THE PERMUTATIONS $123p_4..p_m$ AND $321p_4..p_m$ ARE WILF-EQUIVALENT

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Abstract. Write $p_1, p_2 \ldots p_m$ for the permutation matrix $\delta_{p_i,j}$. Let $S_n(M)$ be the set of $n \times n$ permutation matrices which do not contain the $m \times m$ permutation matrix M as a submatrix. In [?] Simion and Schmidt show bijectively that $|S_n(123)| = |S_n(213)|$. In [?] this was generalised to a bijection between $S_n(12p_3 \ldots p_m)$ and $S_n(21p_3 \ldots p_m)$. In the present paper we obtain a bijection between $S_n(123p_4 \ldots p_m)$ and $S_n(321p_4 \ldots p_m)$.

1. Introduction and main theorem

A *permutation matrix of order* n is a traversal of the n by n square diagram, in other words a placement of n non-attacking rooks on an n by n board. We can easily generalise this definition of a *traversal* to boards of general shape, and will do so shortly.

Given a permutation matrix M of order m, a permutation matrix N of order $n > m$ will be said to *contain* the smaller matrix if there exist two subsets of the index set $[n]$, $R = \{r_1 < r_2 < \cdots < r_m\}$ and $C = \{c_1 < c_2 < \cdots < c_m\}$, such that the submatrix

$$\begin{bmatrix} N[r_1,c_1] & N[r_1,c_2] & \cdots & N[r_1,c_m] \\ N[r_2,c_1] & N[r_2,c_2] & \cdots & N[r_2,c_m] \\ \vdots & \vdots & & \vdots \\ N[r_m,c_1] & N[r_m,c_2] & \cdots & N[r_m,c_m] \end{bmatrix} = M. \tag{1.1}$$

More generally, a traversal L of the *Ferrers board* of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ is an assignment of zeroes and ones to the squares of λ such that each row and column contains exactly one 1. The traversal L will be said to *contain* the matrix M if there exist two subsets of the positive integers $R = \{r_1 < r_2 < \cdots < r_m\}$ and $C = \{c_1 < c_2 < \cdots < c_m\}$, such that *each of the squares* (r_i, c_j) *falls within the board*, and the matrix equation 1.1 holds. The first condition here can be formalised by the two simultaneous inequalities $r_m \leq n$ and $c_m \leq \lambda_{r_m}$, since we need only check that the lower right square is within the board.

If a traversal does not contain a given submatrix M, then it is said to *avoid* M. Let $S_\lambda(M)$ be the set of M-avoiding traversals of the board λ.

If λ is the square board of order n we write simply $S_n(M)$. This is simply a restatement into matrix terms of the theory of *permutations with forbidden subsequences*, which has been developed in [?], [?] and [?, ?, ?]. In order to complete the translation into this language, we abbreviate by $p_1p_2\dots p_m$ the matrix $M_{i,j} = \delta_{p_i,j}$.

Numerous enumerative problems can be cast in this form. For instance, the well-known vexillary permutations (see [?]), which are characterised by the property that their Stanley symmetric function is a Schur function, are exactly those which avoid the submatrix 2143. For another instance from algebraic geometry, the Schubert variety indexed by a permutation w is smooth iff $w \in S_n(3412, 4231)$. [?, ?]. (An equivalent condition is that $l(w)$ is the number of reflections $\leq w$ in the Bruhat order).

