{\alpha_2}+x^\gamma_{\alpha_3}.$$
2. *(Horizontal quadratic polynomials) For any pair of related triangles $\beta,\beta^*\in\mathcal{T}$ and any pair of faces $\gamma,\gamma^*\in\mathcal{F}$ such that $\beta\subset\gamma$ and $\beta^*\subset\gamma^*$, the minors of the matrices*
$$\begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x_{\alpha_1^*} & x_{\alpha_2^*} & x_{\alpha_3^*} \end{pmatrix}, \begin{pmatrix} x^\gamma_{\alpha_1} & x^\gamma_{\alpha_2} & x^\gamma_{\alpha_3} \\ x_{\alpha_1^*} & x_{\alpha_2^*} & x_{\alpha_3^*} \end{pmatrix},$$
$$\begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x^{\gamma^*}_{\alpha_1^*} & x^{\gamma^*}_{\alpha_2^*} & x^{\gamma^*}_{\alpha_3^*} \end{pmatrix}, \begin{pmatrix} x^\gamma_{\alpha_1} & x^\gamma_{\alpha_2} & x^\gamma_{\alpha_3} \\ x^{\gamma^*}_{\alpha_1^*} & x^{\gamma^*}_{\alpha_2^*} & x^{\gamma^*}_{\alpha_3^*} \end{pmatrix}.$$
3. *(Vertical quadratic polynomials) For any triangle $\beta\in\mathcal{T}$ and any pair of faces $\gamma,\gamma^*\in\mathcal{F}$ such that $\beta\subset\gamma$ and $\beta\subset\gamma^*$, the minors of the matrices*
$$\begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x^\gamma_{\alpha_1} & x^\gamma_{\alpha_2} & x^\gamma_{\alpha_3} \end{pmatrix}, \begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x^{\gamma^*}_{\alpha_1} & x^{\gamma^*}_{\alpha_2} & x^{\gamma^*}_{\alpha_3} \end{pmatrix},$$
$$\begin{pmatrix} x^\gamma_{\alpha_1} & x^\gamma_{\alpha_2} & x^\gamma_{\alpha_3} \\ x^{\gamma^*}_{\alpha_1} & x^{\gamma^*}_{\alpha_2} & x^{\gamma^*}_{\alpha_3} \end{pmatrix}.$$

Proof. If $\widetilde{x}$ is in the image of U°, then

$$q^\gamma(\widetilde{x}) = \widetilde{\mathrm{pr}}_\gamma(x) \qquad \text{and} \qquad q^{\gamma^*}(\widetilde{x}) = \widetilde{\mathrm{pr}}_{\gamma^*}(x)$$

By definition of $\widetilde{\mathrm{pr}}$ and q these equations imply that

$$[x^\gamma_\alpha]_{\alpha\in\mathcal{E}(\gamma)} = [x_\alpha]_{\alpha\in\mathcal{E}(\gamma)} \qquad \text{and} \qquad [x^{\gamma^*}_\alpha]_{\alpha\in\mathcal{E}(\gamma^*)} = [x_\alpha]_{\alpha\in\mathcal{E}(\gamma^*)}.$$

This means that there are nonzero $t_\gamma, t_{\gamma^*} \in \mathbf{k}$ such that
(i) $x_{\alpha_j} = t_\gamma x^\gamma_{\alpha_j}$ for $j = 1, 2, 3$, and
(ii) $x_{\alpha^*_j} = t_{\gamma^*} x^{\gamma^*}_{\alpha^*_j}$ for $j = 1, 2, 3$.

The first linear polynomial comes from Proposition 8.5. Substituting (i) into this linear polynomial and factoring our t_γ gives the second linear polynomial.

The first matrix in the horizontal quadratic list comes from Proposition 8.5. Substituting, respectively, (i), (ii), and both (i) and (ii) into this matrix and factoring out, respectively, t_γ, t_{γ^*}, and $t_\gamma t_{\gamma^*}$ from the corresponding minors gives the minors of the remaining three matrices.

Substituting, respectively, (i), (ii), and both (i) and (ii) into the matrix

$$\begin{pmatrix} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \\ x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_3} \end{pmatrix}$$

and factoring out, respectively, t_γ, t_{γ^*}, and $t_\gamma t_{\gamma^*}$ from the corresponding minors gives the minors of the three matrices in the vertical quadratic list. ☐

Example 8.12. When $n = 4$, the vanishing of the polynomials of Proposition 8.11 have the following interpretation in terms of the valued figures of Example 7.17.

The vanishing of the linear forms implies that sum of the values around any triangle (in any face) is zero, when the sum is taken with appropriate signs. In particular, since all of the projective coordinates corresponding to a face cannot simultaneously vanish, this condition implies that if a triangle appears as a separate connected component in Figure 6, then at most one of the values on its edges can be zero.

The vanishing of the quadratic polynomials relates the values on edges in certain pairs of triangles in the figure. There are two types of such pairs: vertical, and horizontal.

We say that two triangles in the figure are a *vertical pair* if they correspond to the same triangle in one of the original hypersimplices. If in Figure 6, we move the entire right half of the figure to the bottom of the left half, we then have three distinct columns. Then two triangles of the figure form a vertical pair if and only if they are in the same column and one is a parallel translate of the other. Moreover, this parallel translation gives a correspondence between the edges of one triangle and the edges of the other. We say that the triangles in a vertical pair *have the same shape on* $\widetilde{x}$ if the triple of values on the edges of one is a scalar multiple (possibly zero) of the corresponding triple of values on the edges of the other. The vanishing of the vertical quadratic polynomials in Proposition 8.11 is precisely the condition that all vertical pairs have the same shape on $\widetilde{x}$.

We say that two triangles in the figure are a *horizontal pair* if they correspond to a pair of related triangles (cf., Example 3.4) in the original hypersimplices. As noted in Example 3.6 two triangles form a related pair in the original hypersimplices if and only if they are in adjacent hypersimplices and one is obtained from the other by a parallel translation and a 180° rotation. In Figure 6, we again move the entire right half of the figure to the bottom of the left half. Then two triangles form a horizontal pair if and only if they are in adjacent columns and one can be obtained from the other by a parallel translation followed by a 180° rotation. Again, this congruence determines a bijection between the edges of one triangle and the edges of the other. We say that the triangles in a horizontal pair *have the same shape on* $\widetilde{x}$ if the triple of values on the edges of one is a scalar multiple (possibly zero) of the corresponding triple of values on the edges of the other. The vanishing of the horizontal quadratic polynomials in Proposition 8.11 is precisely the condition that all horizontal pairs have the same shape on $\widetilde{x}$.

