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The convex hull of the integer points in a large ball

Imre Bárány[1,*] **David G. Larman**[2]

[1] Mathematical Institute of the Hungarian Academy of Sciences, POB 127, 1364 Budapest, Hungary (e-mail: barany@math-inst.hu)
[2] Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK (e-mail: dgl@math.ucl.ac.uk)

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1. The main result

The "integer convex hull" of rB^d, the ball of radius r centred at the origin, is, by definition

$$P_r = \text{conv}(Z^d \cap rB^d),$$

which is clearly a convex polytope. How many vertices does P_r have? Motivation for the question comes from different sources: integer programming (cf. [CHKM] [BHL]), classical enumeration problems ([J],[Sch], or more generally [W],[Vin]), and from the theory of random polytopes (see later). For the case $d = 2$ it is shown in [BB] that

$$0.33r^{2/3} \leq f_0(P_r) \leq 5.55r^{2/3} \tag{1.1}$$

where $f_k(P)$ denotes the number of k–dimensional faces of the polytope P. The limit, as $R \to \infty$, of the average of $r^{-2/3}f_0(P_r)$, on an interval $[R, R+H]$, is determined by Balog and Deshoullier [BD], and turns out to be $3.453\ldots$, (H must be large). Our main result extends (1.1) to any $d \geq 2$ and to any $f_k(P_r)$ with $k = 0, \ldots, d-1$.

Theorem 1. *For every $d \geq 2$ there are constants $c_1(d)$ and $c_2(d)$ such that for all $k \in \{0, \ldots, d-1\}$*

$$c_1(d)r^{d\frac{d-1}{d+1}} \leq f_k(P_r) \leq c_2(d)r^{d\frac{d-1}{d+1}}. \tag{1.2}$$

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Using Vinogradov's $\ll$ notation this can be written as $r^{d\frac{d-1}{d+1}} \ll f_k(P_r) \ll r^{d\frac{d-1}{d+1}}$. Here the implied constants depend only on dimension d; we will keep to this as a convention throughout the paper (unless stated otherwise).

It is the authors' conviction that lattice points and random points, in relation to convex bodies in "general position", behave similarly. Theorem 1 is another confirmation: (1.2) is in complete analogy with random polytopes. To see this, choose $n = \lceil r^d \operatorname{Vol} B^d \rceil$ random, independent, and uniform points from B^d, and let K_n denote their convex hull. Then, according to [BL] and [B], $n^{\frac{d-1}{d+1}} \ll Ef_k(K_n) \ll n^{\frac{d-1}{d+1}}$, where E stands for expectation. But $n^{\frac{d-1}{d+1}} \ll r^{d\frac{d-1}{d+1}} \ll n^{\frac{d-1}{d+1}}$, showing that the convex hull of n random points and the convex hull of the n lattice points lying in rB^d have the same number of k–dimensional faces.

2. The upper bound

The upper bounds in (1.2) follows from a result of Andrews [An] who proved the case $k = 0$ of the following more general

Theorem 2. *Assume $P \subset R^d$ is a lattice polytope with nonempty interior. Then*

$$f_k(P) \ll (\operatorname{Vol} P)^{\frac{d-1}{d+1}}, \tag{2.1}$$

where the implied constant depends only on d.

The result was rediscovered by Arnol'd [Ar] (case $d = 2$), Konyagin and Sevastyanov [KS], case $d \geq 2$, $k = 0$ with indication to any k. W. Schmidt [Sch] proved (2.1) in slightly stronger form. A more general argument of Bárány and Vershik [BV] implies the case $d \geq 2$, $k = 0$. Here we give yet another proof, based on convex geometry and the technique of cap coverings. What we get is a slight improvement over (2.1), which is also indicated in [KS]. A *tower* (or *flag*) of the polytope P is a chain of incident faces $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ with $\dim F_i = i$. Write $T(P)$ for the number of towers of P.

Theorem 3. *Under the previous assumptions*

$$T(P) \ll (\operatorname{Vol} P)^{\frac{d-1}{d+1}}. \tag{2.2}$$

As clearly $f_k(P) \leq T(P)$, (2.2) indeed generalizes (2.1). The proof, however, starts with the case $k = d - 1$ of (2.1) and uses, twice, a trick of Andrews later.

3. Lower bounds and approximation

W. Schmidt [Sch] asked whether the exponent $\frac{d-1}{d+1}$ in (2.1) is best possible (when $d > 2$). In the case $d = 2$ this is clear from [Ar] and [Sch], Arnol'd also indicates the general case. The lower bounds of Theorem 1 show that the exponent in (2.1), and also in (2.2), is best possible. An argument of the first named author (given in [BD]) proves that the average of $f_0(P_r)$, over $r \in [R, R+H]$ is of order $R^{d\frac{d-1}{d+1}}$. This is a weaker, or average, version of the case $k = 0$ of Theorem 1.

The proof of the lower bounds in Theorem 1 is based on a result from the theory of approximation of (smooth) convex bodies by polytopes. To state what we need, write $\mathscr{C}(D)$ for the collection of convex bodies with $\mathscr{C}^2$ boundary and radius of curvature at every point and every direction between $1/D$ and D. (Here $D \geq 1$.) Let $K \in \mathscr{C}(D)$ and assume $P \subset K$ is a convex polytope. Approximation of K by P is measured as the "relative" missed volume, i.e.,

$$\text{appr}(K, P) = \frac{\text{Vol}(K \setminus P)}{\text{Vol}\, K}.$$

The result we need (cf. [G1]) says that for any $K \in \mathscr{C}(D)$ and for any polytope $P \subset K$ having n vertices

$$\text{appr}(K, P) \gg n^{-\frac{2}{d-1}}. \tag{3.1}$$

On the other hand, there is a polytope $P \subset K$ with n vertices satisfying

$$\text{appr}(K, P) \ll n^{-\frac{2}{d-1}}. \tag{3.2}$$

Here $\gg$ and $\ll$ depend on D as well. More precise asymptotic information is available on best approximation (cf. [G2]): the constant is const(d) times the $\frac{d+1}{d-1}$ power of the affine surface area of K. But we won't need this precision.