An important early contribution to the subject is the enumeration of Baxter permutations in [?]. It was later noted by Olivier Guibert (personal communication) that the Baxter permutations could be characterized using forbidden subsequences, using the barred notation of [?]: they are those which avoid both $41\bar{3}52$ and $25\bar{3}14$. Recalling that an element p_i locates a 1 in position p_i, i of the matrix M, a bar over the element p_i means that what is forbidden is the existence of two subsets of the index set $[n]$, $\bar{R} = \{r_1 < \cdots < r_{p_i-1} < r_{p_i+1} < \cdots < r_m\}$ and $\bar{C} = \{c_1 < \cdots < c_{i-1} < c_{i+1} < \cdots < c_m\}$ such that

$$\begin{bmatrix} N[r_1,c_1] & \cdots & N[r_1,c_{i-1}] & 0 & N[r_1,c_{i+1}] & \cdots & N[r_1,c_m] \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ N[r_{p_i-1},c_1] & \cdots & N[r_{p_i-1},c_{i-1}] & 0 & N[r_{p_i-1},c_{i+1}] & \cdots & N[r_{p_i-1},c_m] \\ 0 & & 0 & 1 & 0 & & 0 \\ N[r_{p_i+1},c_1] & \cdots & N[r_{p_i+1},c_{i-1}] & 0 & N[r_{p_i+1},c_{i+1}] & \cdots & N[r_{p_i+1},c_m] \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ N[r_m,c_1] & \cdots & N[r_m,c_{i-1}] & 0 & N[r_m,c_{i+1}] & \cdots & N[r_m,c_m] \end{bmatrix} = M, \tag{1.2}$$

but where $N[i,j] = 0$ for every i, j with $r_{p_i-1} < i < r_{p_i+1}$ and $c_{i-1} < j < c_{i+1}$. For applications of this "barred" notation to a sorting problem and to a bijection with planar maps, see [?].

In the early 1980s Herb Wilf asked when it is true that two forbidden patterns are equally restrictive. Therefore we will say that two matrices M and M' are *Wilf-equivalent* if $|S_n(M)| = |S_n(M')|$ for all positive integers n. It is almost obvious that M and M' are Wilf-equivalent if M' can be obtained from M by an operation of the dihedral group D_4; in this case the elements of $S_n(M')$ are just the elements of $S_n(M)$ after application of the same operation.

In [?], it is shown that the following matrices form a Wilf-equivalent pair:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.3}$$

while in [?] a bijection is given which reëstablishes the quite well-known fact that the following pair is Wilf-equivalent:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.4}$$

Computer studies with forbidden matrices of order ≤ 7 suggest that 1.3 is a very sporadic case, while 1.4 can be generalized considerably. Indeed, it appears that two matrices are Wilf-equivalent if and only if they are related by 1.3, or as in

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix} \tag{1.5}$$

(where I_t is the identity matrix of order t and J_t its non-trivial reflection under the dihedral group), or are equivalent under the dihedral group to two matrices related in one of these ways, or are linked by a chain of such equivalences. This general form collapses to 1.4 if we take $t = 2$ and $A = I_1$.

In [?] the second author established bijectively that two permutations are Wilf-equivalent if they are related by 1.5 with $t = 2$. We reprove this result as the following theorem, using exactly the same proof as in [?] but rephrased in terms of matrices rather than forbidden subsequences.

Theorem 1.6. *Let M and M' be related by*

$$M = \begin{bmatrix} I_2 & 0 \\ 0 & A \end{bmatrix}, M' = \begin{bmatrix} J_2 & 0 \\ 0 & A \end{bmatrix} \tag{1.7}$$

Then $|S_n(M)| = |S_n(M')|$ for all positive integers n.

Two applications of theorem 1.6 together with an operation from the dihedral group establish the following corollary, which was first proved by independent processes in [?].

Corollary 1.8. $|S_n(2143)| = |S_n(1234)|$ *for all positive integers n.*

Corollary 1.8 is of interest because it maps the vexillary permutations $S_n(2143)$ to the permutations with no ascending subsequence of length 4; the latter were asymptotically enumerated by Regev [?].

As the main new result of the present work, we will also prove that two permutations are Wilf-equivalent if they are related by 1.5 for $t = 3$.