9. The Torus Action and the Core

9.1. Recall that the torus T is the subgroup of diagonal matrices in $G = GL_n(\mathbf{k})$. We describe the action of T on our local model U. Given $t \in T$, we let $t_i \in \mathbf{k}^*$ denote the ith diagonal entry of t. For $x \in \mathbf{k}^{\mathcal{E}}$ and $k = 1, \dots, n-1$, we let x_k be the image of x under the projection $\mathbf{k}^{\mathcal{E}} \mapsto \mathbf{k}^{\mathcal{E}_k}$ (corresponding to the inclusion $\mathcal{E}_k \subset \mathcal{E}$). We define an action of T on

$$\mathbf{k}^{\mathcal{E}} = \mathbf{k}^{\mathcal{E}_1} \times \mathbf{k}^{\mathcal{E}_2} \times \cdots \times \mathbf{k}^{\mathcal{E}_{n-1}}$$

by $t \cdot x = ((t_2/t_1)x_1, \dots, (t_n/t_{n-1})x_{n-1})$, and we define an action of T on

$$\mathbf{k}^{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

by $t \cdot \widetilde{x} = (t \cdot x, q(\widetilde{x}))$ (i.e., T acts as just defined on the first factor and acts trivially on the product of projective spaces).

Lemma 9.2. *The isomorphisms $Z(E_*, E_*^{op}) \to U$ and $\widetilde{Z}(E_*, E_*^{op}) \to \widetilde{U}$ are T-equivariant.*

Proof. Let p and $t \cdot p$ in $Z(E_*, E_*^{op})$ have images x and y, respectively, in U. Let $\alpha = \{I, J\}$ with $I < J$ and $k = \|I\| = \|J\|$. Let b and c be constants such that

$$\begin{array}{lcccc} s_I(p) & = & e_k & + & be_{k+1} \\ s_J(p) & = & e_k & + & ce_{k+1} \\ s_{I\cup J}(p) & = & & & e_{k+1} \end{array}$$

modulo the subspace $E_{\{k+2,\dots,n\}}$. Since the T-action is the standard diagonal action on V, we have

$$\begin{array}{lcccc} s_I(t \cdot p) & = & e_k & + & (t_{k+1}/t_k)be_{k+1} \\ s_J(t \cdot p) & = & e_k & + & (t_{k+1}/t_k)ce_{k+1} \\ s_{I\cup J}(t \cdot p) & = & & & e_{k+1} \end{array}$$

modulo the subspace $E_{\{k+2,\dots,n\}}$. Applying Proposition 8.2 to p and $t\cdot p$ gives

$$s_J(p) = s_I(p) + x_\alpha s_{I\cup J}(p) \quad \text{and} \quad s_J(t\cdot p) = s_I(t\cdot p) + y_\alpha s_{I\cup J}(t\cdot p).$$

The first equation implies $x_\alpha = c-b$ while the second implies $y_\alpha = (t_{k+1}/t_k)(c-b)$. Solving, we get

$$y_\alpha = (t_{k+1}/t_k)x_\alpha,$$

proving equivariance for the first isomorphism.

Equivariance for the second isomorphism follows since if $x \in U^\circ$, and $\beta \in \mathcal{E}_k$, then

$$\widetilde{\mathrm{pr}}_\beta(t\cdot x) = [y^\beta_\alpha]_{\alpha\in\mathcal{E}(\beta)} = [y_\alpha]_{\alpha\in\mathcal{E}(\beta)} = [(t_{k+1}/t_k)x_\alpha]_{\alpha\in\mathcal{E}(\beta)} = [x_\alpha]_{\alpha\in\mathcal{E}(\beta)} = \widetilde{\mathrm{pr}}_\beta(x).$$

This means T acts trivially on the image of U° (and therefore on its closure) in $\prod_\beta \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$. □

9.3. We can view the product

$$\mathbf{k}^{\mathcal{E}} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

as a trivial vector bundle over the second factor (the big product of projective spaces) with bundle projection q and fiber

$$\mathbf{k}^{\mathcal{E}} = \mathbf{k}^{\mathcal{E}_1} \times \mathbf{k}^{\mathcal{E}_2} \times \cdots \times \mathbf{k}^{\mathcal{E}_{n-1}}.$$

As such, it has a canonical $(n-1)$-dimensional subbundle $\mathbf{N}$ defined by $x_k \in q^{\gamma_k}(\widetilde{x})$ for all $k = 1,\dots,n-1$ (where $\gamma_k \subset \mathcal{F}_k$ is the maximal face corresponding to the full hypersimplex Δ^n_k as in Example 3.5). Note that the T-action restricts to the bundle $\mathbf{N}$, acting fiberwise, and that the orbits are dense in the fibers.

Let Ξ be the image of $\widetilde{U}$ under the projection

$$q : \mathbf{k}^{\mathcal{E}} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}) \to \prod_{\beta\in\mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

We call it the *core*.

Proposition 9.4. *The variety $\widetilde{U}$ coincides with the restriction $\mathbf{N}|_\Xi$, hence $\widetilde{U}$ is an $(n-1)$-dimensional vector bundle over the core.*

Proof. The image of the map $U^\circ \mapsto \mathbf{k}^{\mathcal{E}} \times \prod_\beta \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$ given by $x \mapsto (x, \widetilde{\mathrm{pr}}(x))$, is contained in the restricted bundle $\mathbf{N}|_\Xi$, hence so is its closure $\widetilde{U}$ (since $\mathbf{N}|_\Xi$ is closed in $\mathbf{k}^{\mathcal{E}} \times \prod_\beta \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$). Let Ξ° be the image of U° in $\prod_\beta \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$ under the map $x \mapsto \widetilde{\mathrm{pr}}`(x)$. Since the torus acts with dense orbits in the fibers of $\mathbf{N}|_{\Xi^\circ}$, the image of U° is dense in $\mathbf{N}|_{\Xi^\circ}$. Since Ξ° is dense in Ξ, the image of U° is dense in $\mathbf{N}|_\Xi$, hence $\widetilde{U} = \mathbf{N}|_\Xi$. □

Example 9.5. In the $n = 4$ case, the torus T is 3-dimensional and acts on the valued figures of Example 7.17 by scaling the edge-values on each of the three hypersimplices in the box and fixing the remaining edge values. A given element of T scales all of the edge values in a given hypersimplex by the same factor, and thus the only points in the closure of a torus orbit that are not in the orbit itself are ones in which all of the edge values of a given hypersimplex (in the box) vanish. We represent such a T-limit point in our figure by replacing the corresponding hypersimplex with a dot (as in Figures 9–13). It follows that the projection to the core $q : \widetilde{U} \to \Xi$ has the effect of replacing all three hypersimplices with dots and leaving the remaining edge values alone.