The proof of the lower bounds is based on

Theorem 4. *For every $d \geq 2$*

$$\text{Vol}(rB^d \setminus P_r) \ll r^{d\frac{d-1}{d+1}}.$$

This implies the case $k = 0$ of Theorem 1: Assume $f_0(P_r) = n$. By (3.1) and Theorem 4

$$n^{-\frac{2}{d-1}} \ll \frac{\text{Vol}(rB^d \setminus P_r)}{\text{Vol}\, rB^d} \ll r^{d\frac{d-1}{d+1}-d} = r^{-\frac{2d}{d+1}}$$

showing that $f_0(P_r) = n \gg r^{d\frac{d-1}{d+1}}$ indeed. On the other hand, $f_0(P_r) \ll r^{d\frac{d-1}{d+1}}$ from Theorem 1 which together with (3.1) imply that

$$r^{-\frac{2d}{d+1}} \ll f_0(P_r)^{-\frac{2}{d-1}} \ll \text{appr}(rB^d, P_r),$$

i.e., P_r is a "best" aproximating polytope to rB^d in the sense of (3.2). So we have

Corollary .

$$f_0(P_r)^{-\frac{2}{d-1}} \ll \text{appr}(rB^d, P_r) \ll f_0(P_r)^{-\frac{2}{d-1}}.$$

A long time ago, C. A. Rogers [R] proved the following analogue of (3.1). For every polytope $P \subset B^d$ with n facets

$$\text{appr}(B^d, P) \gg n^{-\frac{2}{d-1}}. \tag{3.3}$$

From this the case $k = d-1$ of Theorem 1 (the lower bound) follows the same way as above. Cases $k = 1, \dots, d-2$ of Theorem 1 need special, and more involved treatment. The proof would be simpler if, for every convex polytope P, one would have

$$f_k(P) \geq \min\{f_0(P), f_{d-1}(P)\}. \tag{3.4}$$

This would follow from the unimodality conjecture (see [Z]), which is known to be false. But (3.4) may still be true. It is known to hold for simple (and then simplicial) polytopes, see Björner [Bj].

4. Replacing B^d by K

In this section we assume

$$K \in \mathscr{C}(D) \text{ and } 0 \in \text{int}\, K. \tag{4.1}$$

Let P_λ be the integer convex hull of λK, i.e.,

$$P_\lambda = P_\lambda(K) = \text{conv}(Z^d \cap \lambda K).$$

Here λ is large (we keep the letter r for radius of curvature). The questions, and the answers, of the previous sections extend to this case, with the constants implied in $\ll$ depending on d and D:

Theorem 5. *Assume K satisfies (4.1). Then, as $\lambda \to \infty$,*

$$\lambda^{d\frac{d-1}{d+1}} \ll f_k(P_\lambda(K)) \ll \lambda^{d\frac{d-1}{d+1}}. \tag{4.2}$$

We will indicate, after the proofs for B^d, how the extension goes.

The generalization of Rogers' result (3.3) to this case has to be stated and proved separately:

Theorem 6. *Assume K satisfies (4.1) and $P \subset K$ is a polytope with n facets. Then*

$$\text{appr}(K, P) \gg n^{-\frac{2}{d-1}}$$

with the implied constant depending only on d, D.

Again, the proof of the lower bound in Theorem 1 for $k = 1, \dots, d-2$ would be simpler if the following unusual approximation statement were true.

Conjecture. *Assume K satisfies (4.1), $k \in \{1, \dots d-2\}$ and $P \subset K$ is a polytope with $f_k(P) = n$. Then*

$$\text{appr}(K, P) \gg n^{-\frac{2}{d-1}}.$$

5. Proof of Theorem 4

We start by introducing notation and terminology. Let $p \in Z^d$ be a primitive vector, outward normal to the facet $F(p)$ of P_r. The hyperplane $H(p) = \operatorname{aff} F(p)$ cuts off a small cap $C(p)$ from rB^d and

$$Z^d \cap \operatorname{int} C(p) = \emptyset. \tag{5.1}$$

Let $\rho = \rho(p)$ be the radius of the $(d-1)$–ball $H(p) \cap rB^d$ and let $h = h(p)$ be the width, in direction p, of the cap C. Then

$$\rho^2 = (2r - h)h \text{ and so } rh \ll \rho^2 \ll rh. \tag{5.2}$$

Write $|x|$ for the Euclidean length of $x \in R^d$. Letting Area to denote $(d-1)$–dimensional volume, we have

$$\operatorname{Area} F(p) = \ell(p)|p| \ll \rho^{d-1} \tag{5.3}$$

where $\ell(p) > 0$. $|p|$ is, in fact, the determinant of the lattice $Z^d \cap H(p)$. So

$$\ell(p) \in \frac{1}{(d-1)!} Z^d.$$

Lemma 1. *The contribution to* $\operatorname{Vol}(rB^d \setminus P_r)$ *of the caps* $C(p)$ *with* $h(p) \leq r^{-\frac{d-1}{d+1}}$ *is* $\ll r^{d\frac{d-1}{d+1}}$.