Theorem 1.9. *Let M and M' be related by*

$$M = \begin{bmatrix} I_3 & 0 \\ 0 & A \end{bmatrix}, M' = \begin{bmatrix} J_3 & 0 \\ 0 & A \end{bmatrix} \tag{1.10}$$

Then $|S_n(M)| = |S_n(M')|$ for all positive integers n.

The proofs of theorems 1.6 and 1.9 will rely on the following lemmata, the proofs of which will be delayed until the following section.

Lemma 1.11. $|S_\lambda(I_2)| = |S_\lambda(J_2)|$ *for all shapes λ.*

Lemma 1.12. *$|S_\lambda(I_3)| = |S_\lambda(J_3)|$ for all shapes λ.*

Each of these lemmata is of course a special case of the stronger

Conjecture 1.13. *$|S_\lambda(I_t)| = |S_\lambda(J_t)|$ for all $t > 1$ and for all shapes λ.*

Proof. [of 1.6 and 1.9] We actually prove the stronger result that two permutation matrices related as in 1.5 are Wilf-equivalent, making use of conjecture 1.13. Since the lemmata establish the truth of the conjecture when $t = 2, 3$, the theorems follow.

Let M and M' be related as in 1.5. Let N be any element of $S_n(M)$. For convenience, let $Q = [n] \times [n]$ be the set of squares of the $n \times n$ square board. We will call a square $(i, j) \in Q$ *dominant* if there is a submatrix of N isomorphic to A contained in rows $i + 1 \ldots n$ and columns $j + 1 \ldots n$. Clearly, if (i, j) is dominant, then so is (i', j') whenever $i' \leq i$ and $j' \leq j$. Thus the set of dominant squares forms a Ferrers board.

In general, some of the 1s in N will be in dominant squares, and others in nondominant squares. Let $\mathcal{N}(N)$ be the set of nondominant squares containing 1s. It should be clear that if (i, j) is dominant, then it dominates a copy of A whose 1s are contained entirely in squares of $\mathcal{N}(N)$. For consider all of the copies of A in rows $i + 1 \ldots n$ and columns $j + 1 \ldots n$. Look at the smallest row and smallest column coordinates (i', j') of each of these, and choose one such that $i' + j'$ is maximised. If this square is dominant, then we arrive at a contradiction of the maximality of our selection. Thus it is possible to recover the set of dominant squares in N given only $\mathcal{N}(N)$.

Then, for any subset $\mathcal{A}$ of Q, let ${}_{\mathcal{A}}S_n(M)$ be the subset $\{N \in S_n(M) : \mathcal{N}(N) = \mathcal{A}\}$. We will show that $|{}_{\mathcal{A}}S_n(M)| = |{}_{\mathcal{A}}S_n(M')|$ for any $\mathcal{A} \subseteq Q$. (Of course, these sets will be void unless $\mathcal{A}$ is at very least a partial traversal.) Theorem 1.9 then follows immediately by summing over all subsets $\mathcal{A}$ of Q.

Suppose $\mathcal{A}$ is a partial traversal and $|\mathcal{A}| = n - k$. Then to construct an element of ${}_{\mathcal{A}}S_n(M)$ we have to insert the remaining k 1s into the k unclaimed rows and columns. Each of these must go into a dominant square (the set of which can be determined from $\mathcal{A}$ alone by observations above). Since the set of dominant squares forms a standard board, so will the restriction of this set onto the k unclaimed rows and k unclaimed columns. Let the shape of this board, which depends only on $\mathcal{A}$, be $\lambda_{\mathcal{A}}$.