Remark 9.6. Although the core Ξ is defined locally using the models U, we should point out that Ξ is really a global object and does not depend on our choice of coordinates. In fact, Ξ could be defined as either

1. the union of the B-fixed points in $\widetilde{Z}(E_*)$, or
2. the full inverse image in $\widetilde{Z}(E_*)$ of the "most degenerate point" of $Z(E_*)$ (that is, the point p such that $\pi_I(p) = \pi_J(p)$ for all $I, J \subset [\![n]\!]$ with $\|I\| = \|J\|$).

Moreover, one can show that B acts trivially on Ξ, and that if $\mathcal{C} \subset \widetilde{\mathcal{Z}}$ is the union of the G-orbits of points in Ξ, then the restriction $\pi : \mathcal{C} \to \mathbb{F}$ is a trivial bundle with fiber Ξ.

Remark 9.7. With the T-action in hand, we can now explain the difference between our compactification and Schubert's in the case $n = 3$. Recall from the introduction that in both cases the fiber over the unique singular point of U is a $\mathbb{P}^1$. Then in Schubert's space, the torus acts on this $\mathbb{P}^1$ with two isolated fixed points, but in our space, the torus fixes the whole $\mathbb{P}^1$. In fact, one can show that in this case $\widetilde{\mathcal{Z}}$ is isomorphic to the Fulton-MacPherson space $\mathbb{P}^2[3]$, which first appeared in the literature as an auxiliary compactification of Roberts-Speiser [14].

10. Nonsingularity

10.1. For a point $\widetilde{p}$ in $\widetilde{\mathcal{Z}}$, we let p be its image in $\mathcal{Z}$. We then define the *number of k-planes in $\widetilde{p}$*, denoted $n_k(\widetilde{p})$, to be the cardinality of the subset

$$\{\pi_I(p) \mid I \subset [\![n]\!],\ \|I\| = k\} \subset \mathbb{G}_k.$$

Definition 10.2. The point $\widetilde{p} \in \mathcal{Z}$ is called *generic* (resp. *toroidal*) if $n_k(\widetilde{p}) \geq 2$ (resp. $n_k(\widetilde{p}) = 2$) for all $k = 1, \dots, n-1$.

Since the map $q : \widetilde{U} \to \Xi$ is, by Proposition 9.4, a locally trivial vector bundle, if any point in a given fiber is a smooth point, then every point in that fiber will be a smooth point. Thus, to show that $\widetilde{U}$ is smooth, it suffices to show that every fiber contains a smooth point. We call a point $\widetilde{x} \in \widetilde{U}$ generic (resp. toroidal) if it is the image of a generic (resp. toroidal) point $\widetilde{p}$ in $\widetilde{Z}(E_*, E_*^{op})$. The following lemma implies that we need only check smoothness at generic points of $\widetilde{U}$.

Lemma 10.3. *The generic points of $\widetilde{U}$ are dense in each fiber of $\widetilde{U} \to \Xi$.*

Proof. For any fiber, we can choose a point $\widetilde{x}$ such that $T \cdot \widetilde{x}$ is dense in the fiber. Looking at the T-action in 9.1, we see that this implies $x_k \neq 0$ for each $k = 1, \dots, n-1$. But if $\widetilde{p}$ is the point in $\widetilde{Z}(E_*, E_*^{op})$ corresponding to $\widetilde{x}$, and p is its image in $Z(E_*, E_*^{op})$, this means that for each k, we can find an $\alpha \in \mathcal{E}_k$ such that $\chi_\alpha(p) = x_\alpha \neq 0$. Letting $\alpha = \{I, J\}$, we then have $\pi_I(p) \neq \pi_J(p)$, hence $n_k(\widetilde{x}) \geq 2$. Since n_k is constant on T-orbits, every point of $T \cdot \widetilde{x}$ is generic. □

10.4. Recall that the group B acts on $\widetilde{Z}(E_*)$ and takes smooth points to smooth points. Since the set of smooth points in a variety is open, if any point in the closure of a B-orbit is smooth, then the entire B-orbit must be smooth as well. The following proposition implies, therefore, that if every toroidal point in $\widetilde{U}$ is smooth, so is every generic point. Note that since B does not act on the chart $\widetilde{Z}(E_*, E_*^{op})$, we have to be a little bit careful (but not very) about using this action to study the local model.

Proposition 10.5. *If $\widetilde{p}$ is a generic point in the chart $\widetilde{Z}(E_*, E_*^{op})$, then there exists a toroidal point in the closure of the orbit $B \cdot \widetilde{p}$ that is also in the chart $\widetilde{Z}(E_*, E_*^{op})$.*

To prove the proposition we require some lemmas.

Lemma 10.6. *Let E, F, and H be three distinct elements of the Grassmannian $\mathbb{G}_k$. Assume further that E is the B-fixed subspace $E_k = E_{\{1,\dots,k\}}$. Then there exists a one-parameter subgroup $\mu : \mathbf{k}^\times \to B$ such that*

$$\lim_{t\to 0} \mu(t) \cdot F = E \qquad \textit{and} \qquad \lim_{t\to 0} \mu(t) \cdot H \neq E.$$

Proof. Let $z \in H$ be a point not in E or F, and let E_i be the i-dimensional subspace of the fixed flag E_*. Then there is a unique j such that

$$z \in E_i, \quad i \leq j, \tag{6}$$

$$z \notin E_i, \quad i > j. \tag{7}$$

Note that $j > k$, for otherwise $H = E$.