Proof. Everything that is contained in such a $C(p)$ is also contained in

$$rB^d \setminus (r - r^{-\frac{d-1}{d+1}})B^d$$

whose volume is just $\left(r^d - (r - r^{-\frac{d-1}{d+1}})^d\right) \operatorname{Vol} B^d \ll r^{d\frac{d-1}{d+1}}$. □

From now on we can only consider facets $F(p)$ with

$$h(p) \geq r^{-\frac{d-1}{d+1}}. \tag{5.4}$$

We are going to use the Flatness Theorem (cf. [K], [KL]) saying that the lattice width of a lattice point free convex body (in R^d) is at most $c_0 d^2$ where c_0 is a universal constant. Applying this to $C(p)$, or rather to $\operatorname{int} C(p)$ which is lattice point free by (5.1), we get a primitive vector $q \in Z^d$ such that

$$\max\{q(x-y)\, x, y \in C(p)\} \leq c_0 d^2. \tag{5.5}$$

Case 1: when $h(p) \leq c_0 d^2 |p|^{-1}$. In this case p is a flatness direction for $C(p)$ (since consecutive lattice hyperplanes with normal p are at distance $|p|^{-1}$ apart). Then $\rho^2 \ll rh \ll r|p|^{-1}$ and

$$\operatorname{Area} F(p) = \ell(p)|p| \ll \rho^{d-1} \ll (r|p|^{-1})^{\frac{d-1}{2}},$$

implying

$$\ell(p) \ll r^{\frac{d-1}{2}} |p|^{-\frac{d+1}{2}}.$$

As $\ell(p) \geq \frac{1}{(d-1)!}$ we get $|p| \ll r^{\frac{d-1}{d+1}}$. We write $b = b(d)$ for the implied constant. The lost volume in Case 1 is

$$\ll \sum_p \operatorname{Area} F(p) h(p) \ll \sum_p \ell(p) \ll \sum_{|p| \leq br^{\frac{d-1}{d+1}}} r^{\frac{d-1}{2}} |p|^{-\frac{d+1}{2}}$$

$$(5.6) \qquad \ll r^{\frac{d-1}{2}} \int_0^{br^{\frac{d-1}{d+1}}} x^{-\frac{d+1}{2}} x^{d-1} dx \ll r^{d\frac{d-1}{d+1}},$$

as a simple computation reveals.

Case 2: when $h(p) > c_0 d^2 |p|^{-1}$. Then some $q \in Z^d$, distinct from p, is the flatness direction of $C(p)$.

Assume $C(p)$ is between hyperplanes $qx = \ell_1$ and $qx = \ell_2$ with $0 < \ell_1 < \ell_2 \leq |q|r$ and $\ell_2 - \ell_1 \leq c_0 d^2$. Set $k_i = |q|r - \ell_i$ and $x_i = k_i/|q|$, $(i = 1, 2)$. Consider the two–dimensional plane containing $0, q$, and the centre of $C(p)$. We show first, assuming $x_2 > 0$, that ϕ (see the figure) gets small as r gets large. Indeed, using (5.4)

$$\sin \phi = \frac{x_1 - x_2}{2\rho} = \frac{x_1 - x_2}{2\sqrt{(2r - h)h}} \leq \frac{k_1 - k_2}{2|q|\sqrt{rh}} \leq \frac{c_0 d^2}{2|q|\sqrt{r \cdot r^{-\frac{d-1}{d+1}}}} \ll r^{-\frac{1}{d+1}}$$

since $|q| \geq 1$.

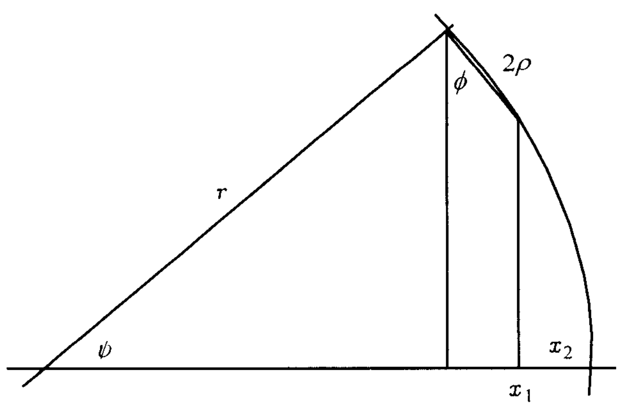

Fig. 1.

As ϕ and ψ (see the figure) are almost equal, (5.6) implies

$$(5.7) \qquad x_1 = r(1 - \cos \psi) \leq r \sin^2 \phi \ll r^{\frac{d-1}{d+1}}.$$

We can estimate ρ from the figure, again. As $\cos \phi > 1/2$ for large enough r, we get

$$\rho < \sqrt{(2r-x_1)x_1} - \sqrt{(2r-x_2)x_2} = \frac{(2r-x_1)x_1 - (2r-x_2)x_2}{\sqrt{(2r-x_1)x_1} + \sqrt{(2r-x_2)x_2}}$$

$$(5.8) \quad \leq \frac{(2r-x_1-x_2)(x_1-x_2)}{\sqrt{r}(\sqrt{x_1}+\sqrt{x_2})} \leq 2\sqrt{r}\frac{k_1-k_2}{|q|}\frac{\sqrt{|q|}}{\sqrt{k_1}+\sqrt{k_2}} \ll \sqrt{\frac{r}{|q|k_1}}.$$