We obtain an element of ${}_{\mathcal{A}}S_n(M)$ by inserting k 1s into the board $\lambda_{\mathcal{A}}$ in such a way that a traversal results which does not contain an I_t. If an I_t were formed, it is easy to see that a submatrix isomorphic to M would then appear in the resulting traversal of the n by n board. Conversely, the only way an M might arise would be to have an I_t falling entirely in dominant squares. We conclude that $|{}_{\mathcal{A}}S_n(M)| = |S_{\lambda_{\mathcal{A}}}(I_t)|$. In exactly the same way, $|{}_{\mathcal{A}}S_n(M')| = |S_{\lambda_{\mathcal{A}}}(J_t)|$. The lemma now gives us that $|{}_{\mathcal{A}}S_n(M)| = |{}_{\mathcal{A}}S_n(M')|$, completing the proof of the theorem. $\square$

2. The proof of the lemmata

The purpose of this section is to prove 1.11 and 1.12.

Proof. [of 1.11] The lemma is a statement about traversals of boards of standard shape $\lambda = (\lambda_1, \lambda_2 \dots \lambda_k)$. By the definition of a traversal, any board having a traversal must have its width λ_1 equal to its height k. Also, if λ does not dominate the k-staircase, then λ will have no traversals at all, and the number of these satisfying any further condition will always be 0. So we can restrict our attention to boards falling between the staircase and the square:

$$(k, k-1 \dots 3, 2, 1) \leq_Y \lambda \leq_Y (k, k \dots k, k, k) \tag{2.1}$$

where $\leq_Y$ refers to the partial order of Young's lattice.

We will actually prove that $|S_\lambda(I_2)| = |S_\lambda(J_2)| = 1$ for all such shapes λ. That is, there is a unique traversal of λ which avoids I_2 and a unique traversal which avoids J_2.

The unique traversal avoiding I_2 is in fact the off-diagonal of the shape λ. Any other traversal L would locate two 1s in positions (i, j) and (i', j') with $i < i'$ and $j < j'$. But since positions (i, j') and (i', j) both fall within the shape and contain 0s, L would then contain an I_2.

Although it is most simply described as the off-diagonal traversal, the unique traversal avoiding I_2 can also be thought of as being built up row by row, starting at the bottom row k, by always locating the 1 for each row in the leftmost unoccupied column of the row. The unique J_2-avoiding traversal can likewise be built from the bottom up, always locating the 1 in the *rightmost* unoccupied column of the row. This process is well-defined and will always succeed in constructing exactly one traversal. As each 1 is entered, say in position (i, j'), we check whether it might be the upper 1 in a forbidden J_2. If it is, then there is already a 1 in position (i', j) with $i < i'$ and $j < j'$. Furthermore there is a 0 in position (i', j'), which falls within the board. But then we should have placed a 1 in (i', j') because it was an available position in the same row but to the right of (i, j'). Conversely, if we ever fail to place a 1 in the rightmost available column, then *wherever* the one falls higher up in that column it will lead to a forbidden J_2. □

The proof of lemma 1.12 will itself require a sublemma, which we call the *sliding lemma*, and state and prove first.

Lemma 2.2. *Let $\lambda = (\lambda_1, \lambda_2 \dots \lambda_k)$ be a standard board. Let i, a, b be integers $1 < i < \lambda_k$ and $0 \leq a < b \leq (\lambda_k - i)$. Let $S_\lambda^{[i;d]}(J_3)$ be the subset of $S_\lambda(J_3)$ which is J_2-free on the $(k \times i)$ strip comprised of columns $(d+1) \dots (d+i)$. Then $|S_\lambda^{[i;a]}(J_3)| = |S_\lambda^{[i;b]}(J_3)|$.*

Proof. It suffices to show that the $(k \times i)$ strip in question can be slid by *one* position, i.e. that $|S_\lambda^{[i;a]}(J_3)| = |S_\lambda^{[i;a-1]}(J_3)|$. We establish a bijection. Let L be an element of $|S_\lambda^{[i;a]}(J_3)|$. Suppose the 1s of L in the relevant columns are in positions (r_0, a) and $(r_1, a+1), (r_2, a+2), \dots (r_i, a+i)$. Since the strip of columns $(a+1)\dots(a+i)$ is J_2-free, we must have $r_1 < r_2 < \cdots < r_i$. Suppose $r_j < r_0 < r_{j+1}$. Then we claim that L' is an element of $|S_\lambda^{[i;a-1]}(J_3)|$ if L' has 1s in positions $(r_1, a), (r_2, a+1), \dots, (r_{j-1}, a+j-2), (r_0, a+j-1), (r_{j+1}, a+j), \dots, (r_i, a+i-1)$ and $(r_j, a+i)$ and outside these columns has 1s in the same positions as L. (The cases where $r_0 < r_1$ and $r_0 > r_i$ are similar.)