Now choose an independent set $w_{k+1}, \dots, w_n \in \mathbf{k}^n$ as follows. If $i = j$, then set $w_i = z$. Otherwise generically choose $w_i \in E_i \setminus E_{i-1}$ and not in F or H. Let W be the span of $\{w_i\}$. Note that $\dim W = n - k$, and $E \cap W = F \cap W = \{0\}$, but $H \cap W \neq \{0\}$.

Let $\mu \colon \mathbf{k}^\times \to G$ be a one-parameter subgroup that scales in E with a negative weight and scales in W with a positive weight. Then $\lim_{t\to 0} \mu(t) \cdot F = E$, but $\lim_{t\to 0} \mu(t) \cdot H \neq E$.

Now to complete the proof, we must show that this one-parameter subgroup is a subgroup of B. But it is clear from the way that the w_i were chosen that the flag E_* is preserved. Thus $\mu(\mathbf{k}^\times) \subset B$.

□

Lemma 10.7. *Let $\widetilde{p}$ and $\widetilde{s}$ be points in the chart $\widetilde{Z}(E_*, E_*^{op})$. If $\widetilde{p}$ is a generic point and $\widetilde{s}$ is in the closure of the orbit $B \cdot \widetilde{p}$, then there exists a generic point $\widetilde{r}$ in the closure of $B \cdot \widetilde{p}$ such that (1) $\widetilde{s}$ is in the closure of $T \cdot \widetilde{r}$ and (2) $\widetilde{r}$ is in the chart $\widetilde{Z}(E_*, E_*^{op})$.*

Proof. Let W be the set of generic points in the orbit $(B \cdot \widetilde{p})$. It is obviously an open subset, hence $\widetilde{s}$ is in its closure. But since the T-orbit of any point in W is also in W, the entire fiber $q^{-1}(q(\widetilde{s}))$ must also be in the closure of W. Letting $\widetilde{r}$ be a generic point in this fiber completes the proof. □

Lemma 10.8. *Suppose $\widetilde{r}$ and $\widetilde{s}$ are points in the chart $\widetilde{Z}(E_*, E_*^{op})$. If $\widetilde{s}$ is in the closure of $T \cdot \widetilde{r}$, and $\widetilde{s}$ has more than one k-plane, then $\widetilde{s}$ and $\widetilde{r}$ have the same number of k-planes.*

Proof. A look at the torus action (e.g., in the local model in 9.1) shows that passing to a point in the chart which is in the closure of an orbit either does not change the number of k-planes or drops it all the way down to one. □

Proof of the proposition. We use the lemmas above to collapse the configuration associated to $\widetilde{p}$ so that only two subspaces of each dimension remain. We use implicitly that in passing to a point in the closure of a B-orbit, the number of planes in any given dimension cannot increase (if an incidence condition holds on a B-orbit, it must hold on its closure).

By assumption, not all lines of $\widetilde{p}$ coincide. If there are more than two distinct lines, we can use Lemma 10.6 to find a point $\widetilde{o}$ in the closure of $B \cdot \widetilde{p}$ such that $\widetilde{o}$ has fewer, but at least two, lines. The entire B-orbit of $\widetilde{o}$ must, therefore, be in the closure of $B \cdot \widetilde{p}$, and since B acts transitively on the covering $\{Z(E_*, A_*) \mid A_* \in \mathbb{F}(E_*)\}$, we can find some B-translate $\widetilde{s}$ of $\widetilde{o}$ which is (1) in the closure of $B \cdot \widetilde{p}$ and (2) in the chart $\widetilde{Z}(E_*, E_*^{op})$. Since the number of k-planes is constant on B-orbits, $\widetilde{s}$ will have the same number of lines as $\widetilde{o}$. By Lemma 10.7, we can then find a generic point $\widetilde{r}$ in the closure of $B \cdot \widetilde{p}$ such that (1) $\widetilde{s}$ is in the closure of $T \cdot \widetilde{r}$, and (2) $\widetilde{r}$ is in the chart. By Lemma 10.8, we know that $\widetilde{r}$ has the same number of lines as $\widetilde{s}$, i.e., at least two but fewer than $\widetilde{p}$. Since any point in the closure of $B \cdot \widetilde{r}$ will also be in the closure of $B \cdot \widetilde{p}$, we can repeat this procedure until we find a generic point $\widetilde{p}_1$ which is (1) in the closure of $B \cdot \widetilde{p}$, (2) in the chart, and (3) has exactly two lines.

Now proceed by induction to construct $\widetilde{p}_2, \dots, \widetilde{p}_{n-1}$, such that for each i, $\widetilde{p}_i$ is a generic point which is (1) in the closure of $B \cdot \widetilde{p}_{i-1}$ (hence, in the closure of $B \cdot \widetilde{p}$), (2) is in the chart, and (3) has exactly two k-planes for $1 \le k \le i$. The point $\widetilde{p}_{n-1}$ is therefore a toroidal point in the closure of $B \cdot \widetilde{p}$ that is in the chart $\widetilde{Z}(E_*, E_*^{op})$. □

10.9. By Proposition 10.5 any generic point of $\widetilde{U}$ has a toroidal point in the closure of its B-orbit that is also in $\widetilde{U}$. Thus, if every toroidal point in the local model $\widetilde{U}$ is smooth, then so is every generic point in $\widetilde{U}$, and hence by Lemma 10.3, so is every point of $\widetilde{U}$. But to check smoothness of a toroidal point in $\widetilde{U}$ it suffices to check smoothness of its image in the core. We will call a point $z \in \Xi$ a *special point* if it is the image of a toroidal point (i.e., if $z = q(\widetilde{x})$ for some toroidal point $\widetilde{x}$.). Combining the results above, and using the fact that $\widetilde{U}$ is a vector bundle over the core, we have the following theorem.

Theorem 10.10. *The map $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is a smooth G-equivariant resolution if and only if every special point is a smooth point of the core.*

11. The Case of Tetrahedra

11.1. Now for the rest of the article, we will fix $G = GL_4(\mathbf{k})$. We will classify the special points and apply the Jacobian condition for nonsingularity. Much of our calculation will rely on the figures of Example 7.17.

Using the restrictions on these figures introduced in Example 8.12, we first determine which subsets of the various coordinates can be simultaneously zero. We will indicate a zero coordinate in a valued figure by drawing the corresponding edge as a thick line (i.e., in bold). Figures 7 and 8 show the possible combinations of vanishing coordinates for any vertical or horizontal pair.