The same estimate follows directly when $x_2 = 0$. From this $h \ll \rho^2 r^{-1} \ll (|q|k_1)^{-1}$. Now (5.4) shows $k_1|q| \ll r^{\frac{d-1}{d+1}}$. Set now $k = \lceil k_1 \rceil$. As p is not a flatness direction, $1 \leq k_1 - k_2 \leq k_1$. So $k \geq 1$ and

$$k|q| \ll r^{\frac{d-1}{d+1}}.$$

Collect the $F(p)$ with fixed flatness direction q and fixed k into groups. The missed volume in the corresponding caps is

$$(5.9) \qquad \ll \sum \operatorname{Area} F(p)h(p) \leq S \max h(p)$$

where S is the surface area of rB^d between hyperplanes $qx = \ell_1$ and $qx = \ell_2$. Since ϕ is small,

$$S \leq 2\left([(2r-x_1)x_1]^{\frac{d-1}{2}} - [(2r-x_2)x_2]^{\frac{d-1}{2}}\right) \operatorname{Area} B^{d-1}$$

$$\ll (\sqrt{(2r-x_1)x_1} - \sqrt{(2r-x_2)x_2})[(2r-x_1)x_1]^{\frac{d-2}{2}} \ll \sqrt{\frac{r}{|q|k}}\left(\frac{rk}{|q|}\right)^{\frac{d-2}{2}}.$$

where we used the second half of (5.8). Evidently $\max h(p) \leq \rho^2/r \ll (|q|k)^{-1}$. We continue (5.9):

$$\ll \frac{1}{|q|k}\sqrt{\frac{r}{|q|k}}\left(\frac{rk}{|q|}\right)^{\frac{d-2}{2}} = r^{\frac{d-1}{2}}|q|^{-\frac{d+1}{2}}k^{\frac{d-5}{2}}.$$

This is to be summed for all $k = 1, 2, \ldots$ and $q \in Z^d$ primitive with $k|q| \leq R$ where $R \ll r^{\frac{d-1}{d+1}}$. Then the total missed volume is

$$\ll r^{\frac{d-1}{2}} \sum_{k=1}^{R} \sum_{q \in Z^d}^{\frac{R}{k}} |q|^{-\frac{d+1}{2}} k^{\frac{d-5}{2}} \ll r^{\frac{d-1}{2}} \sum_{k=1}^{R} \int_{x \in R^d,\ |x| \leq \frac{R}{k}} |x|^{-\frac{d+1}{2}} k^{\frac{d-5}{2}} dx$$

$$\ll r^{\frac{d-1}{2}} \sum_{k=1}^{R} k^{\frac{d-5}{2}} \int_0^{\frac{R}{k}} t^{d-1} t^{-\frac{d+1}{2}} dt \ll r^{\frac{d-1}{2}} \sum_{k=1}^{R} k^{\frac{d-5}{2}} \left(\frac{R}{k}\right)^{\frac{d-1}{2}}$$

$$(5.10) \quad = r^{\frac{d-1}{2}} R^{\frac{d-1}{2}} \sum_{k=1}^{R} k^{-2} \ll (rR)^{\frac{d-1}{2}} \ll r^{d\frac{d-1}{d+1}},$$

as one can check easily. □

Remark 1. This proof shows the inequality $f_0(P_r) \ll r^{d\frac{d-1}{d+1}}$ (from Theorem 1) directly. Actually, it shows the stronger result that

$$|\partial P_r \cap Z^d| \ll r^{d\frac{d-1}{d+1}}.$$

To see this one has to use the simple fact

$$|F(p) \cap Z^d| \ll \frac{\operatorname{Area} F(p)}{|p|}$$

valid for every facet $F(p)$ of P_r. This gives, in Case 1,

$$\sum_p |F(p) \cap Z^d| \ll \sum_p \frac{\operatorname{Area} F(p)}{|p|} \ll \sum_p \frac{\rho(p)^{d-1}}{|p|} \ll \sum_p r^{\frac{d-1}{2}} |p|^{-\frac{d+1}{2}},$$

which is $\ll r^{d\frac{d-1}{d+1}}$, according to (5.6). Case 2 is even simpler. Then

$$|F(p) \cap Z^d| \ll \frac{\operatorname{Area} F(p)}{|p|} \ll \operatorname{Area} F(p) h(p) \ll \operatorname{Vol} C(p)$$

and (5.9), (5.10) can be applied.

Remark 2. An essentially identical proof works when B^d is replaced by K satisfying (4.1). The main difference is that $H(p) \cap \lambda K$ is not a ball. But it is very close to an ellipsoid (since $h(p)$ is very small, less than $\lambda^{-\frac{d-1}{d+1}}$: this is shown by Lemma 1). This ellipsoid is sandwiched between two concentric balls of radii $\sqrt{\frac{\lambda h}{D}}$ and $\sqrt{2\lambda h D}$. This shows that the corresponding ρ and $\operatorname{Area} F(p)$ can be bounded as in (5.2) and (5.3) with the implied constants depending on D as well.

We elaborate on how to deal with ϕ and ψ on the figure. Let $y \in \partial K$ be the point where the outer normal to K is q. Then the figure shows the intersection of P_λ with the two–plane H parallel with q, containing the centre of $C(p)$ and the point λy. Write r for the radius of curvature at λy of $H \cap \lambda K$. Clearly, r/λ is between $1/D$ and D. The boundary of $H \cap \lambda K$, in a neighbourhood of λy is very close to the circle of radius r with centre $\lambda y - rq/|q|$. Now ϕ and ψ are the same as on the figure and the estimation of $\sin\phi$ and x_1 works the same way. (h on the figure may be different from the depth of the cap $C(p)$ but their ratio is bounded as a function of D.)