It is obvious by construction that L' is J_2-free in the required columns $a, \dots, (a+i-1)$. To see that L' is J_3-free, consider the various ways in which J_3 might appear as a submatrix of L'. If none of the columns $a, \dots, (a+i)$ are involved, then exactly this same submatrix appears in L, which we know to be J_3 free. If one of these columns is involved, we can replace it by the column in L having a 1 in the same row.

Finally, we remark that this is a bijection since L can be recovered from L'. □

Proof. [of 1.12] As in the proof of 1.11 we let λ be an arbitrary board satisfying inequalities 2.1.

Let $S_\lambda(I_3)$ ($S_\lambda(J_3)$) be the set of traversals of the board λ which do not contain I_3 (J_3) as a submatrix. We will count the elements of $S_\lambda(I_3)$ according to the position of the 1 in row k, which must appear in exactly one of the columns $1\dots\lambda_k$. Suppose this is j, then deletion of row k and column j will result in a traversal of the shape $\bar\lambda = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{k-1} - 1)$. Note that this shape is independent of j and fulfils the conditions of equation 2.1 for $k-1$. We will thus be performing an induction on k by transferring a certain set of equations from $\bar\lambda$ to λ. We now define the parameters which will be equated:

Let ${}_\lambda f_i^I$ (for $1 \le i \le \lambda_k$) be the number of traversals in $S_\lambda(I_3)$ which have a 1 in box (k, i).

Let ${}_\lambda f_i^J$ (for $1 \le i \le \lambda_k$) be the number of traversals in $S_\lambda(J_3)$ which have a 1 in box $(k, \lambda_k + 1 - i)$.

Let ${}_\lambda g_i^I$ (for $0 \le i \le \lambda_k$) be the number of traversals in $S_\lambda(I_3)$ which are $I_2 - free$ in the $(k \times i)$ strip comprised of columns $1\dots i$.

Let ${}_\lambda g_i^J$ (for $0 \le i \le \lambda_k$) be the number of traversals in $S_\lambda(J_3)$ which are $J_2 - free$ in the $(k \times i)$ strip comprised of columns $\lambda_k - i + 1 \dots \lambda_k$.

We note that for $\lambda = (1)$, each of $S_\lambda(I_3)$ and $S_\lambda(J_3)$ consists of a single traversal, and that ${}_{(1)}f_1^I = {}_{(1)}f_1^J = 1$ while ${}_{(1)}g_0^I = {}_{(1)}g_0^J = 1$ and ${}_{(1)}g_1^I = {}_{(1)}g_1^J = 1$. We will use this as the basis of an inductive proof that these equivalences remain valid for all shapes λ satisfying the constraints 2.1 and for all appropriate subscripts i. That is to say, for all such λ,

$$ {}_\lambda f_i^I = {}_\lambda f_i^J \quad \text{for all} \quad 1 \le i \le \lambda_k \tag{2.3} $$

$$ {}_\lambda g_i^I = {}_\lambda g_i^J \quad \text{for all} \quad 0 \le i \le \lambda_k \tag{2.4} $$

This will complete the proof of the lemma, as

$$|S_\lambda(I_3)| = \sum_{i=1}^{\lambda_k} {}_\lambda f_i^I$$

$$|S_\lambda(J_3)| = \sum_{i=1}^{\lambda_k} {}_\lambda f_i^J$$

Note that only the equivalence of the values of f^I and f^J asserted in equation 2.3 is needed to complete the proof of the lemma. The introduction of the parameters g^I and g^J and the assertion of equation 2.4 amounts to a strengthening of the induction hypothesis.