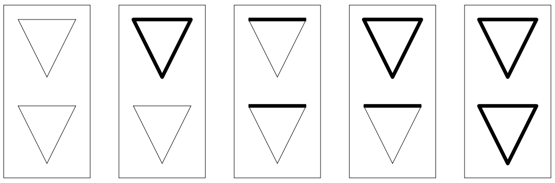

Figure 7. Possible vertical pairs of triangles.

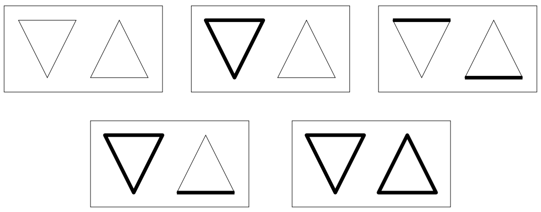

Figure 8. Possible horizontal pairs of triangles.

11.2. We now consider which valued figures correspond to special points, by first determining which figures correspond to toroidal points.

The number of k-planes for a point $\widetilde{x} \in \widetilde{U}$ can be determined as follows from the valued figure for $\widetilde{x}$. We assume all of the zero values in the figure are indicated by bold edges. Then $\widetilde{x}$ determines a *bold subgraph* of each face by taking all of the vertices, but just the bold edges. The number of connected components in the bold subgraph of the rank-k hypersimplex (in the box) is precisely the number of k-planes in $\widetilde{x}$. In fact, since the vertices of the rank-k hypersimplex represent the projections $\{\pi_I(p) \mid \|I\| = k\}$, the bold subgraph indicates exactly which k-planes coincide on $\widetilde{x}$.

Since there are four lines, six 2-planes, and four 3-planes for a point in U°, we see that a toroidal point, up to symmetry, must have its subspaces partitioned as follows:

1. The lines must collapse together as $(3,1)$ or $(2,2)$. (The notation $(p.q)$ means that the two distinct lines are the image of p and q lines, where $p+q=4$.)
2. The 2-planes must collapse together as $(5,1)$, $(4,2)$, or $(3,3)$.
3. The 3-planes, like the lines, must collapse together as $(3,1)$ or $(2,2)$.

In fact, these partitions must persist in the maximal faces when we project to the core.

Lemma 11.3. *Let $\widetilde{x}$ be a toroidal point, and let $\Gamma_k(\widetilde{x})$ be the bold subgraph of the hypersimplex Δ^4_k (in the box) determined by $\widetilde{x}$. Then the bold subgraph of the hypersimplex Δ^4_k (below the box) determined by the special point $q(\widetilde{x})$ coincides with $\Gamma_k(\widetilde{x})$.*

With these figures in hand, it is a pleasant combinatorial exercise to compute the special points. We start by choosing various 2-component bold subgraphs in the three maximal faces (outside the box). These subgraphs must be consistent with the allowable horizontal pairs in Figure 8. We then extend these to bold subgraphs of the remaining triangular components, such that only the allowable vertical and horizontal pairs appear. We summarize the results in Figures 9–13. Note that all of the diagrams correspond to isolated points in the core, except for Figure 13, which corresponds to a family of special points. Note also that each of the figures actually corresponds to an orbit of special points under the symmetric group action, and that reversing the role of lines and 3-planes in Figure 10 gives a dual configuration which we have not included in the list. Clearly it suffices to verify smoothness just for the cases listed. For an explanation of the names of the figures, see [2].

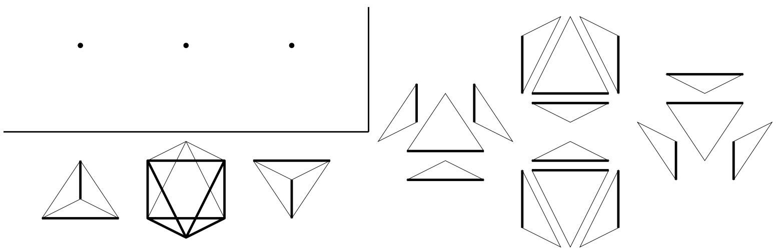

FIGURE 9. *DDE*.

11.4. We are now ready to prove that the special points are smooth points of the core.

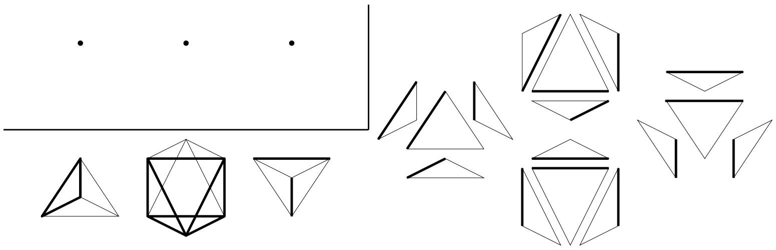

FIGURE 10. CDE.

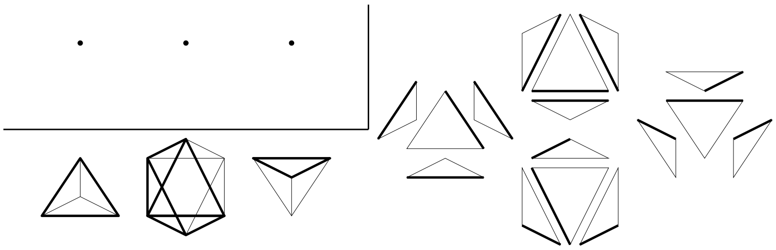

FIGURE 11. CC^*E.

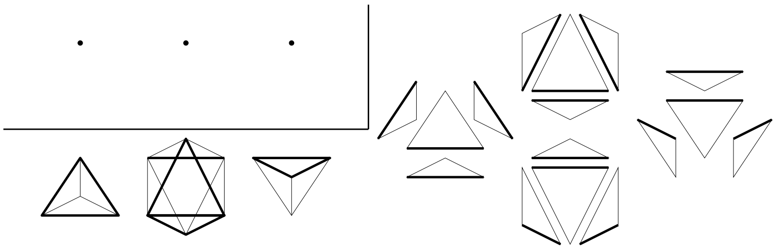

FIGURE 12. $CC^*_{nop}D$.