6. Auxiliary results

Let K be a convex body in R^d. For $x \in K$ and $\lambda > 0$ define

$$M_K(x, \lambda) = x + \lambda\{(K - x) \cap (x - K)\}.$$

This is the M–region introduced by Macbeath [M] in 1953. We define two functions $u, v\ K \to R$ by

$$(6.1) \qquad u(x) = u_K(x) = \operatorname{Vol} M_K(x, 1)$$

$$(6.2) \qquad v(x) = v_K(x) = \min\{\operatorname{Vol}(K \cap H)\ x \in H,\ H \text{ is a halfspace}\}.$$

The set $K(v \geq t) = \{x \in K\ v(x) \geq t\}$ is evidently convex. So is $K(u \geq t)$ (see [M]) but we will not need this. It follows from the existence of the Löwner–John ellipsoid that $K(v \geq t)$ is nonempty when $t < \frac{1}{2d!} \operatorname{Vol} K$.

Several properties of these functions, their level sets, and of the M–regions are established in [M], [ELR], [BL], [B]. We list those that will be needed later.

Lemma A. *([ELR]) If $M(x, 1/2) \cap M(y, 1/2) \neq \emptyset$, then $M(x, 1) \subset M(y, 5)$.*

Lemma B. *(simple) $u(x) \leq 2v(x)$.*

Lemma C. *([BL]) If $v(x) \leq (2d)^{-2d} \operatorname{Vol} K$, then $v(x) \leq (3d)^d u(x)$.*

Lemma D. *([B]) $K(v \geq t)$ contains no line segment on its boundary (provided $t > 0$).*

Lemma E. *([ELR],[B]) Let C be a cap, i.e., $C = K \cap H$ with some halfspace H. If $\varepsilon < (2d)^{-2d}$ and $C \cap K(v \geq \varepsilon \operatorname{Vol} K)$ is a single point x, then $C \subset M(x, 3d)$ and $\varepsilon \operatorname{Vol} K \leq \operatorname{Vol} C \leq d\varepsilon \operatorname{Vol} K$.*

Lemma F. *([BL]) For every convex body $K \subset R^d$*

$$\operatorname{Vol} K(v \leq \varepsilon \operatorname{Vol} K) \ll \varepsilon^{\frac{2}{d+1}} \operatorname{Vol} K$$

with the implied constant depending only on d.

When $K \in \mathscr{C}(D)$ and x is close to the boundary of K, $u(x), v(x)$ are easy to estimate. For instance, as we saw it in Remark 2, the boundary of K is very close to an ellipsoid E in the vicinity of x, and for ellipsoids $u_E(x)$ and $v_E(x)$ are simple to determine, and $u_E(x) = 2v_E(x)$. It follows that, writing $h = h(x)$ for the width of the cap $K \cap H$ giving the minimum in (6.2)

$$h^{\frac{d+1}{2}} \ll u_K(x) \ll v_K(x) \ll h^{\frac{d+1}{2}} \tag{6.3}$$

with the implied constants depending only on d, D.

7. Proof of Theorems 2 and 3

Set $\operatorname{Vol} P = V$ and define, with clear anticipation, $\varepsilon = [3(15d)^d d! V]^{-1}$. Let F be a facet of P (with outer normal p). Let x_F be the point on the boundary of $P(v \geq \varepsilon V)$ where the outer normal coincides with p. According to Lemma D, x_F is unique. Let $C(x_F) = P \cap \{x\ p(x - x_F) \geq 0\}$.

Claim. For distinct facets F and G of P

$$M(x_F, 1/2) \cap M(x_G, 1/2) = \emptyset.$$

Proof. According to Lemma E

$$\varepsilon V \leq \operatorname{Vol} C(x_F) \leq d\varepsilon V \text{ and } C(x_F) \subset M(x_F, 3d).$$

Assume $M(x_F, 1/2) \cap M(x_G, 1/2) \neq \emptyset$. Lemma A shows then, that $M(x_F, 1) \subset M(x_G, 5)$, and so

$$F \subset C(x_F) \subset M(x_F, 3d) \subset M(x_G, 15d).$$

Since $G \subset C(x_G) \subset M(x_G, 3d) \subset M(x_G, 15d)$ as well, $M(x_G, 15d)$ contains $d+1$ affinely independent lattice points: d from G and at least one more form F. The volume of their convex hull is at least $1/d!$. Thus by Lemma B

$$\frac{1}{d!} \leq \operatorname{Vol} M(x_G, 15d) \leq (15d)^d u(x_G) \leq (15d)^d \cdot 2\varepsilon V = \frac{2}{3d!},$$

a contradiction. □

So the M–regions $M(x_F, 1/2)$ are pairwise disjoint. $P(v \leq \varepsilon V)$ contains half of each: the half cut off by the halfspace $p(x - x_F) \geq 0$. Then by Lemma F (which is a version of the affine isoperimetric inequality)

$$\sum_F \frac{1}{2} \operatorname{Vol} M(x_F, \frac{1}{2}) \leq \operatorname{Vol} P(v \leq \varepsilon V) \ll \varepsilon^{\frac{2}{d+1}} V \ll V^{\frac{d-1}{d+1}}.$$