We now establish equations 2.3 and 2.4 for an arbitrary shape λ satisfying 2.1 for $k > 1$. Assume that equations 2.3 and 2.4 hold for all shapes on fewer than k rows, and so in particular for the shape $\bar{\lambda}$.

First note that

$$ {}_\lambda f_i^I = {}_{\bar{\lambda}} g_{i-1}^I \quad \text{for all} \quad 1 \le i \le \lambda_k. \tag{2.5}$$

This is so because if a 1 appears in box (k, i), the strip above and to the left of this box must be I_2-free if the whole traversal is to be I_3-free. If $\lambda_{k-1} = \lambda_k$ we can argue in the same way that

$$ {}_\lambda f_i^J = {}_{\bar{\lambda}} g_{i-1}^J \quad \text{for all} \quad 1 \le i \le \lambda_k, \tag{2.6}$$

while if $\lambda_{k-1} < \lambda_k$, we must invoke the sliding lemma 2.2 to reach the same end. The right-hand sides of equations 2.5 and 2.6 are equal by the induction hypothesis, hence the left-hand sides are equal.

Next we count ${}_\lambda g_i^I$, the number of traversals in $S_\lambda(I_3)$ which are I_2-free in the first i columns, for $0 \le i \le \lambda_k$. We count these according to the position of the 1 in row k. If the 1 is in box $(k, 1)$, then we have ${}_{\bar{\lambda}} g_{i-1}^I$ possibilities, as deleting row k and column 1 will leave a traversal of $\bar{\lambda}$ which is I_2-free in the first $i-1$ columns. The 1 cannot appear in box (k, j) for $2 \le j \le i$, as then it would combine with the 1 in column 1 to give a forbidden I_2. Finally, we count all λ-traversals in $S_\lambda(I_3)$ with a 1 in one of the positions $i+1, \ldots, \lambda_k$; as these have no I_3 they will have no forbidden I_2. Combining these statements, we have:

$$\begin{aligned}
{}_\lambda g_i^I &= {}_{\bar{\lambda}} g_{i-1}^I + \sum_{j=i+1}^{\lambda_k} {}_\lambda f_j^I \\
&= {}_{\bar{\lambda}} g_{i-1}^I + \sum_{j=i+1}^{\lambda_k} {}_{\bar{\lambda}} g_{j-1}^I \\
&= \sum_{j=i}^{\lambda_k} {}_{\bar{\lambda}} g_{j-1}^I
\end{aligned}$$

The simplification here is made using equation 2.5. To allow for $i = 0$, make the definition ${}_\lambda g_{-1}^I = 0$.

Finally, we pursue the argument of the preceding paragraph to count ${}_{\lambda}g_i^J$. Here we are interested in the number of J_3-free λ-traversals which are J_2-free in the *rightmost* i full columns. If there is a 1 in box (k, λ_k), we count ${}_{\bar\lambda}g_{i-1}^J$ possibilities. If there is a 1 in any of the rightmost i boxes in row k, it will combine with the 1 in column λ_k to give a J_2. Finally, any traversal in $S_\lambda(J_3)$ having a 1 in box (k, j) for $j \leq \lambda_k - i$ is acceptable. Hence:

$$\begin{aligned} {}_{\lambda}g_i^J &= {}_{\bar\lambda}g_{i-1}^J + \sum_{j=i+1}^{\lambda_k} {}_{\lambda}f_j^J \\ &= \sum_{j=i}^{\lambda_k} {}_{\bar\lambda}g_{j-1}^J \end{aligned}$$

Again, we apply the induction hypothesis to the right-hand sides to see the equivalence of the left-hand sides. This completes the inductive step and hence the lemma. □

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