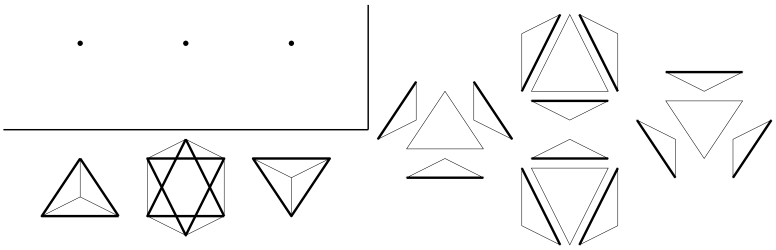

FIGURE 13. $CC^*_{op}D$.

Let $\xi = q(\widetilde{x}) \in \Xi$ be one of the special points in Figures 9–13. We fix an affine neighborhood of ξ in the variety

$$\prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$$

as follows. For each face $\beta \in \mathcal{H}$, we selecting one $\alpha(\beta) \in \mathcal{E}(\beta)$ such that $x^{\beta}_{\alpha(\beta)} \neq 0$, and we set this coordinate equal to 1. We indicate this choice in our figures by placing a 1 on the edge of the face β corresponding to $\alpha(\beta)$. Having done this, the other variables $\{x^{\beta}_{\alpha} \mid \alpha \neq \alpha(\beta)\}$ form a system of local parameters at ξ in $\prod_{\beta} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)})$.

Let Ω^1 be the $\mathbf{k}$-vector space of differentials of Ξ at the point ξ. The dimension of this vector space is at least the dimension of Ξ, and ξ is a smooth point if and only if the dimensions are equal. Since when $n = 4$, $\widetilde{U}$ is 6-dimensional and is a 3-dimensional vector bundle over the core Ξ, we know that Ξ is a 3-dimensional subvariety of

$$\prod_{\beta \in \mathcal{H}} \mathbb{P}(\mathbf{k}^{\mathcal{E}(\beta)}).$$

Thus, we need to show that Ω^1 has at most 3 independent differentials.

Recall (cf. [5]) that

$$\Omega^1 = \left(\bigoplus \mathbf{k} \cdot dx^{\beta}_{\alpha} \right) / J,$$

where the sum is taken over all pairs (α, β) where $\alpha \in \mathcal{E}(\beta) \smallsetminus \{\alpha(\beta)\}$, and J is the subspace generated by the differentials (evaluated at ξ) of all functions vanishing on Ξ. To study this quotient, we will develop some simple combinatorial rules from our equations for computing with differentials. The following lemma is easy to prove through a direct computation:

Lemma 11.5. *Suppose that $x^{\beta_1}_{\alpha_1} x^{\beta_2}_{\alpha_2} - x^{\beta_1}_{\alpha_2} x^{\beta_2}_{\alpha_1}$ vanishes on Ξ.*

1. *If $x^{\beta_1}_{\alpha_1} = x^{\beta_1}_{\alpha_2} = x^{\beta_2}_{\alpha_1} = 0$ and $x^{\beta_2}_{\alpha_2} \neq 0$, then $dx^{\beta_1}_{\alpha_1} = 0$.*
2. *If $x^{\beta_1}_{\alpha_1} = x^{\beta_2}_{\alpha_1} = 0$ and $x^{\beta_2}_{\alpha_2} = x^{\beta_1}_{\alpha_2} \neq 0$, then $dx^{\beta_1}_{\alpha_1} = dx^{\beta_2}_{\alpha_1}$.*

We apply Lemma 11.5 to any horizontal or vertical pair of triangles which has bold edges corresponding to the 3rd or 4th possibility in Figures 7 and 8. We encode the results in our figures using the notation of Figure 14. The 0 means that the differential of the variable corresponding to the edge is 0, and the two das indicate that the two differentials coincide in Ω^1.

FIGURE 14. Rules for differentials.

Now can now verify nonsingularity for the isolated special points.

Lemma 11.6. *The isolated special points are smooth points of Ξ.*

Proof. The computations for the isolated points are very similar. Hence we will explain the computation for DDE and will leave the other isolated points to the reader.

We choose an affine neighborhood of DDE by assigning the value 1 to exactly one thin edge in each of the 19 components in Figure 9. The vanishing of the differentials of the linear polynomials (being in J) then implies that the differential corresponding to any other thin edge is a linear combination of differentials corresponding to thick edges. Thus, Ω^1 is generated by the dx^{β}_{α}, where α is a thick edge of the component β.

Next consider the hypersimplex connected components of Figure 9. Applying Lemma 11.5 results in Figure 15. We have found three independent differentials da, db, and dc, and the other differentials are 0.

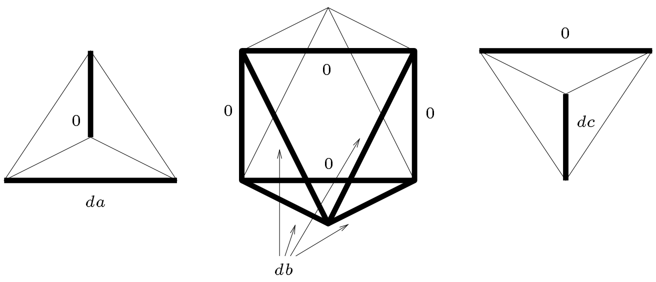

FIGURE 15. Three differentials for DDE.

Now consider the other connected components of Figure 9. Using Lemma 11.5 we see that the remaining differentials are either 0 or are equal to da, db, or dc. The result is summarized in Figure 16. Hence Ω^1 is 3-dimensional, and DDE is a nonsingular point of Ξ.