On the other hand, by Lemma C

$$\operatorname{Vol} M(x_F, 1/2) = 2^{-d} u(x_F) \geq 2^{-d}(3d)^{-d} v(x_F) \geq (6d)^{-d} \varepsilon V \gg 1.$$

This clearly implies

$$f_{d-1}(P) \ll V^{\frac{d-1}{d+1}} = (\operatorname{Vol} P)^{\frac{d-1}{d+1}}.$$

From this we show, using an idea of Andrews, that $f_0(P) \ll (\operatorname{Vol} P)^{\frac{d-1}{d+1}}$. Let z be a vertex of P with neighbouring vertices $w_1, \dots, w_n$. Define

$$P_z = \operatorname{conv}\{\cup_1^n \{\frac{2}{3} z + \frac{1}{3} w_i + \lambda(w_i - z) : \lambda \geq 0\}\}.$$

As $z \notin P_z$, there is a facet F_z of P_z separating them. This facet is of the form $\operatorname{conv}\{\frac{2}{3}z + \frac{1}{3}w_i : \text{ some } i\}$. Set $Q = \cap P_z$ for all vertices z of P. Then F_z is a facet of Q as well and $F_z \neq F_y$ for distinct z, y. Q is a lattice polytope in $\frac{1}{3}Z^d$ so

$$f_0(P) \leq f_{d-1}(Q) \ll (\operatorname{Vol} Q)^{\frac{d-1}{d+1}} \ll (\operatorname{Vol} P)^{\frac{d-1}{d+1}}.$$

We are now in a position to prove Theorem 3.

Proof of Theorem 3. We are going to define a polytope $Q \subset P$ which is a lattice polytope in $\frac{1}{s(d)}Z^d$ (where $s(d)$ depends only on d), and a map f from the towers of P to the vertices of Q that maps distinct towers to distinct vertices. This will show

$$T(P) \leq f_0(Q) \ll (s^d \operatorname{Vol} Q)^{\frac{d-1}{d+1}} \ll (\operatorname{Vol} P)^{\frac{d-1}{d+1}}.$$

The proof is by induction and we start with $d = 2$. The vertices of P are $z_1, \dots, z_n$ in this order. The vertices of Q will be

$$\frac{2}{3}z_i + \frac{1}{3}z_{i+1}, \text{ and } \frac{1}{3}z_i + \frac{2}{3}z_{i+1} \text{ for } i = 1, \dots, n.$$

The towers of P are $z_i, \{z_i, z_{i+1}\}$ and $z_{i+1}, \{z_i, z_{i+1}\}$. Define

$$f(z_i, \{z_i, z_{i+1}\}) = \frac{2}{3}z_i + \frac{1}{3}z_{i+1} \text{ and } f(z_{i+1}, \{z_i, z_{i+1}\}) = \frac{1}{3}z_i + \frac{2}{3}z_{i+1}.$$

This is evidently fine; we get $s(2) = 3$.

Now for $d \geq 3$. For every facet F of P the inductional hypothesis guarantees the existence of a lattice polytope $Q^F \subset F$ (in the lattice $\frac{1}{s(d-1)}Z^d \cap \operatorname{aff} F$) and a mapping

$$f^F \ \{\text{towers of } F\} \to \{\text{vertices of } Q^F\}.$$

Make sure, by contracting Q^F suitably if necessary, that $Q^F \cap Q^G = \emptyset$ for distinct facets F, G. It is not hard to see that one can take, as centre of contraction, a point from $\frac{1}{ds(d-1)}Z^d \cap \operatorname{conv} F$. Contraction by the factor $1/2$ suffices so Q^F is a lattice polytope in the lattice $\frac{1}{2ds(d-1)}Z^d \cap \operatorname{aff} F$. Set

$$Q = \operatorname{conv}(\cup_F Q^F),$$

Q is a $\frac{1}{s(d)}Z^d$–lattice polytope (with $s(d) = 2ds(d-1)$), contained in P. To define f let $T_0 \subset T_1 \subset \cdots \subset T_{d-1}$ be a tower of P. Then $T_{d-1} = F$ for some facet F. Define

$$f(T_0, \dots, T_{d-1}) = f^F(T_0, \dots, T_{d-2}) \in \operatorname{vert} Q^F \subset \operatorname{vert} Q. \quad \square$$

8. Proof of Theorem 6

In this section the implied constants depend on d and D as well. We assume $\operatorname{Vol} K = 1$. Then $\operatorname{Area} \partial K \gg 1$.

Let F be a facet of P and denote by x_F the point where the function v_K is maximal on $\operatorname{aff} F$. Note that x_F need not be contained in F. But the cap $C(x_F)$ cut off from K by $\operatorname{aff} F$ must have small ($\ll n^{-\frac{2}{d+1}}$) volume as otherwise there is nothing to prove. Write h_F for the *depth* of the facet F in K; this is the same as the width of the cap $C(x_F)$. As $K \in \mathscr{C}(D)$ and h_F is small, (6.3) applies yielding

(8.1) $$h_F^{\frac{d+1}{2}} \ll u(x_F) \ll v(x_F) \ll h_F^{\frac{d+1}{2}}.$$

Similarly,

(8.2) $$h_F^{\frac{d-1}{2}} \ll \operatorname{Area}(K \cap \operatorname{aff} F) \ll \operatorname{Area}(M(x_F, 1) \cap \operatorname{aff} F) \ll h_F^{\frac{d-1}{2}}.$$