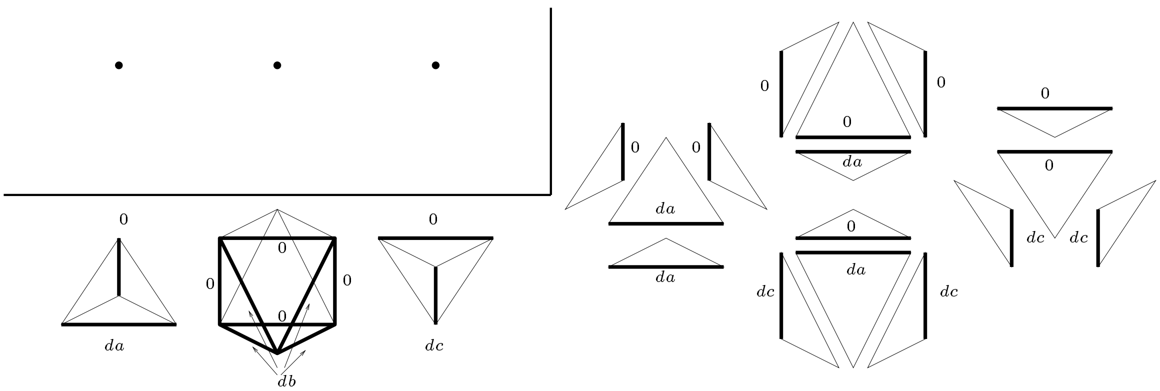

FIGURE 16. All differentials for DDE.

□

11.7. To complete the proof of our main theorem, we must investigate the point $CC^*_{op}D$. It turns out that Lemma 11.5 is not sufficient to verify nonsingularity: there are polynomials that vanish on Ξ which are not in the ideal generated by the polynomials of Proposition 8.11.

To see how these additional polynomials arise, we note that the variety $\mathcal{Z}$ is an irreducible subvariety of the *incidence variety* $\mathcal{Y}$ defined by

$$\mathcal{Y} = \{p \in \mathcal{X} \mid \pi_I(p) \subset \pi_J(p) \text{ for all } I \subset J\}.$$

As in Section 4, there is a fibration $\mathcal{Y} \to \mathbb{F}$, with fiber $Y(F_*)$ consisting of those configurations with a given fixed flag F_*. As in Section 5, for any A_* in general position to F_*, we let $Y(F_*, A_*)$ be the set of all $p \in Y(F_*)$ such that $\pi_I(p)$ is in general position relative to A_* for all $I \subset [\![n]\!]$. Then the closed embedding of Lemma 7.9 actually defines an embedding

$$Y(F_*, A_*) \to \mathbf{k}^{\mathcal{E}}$$

and all of the equations of Proposition 8.5 hold on the image.

For $n = 4$, the variety $Y(F_*, A_*)$ has three irreducible components. One component has an open set consisting of subspaces arranged to form nondegenerate tetrahedra. The other two components parameterize the following configurations in $\mathbf{k}^4$:

1. Four 3-planes containing a 2-plane, which contains four lines.
2. Six 2-planes containing a line and contained in a 3-plane.

Figure 17 shows general points in these components as configurations in $\mathbb{P}^3$.

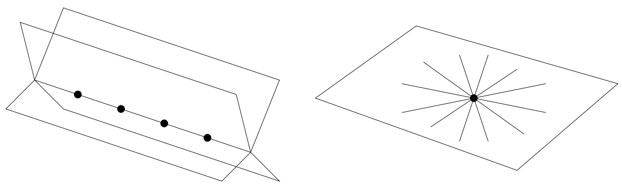

FIGURE 17. Other components of the incidence variety for $n = 4$.

The extra equations that we need describe which of the configurations in Figure 17 can occur as the degeneration of a tetrahedron. In the following lemma, we avoid cumbersome notation by omitting the braces on sets indexing the edge variables.

Lemma 11.8. *For any distinct elements i, j, k, l in $\{1, 2, 3, 4\}$, the cubic polynomial*

$$x_{ij,il}x_{ik,kl}x_{jk,jl} - x_{jk,kl}x_{ij,jl}x_{ik,il}$$

and the quartic polynomial

$$x_{i,j}x_{k,l}x_{ijk,ikl}x_{ijl,jkl} - x_{i,k}x_{j,l}x_{ikl,jkl}x_{ijk,ijl}$$

both vanish if $x \in U$.

Proof. It suffices to prove that the polynomials vanish for any $x \in U^\circ$, hence we can assume all of the edge variables are nonzero.

To prove that the quartic vanishes, it suffices by symmetry to prove that $acfh - bdeg$ vanishes for the edge variables as in Figure 18. Consider the four triangles surrounding the top vertex of the octahedron. The left and right triangles each have the same shape as a triangles in the first hypersimplex, giving the equations

$$\frac{a}{b} = \frac{r}{s} \qquad \text{and} \qquad \frac{c}{d} = \frac{t}{u}.$$

Likewise, the front and back triangles each have the same shape as a triangle in the third hypersimplex, giving

$$\frac{f}{g} = \frac{s}{t} \qquad \text{and} \qquad \frac{h}{e} = \frac{u}{r}.$$

Multiplying, we have

$$\frac{a}{b}\frac{c}{d}\frac{f}{g}\frac{h}{e} = \frac{r}{s}\frac{t}{u}\frac{s}{t}\frac{u}{r} = 1,$$

and clearing denominators gives the quartic.

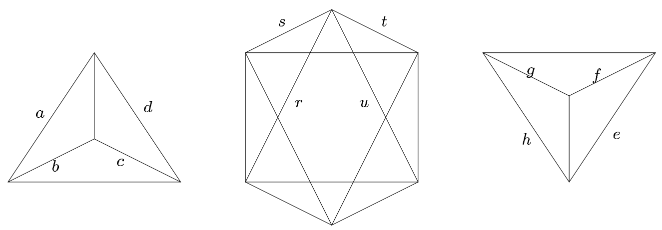

FIGURE 18.

The proof that the cubics vanish is similar. The six exterior edges of the octahedron correspond in pairs to the three edges surrounding the center vertex of the first tetrahedron. Multiplying the ratios in such a way that the edge variables in the tetrahedron cancel gives the cubic equation. $\square$

Remark 11.9. All the bundles and regular sections of Section 3 are defined over all of $\mathcal{Y}$, and not just the component $\mathcal{Z}$. Hence we can use our setup to define a "resolution" of any closed irreducible G-invariant subvariety $\mathcal{Y}_0$ of $\mathcal{Y}$. In general, however, the ambient variety will not be all of $\mathbf{E}_\#$. If for some section $\mathbf{u}_\beta$, the image $\mathbf{u}_\beta(\mathcal{Y}_0)$ is contained in the center $\mathbf{D}_\beta$ of the blow-up $(\mathbf{E}_\beta)_\# \to \mathbf{E}_\beta$, then $\mathbf{u}_\beta$ will not induce a rational map $\mathcal{Y}_0 \dashrightarrow \mathbf{E}_\#$. To avoid this, we let $\mathcal{H}_0$ denote the set of all faces $\beta \in \mathcal{H}$ such that $\mathbf{u}_\beta(p) \notin \mathbf{D}_\beta$ for some (hence, almost every) $p \in \mathcal{Y}_0$, and define $(\mathbf{E}_\#)_0 \to \mathcal{Y}_0$ to be the bundle $\prod_{\beta \in \mathcal{H}_0} (\mathbf{E}_\beta)_\#$. Then there is an induced rational section $\mathcal{Y}_0 \dashrightarrow (\mathbf{E}_\#)_0$, and we define $\widetilde{\mathcal{Y}}_0$ to be the closure of the image. We do not know if the spaces $\widetilde{\mathcal{Y}}_0$ are all nonsingular, even for $n = 4$.

11.10. We now consider the family of special points $CC^*_{op}D$ in Figure 13. Choose a point ξ in $CC^*_{op}D$. To get an affine neighborhood of this point, we choose one thin edge in each component and set it equal to one. At the point ξ, the linear relations on the triangles then imply that all of the thin edges *except those in the four thin triangles* will also have value one. The linear relations and horizontal quadratic relations on the thin triangles imply that we can find constants x and y in $\mathbf{k}$ such that the edge values are as in Figure 19 (up to the choice of the edges with values 1). Hence, a priori, there is a 2-dimensional family of such special points.

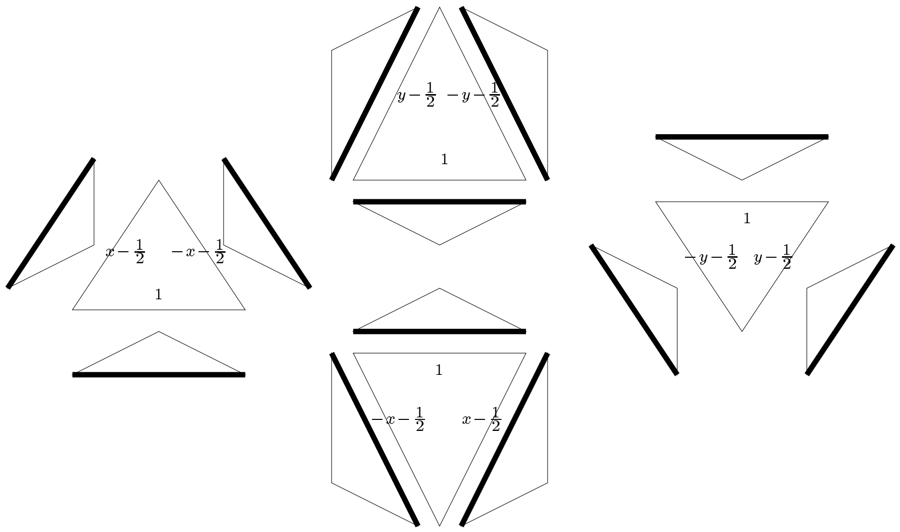

FIGURE 19. A typical point in $CC^*_{op}D$.

The quartic relations from Lemma 11.8, however, together with the fact that vertical pairs of triangles have the same shape on ξ imply that the two parameters x and y satisfy a linear equation. For the choice of parameters in Figure 19, for example, this relation is

$$1 \cdot (x - \frac{1}{2}) \cdot 1 \cdot 1 = 1 \cdot (y - \frac{1}{2}) \cdot 1 \cdot 1, \quad \text{or} \quad x = y.$$

Thus the family of special points in $CC^*_{op}D$ is, in fact, one dimensional. We now finish the proof of the main theorem.

Lemma 11.11. *The special points in $CC^*_{op}D$ are all smooth points of the core.*

Proof. As in the case of the isolated special points, the differentials on all thin edges except for those in the four thin triangles can be expressed as linear combinations of the differentials on bold edges. And the differentials on edges of the thin triangles can be expressed as linear combinations of dx and dy. Thus, using Lemma 11.5, we can find 8 differentials that span Ω^1 (Figure 20):

$$da, db, dc, da^*, db^*, dc^*, dx, dy.$$

Note that the span of these is at most 5-dimensional, because of the relations $da + db + dc = da^* + db^* + dc^* = 0$ coming from the differentials of the linear polynomials for the corresponding triangles, and the relation $dx = dy$ coming from the linear relation between x and y.

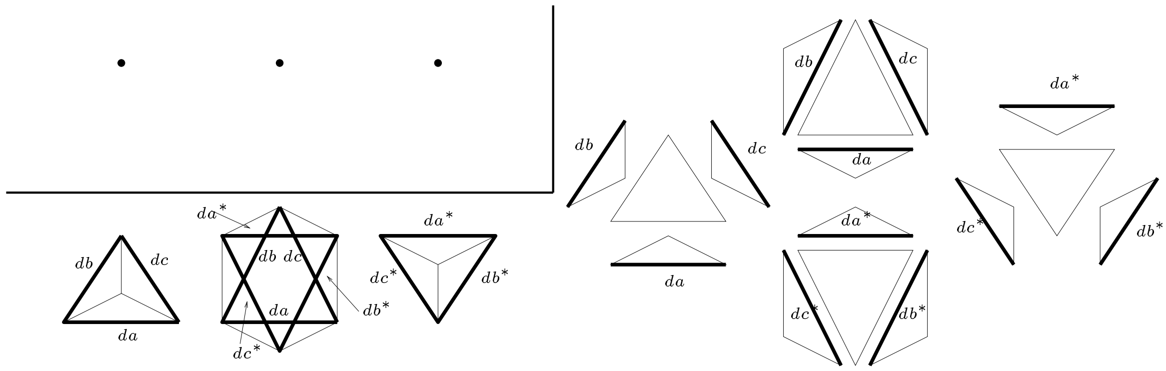

FIGURE 20. Differentials for $CC^*_{op}D$.

To finish, we claim that the spans of da, db, dc and da^*, db^*, dc^* are each 1-dimensional. Indeed, one of the vertical quadratic relations which say that the the front face of the octahedron in Figure 20 has the same shape as the corresponding face in Figure 19 implies

$$db - (x - \frac{1}{2})\, dc = 0.$$

This relation, and a similar one involving db^* and dc^* shows that $\dim \Omega^1 = 3$. This completes the proof of the main theorem. $\square$

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