Choose a system $y_1, \dots, y_m \in \{x_F \ F \text{ a facet}\}$, maximal with respect to the condition that for distinct i, j

$$M(y_i, 1/2) \cap M(y_j, 1/2) = \emptyset.$$

Half of each $M(y_i, 1/2)$ is contained in $K \setminus P$. So with (8.1) we get

$$(8.3) \qquad \sum_1^m h_i^{\frac{d+1}{2}} \ll \sum_1^m \frac{1}{2} \operatorname{Vol} M(y_i, \frac{1}{2}) \le \operatorname{Vol}(K \setminus P).$$

On the other hand, by Lemma A, for every facet F of P there is an i such that $M(x_F, 1) \subset M(y_i, 5)$. In this case the outer unit normals to the facets F and $F(y_i)$ cannot differ much. Then S_i, the total $(d-1)$–volume of the projections of all such facets F onto $\operatorname{aff} F(y_i)$ is essentially equal to the $(d-1)$–volume of these facets. So we get, using (8.2) as well,

(8.4)

$$\operatorname{Area} \partial P = \sum_F \operatorname{Area} F \ll \sum_1^m S_i \le \sum_1^m \operatorname{Area}[\operatorname{aff} F(y_i) \cap M(y_i, 5)] \ll \sum_1^m h_i^{\frac{d-1}{2}}.$$

Of course, $\operatorname{Area} \partial P \gg 1$. We combine (8.3), (8.4), and the inequality between the $\frac{d-1}{2}$ and $\frac{d+1}{2}$ means:

$$(8.5) \quad \left(\frac{1}{m}\right)^{\frac{2}{d-1}} \ll \left(\frac{\sum h_i^{\frac{d-1}{2}}}{m}\right)^{\frac{2}{d-1}} \le \left(\frac{\sum h_i^{\frac{d+1}{2}}}{m}\right)^{\frac{2}{d+1}} \ll \left(\frac{\operatorname{Vol}(K \setminus P)}{m}\right)^{\frac{2}{d+1}}.$$

This gives

$$\operatorname{appr}(K, P) = \frac{\operatorname{Vol}(K \setminus P)}{\operatorname{Vol} K} \gg m^{1-\frac{d+1}{d-1}} = m^{-\frac{2}{d-1}} \ge n^{-\frac{2}{d-1}},$$

since $n \ge m$. □

Remark 3. The proof works even if the maximal system $y_1, \dots, y_m$ is chosen from a subset of the facets, if the total $(d-1)$–volume of these facets is $\gg 1$. This observation will be used in the next section.

9. Lower bounds for $k = 1, \dots, d-2$

We show first that most of the surface area of P_r comes from facets whose depth h is between $b_1 r^{-\frac{d-1}{d+1}}$ and $b_2 r^{-\frac{d-1}{d+1}}$ where $b_1 < 1$ is small, $1 < b_2$ is large.

Lemma 2. *The contribution to the surface area of P_r of the facets with $h \le b_1 r^{-\frac{d-1}{d+1}}$ is $\ll b_1^{\frac{d-1}{2}} r^{d-1}$.*

Proof. The surface area of $F(p)$ with $h = h(p) \le b_1 r^{-\frac{d-1}{d+1}}$ is at most

$$\rho^{d-1} \operatorname{Area} B^{d-1} \ll (rh)^{\frac{d-1}{2}} \ll b_1^{\frac{d-1}{2}} r^{\frac{d-1}{d+1}}.$$

The total number of facets is $\ll r^{d\frac{d-1}{d+1}}$, so the surface area in question is indeed

$$\ll b_1^{\frac{d-1}{2}} r^{\frac{d-1}{d+1}} r^{d\frac{d-1}{d+1}} = b_1^{\frac{d-1}{2}} r^{d-1}. \qquad \square$$

Lemma 3. *The contribution to the surface area of P_r of the facets with $h \geq b_2 r^{-\frac{d-1}{d+1}}$ is $\ll b_2^{-1} r^{d-1}$*

Proof. Define $D(p)$ as the set of points $x \in rB^d$ such that the segment $[0,x]$ intersects the facet $F(p)$. Clearly, the $D(p)$ are pairwise internally disjoint and their union is $rB^d \setminus P_r$. Let $y \in F(p)$ be the point closest to x_p, the centre of the cap $C(p)$. Let $m(p)$ denote the length of the longest segment parallel with p that is contained in $D(p)$. Clearly, this segment starts at y.

Claim. $m(p) \gg h(p)$

The claim implies the Lemma as follows. The halfline starting at the origin and containing y intersects the boundary of rB^d at y'. So $\mathrm{conv}(F(p) \cup \{y'\}) \subset D(p)$ and its volume equals $\frac{1}{d}$ Area $F(p)$ times the p–component of the vector $y' - y$. The latter is at least $\frac{1}{2}m(p)$ since p is almost parallel with $y' - y$. So, using Theorem 4,

$$\begin{aligned} r^{d\frac{d-1}{d+1}} &\gg \mathrm{Vol}(rB^d \setminus P_r) \geq \sum_{\text{all } p} \mathrm{Vol}\, D(p) \\ &\geq \sum_{\text{all } p} \frac{1}{2d} m(p)\, \mathrm{Area}\, F(p) \gg \sum_{h(p) \geq b_2 r^{-\frac{d-1}{d+1}}} h(p)\, \mathrm{Area}\, F(p) \\ &\gg b_2 r^{-\frac{d-1}{d+1}} \sum_{h(p) \geq b_2 r^{-\frac{d-1}{d+1}}} \mathrm{Area}\, F(p), \end{aligned}$$

which proves the Lemma.

Now for the claim. Set $\rho = \rho(p)$, $m = m(p)$, etc, and $\rho_1 = |y - x_p|$. If $\rho_1 \leq \rho\sqrt{1 - \frac{1}{d-1}}$, then

$$m \geq \frac{\rho - \rho_1}{\rho} h \geq \left(1 - \sqrt{1 - \frac{1}{d-1}}\right) h \geq \frac{1}{2(d-1)} h.$$

and we are done. So suppose $\rho_1 > \rho\sqrt{1 - \frac{1}{d-1}}$.

Write B_0 for the $(d-1)$–ball $rB^d \cap \mathrm{aff}\, F(p)$. Let C denote the $(d-1)$–cap cut off from B_0 by the hyperplane orthogonal to $y - x_p$ and passing through y. The diameter of C is $2\sqrt{\rho^2 - \rho_1^2} < \frac{2}{\sqrt{d-1}}\rho$. C contains $F(p)$ and so it contains d affinely independent vectors $v_1, \ldots, v_d \in Z^d$. The hyperplane $\mathrm{aff}\, F(p)$ is then covered by lattice translates of the parallelotope spanned by $v_2 - v_1, \ldots, v_d - v_1$ and x_p is contained in one of the translates. As it is well–known, this translate has a vertex at distance at most $\frac{1}{2}\sqrt{d-1} \max |v_i - v_1| \leq \frac{1}{2}\sqrt{d-1}\, \mathrm{diam}\, C < \rho$ from x_p. So this vertex is in B_0 and consequently in $F(p)$. Then it cannot be closer to x_p than ρ_1, the shortest distance between x_p and $F(p)$:

$$\rho_1 \leq \frac{1}{2}\sqrt{d-1} \max |v_i - v_1| \leq \frac{1}{2}\sqrt{d-1}\, \mathrm{diam}\, C = \sqrt{d-1}\sqrt{\rho^2 - \rho_1^2}.$$

This shows $\rho_1 \le \rho\sqrt{1-\frac{1}{d}}$ and the previous argument applies again:

$$m \ge \frac{\rho-\rho_1}{\rho}h \ge \left(1-\sqrt{1-\frac{1}{d}}\right)h \ge \frac{1}{2d}h.$$

□

Now choose a small $b_1 = b_1(d)$ and a large $b_2 = b_2(d)$ so that half of the surface area of P_r comes from facets $F(p)$ satisfying

$$b_1 r^{-\frac{d-1}{d+1}} \le h(p) \le b_2 r^{-\frac{d-1}{d+1}}.$$

Write $\mathcal{F}$ for the collection of these facets. We apply the proof method of Theorem 6, this time with rB^d instead of K. So choose a system $F_1, \dots, F_m$ of facets (from $\mathcal{F}$) maximal with respect to the condition that

$$M(y_i, 1/2) \cap M(y_j, 1/2) = \emptyset,$$

where y_i is the point where v is maximal on aff F_i. The previous proof, combined with Remark 3, gives

$$m \gg r^{d\frac{d-1}{d+1}}.$$

Now define

$$\mathcal{F}_j = \{F_i \in \mathcal{F} : 2^j r^{-\frac{d-1}{d+1}} \le h_i < 2^{j+1} r^{-\frac{d-1}{d+1}}\}.$$

Clearly $\log b_1 \le j \le \log b_2$ implying the existence of a j such that

$$\mathcal{F}_j \ge \left(\log\frac{b_1}{b_2}\right)^{-1} m \gg r^{d\frac{d-1}{d+1}}.$$

Fix such a j.

Now let L be a k–face of P_r and fix a point $x_L \in L$. If $L \subset F_i$ for some $F_i \in \mathcal{F}_j$, then the cap $C(y_i)$ lies in a ball with centre x_L and radius $2^{\frac{j}{2}+2} r^{\frac{1}{d+1}}$. Indeed, as $x_L \in L \subset F_i \subset C(y_i)$, the distance between x_L and y_i is at most ρ_i. The diameter of $C(y_i)$ is

$$2\rho_i = 2\sqrt{(2r-h_i)h_i} \le 2\sqrt{2r\cdot 2^{j+1} r^{-\frac{d-1}{d+1}}} = 2^{\frac{j}{2}+2} r^{\frac{1}{d+1}}.$$

Consider now the M–regions $M(y_i, 1/2)$ for i with $F_i \in \mathcal{F}_j$. Since they are pairwise disjoint, so are their intersections with the sphere S_R of radius $R = r - \frac{9}{8}2^j r^{-\frac{d-1}{d+1}}$, centred at the origin. A straightforward, if tedious, computation shows that $S_R \cap M(y_i, 1/2)$ contains a spherical cap of radius $2^{\frac{j}{2}-1} r^{\frac{1}{d+1}}$. These caps are all contained in the intersection of S_R with the ball of radius $2^{\frac{j}{2}+2} r^{\frac{1}{d+1}}$ (centred at x_L). An easy computation shows that there are at most 8^{d-1} such caps. This implies that at most 8^{d-1} facets from $\mathcal{F}_j$ contain L. So the total number of k–faces is at least $8^{-(d-1)}|\mathcal{F}_j| \gg m \gg r^{d\frac{d-1}{d+1}}$. □

Remark 4. The extension of this estimate to $K \in \mathcal{C}(D)$ from B^d is similar to the one outlined in Remark 2. Details are left to the reader.

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