The Complex of Maximal Lattice Free Simplices

by

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I. Introduction

We study the global topological structure of a simplicial complex arising naturally in the study of integer programming. The simplices in the complex consist of maximal lattice free bodies defined in the following way. Let Z^n be the lattice of integers in R^n, and let A be an $(n+1) \times n$ matrix, with ith row denoted by a_i, and which satisfies the two conditions:

A1. There is a <u>unique</u> (up to a positive multiple) strictly positive $(n+1)$ row-vector π such that $\pi A = 0$, and

A2. If, for any index i, $a_i \cdot z = 0$ for some integral z, then $z = 0$.

Condition 1. implies that the $n \times n$ minors of A are non-singular and that for any b the body $\{x \mid Ax \leq b\}$ is bounded. Since, for our purposes, the rows of A can be normalized arbitrarily, there is no loss in generality in assuming that $\pi = 1 = (1, \ldots, 1)$. Condition 2. asserts that for any b_i the hyperplane $a_i \cdot x = b_i$ contains at most one lattice point in Z^n. This condition can be relaxed so that small perturbations of A are also admissible.

A <u>maximal lattice free body</u> is a body of the form $K = \{x \mid Ax \leq b\}$ containing no lattice points in its interior, and such that any closed

convex body which properly contains K does have a lattice point in its interior. It follows from A2. that for each i there is a single lattice point z^i on the hyperplane $a_i.x = b_i$, that these lattice points are distinct for different i, and that $a_i.z^j < a_i.z^i$ for $j \neq i$. Moreover there is no lattice point z such that $a_i.z < a_i.z^i$ for $i = 0,1,...,n$.

The abstract simplicial complex $K(A)$ consists of all of the formal simplices $(z^0,z^1,...,z^n)$, oriented in this fashion, and all of their subsimplices. The complex contains an infinite number of simplices, since an arbitrary lattice translate of a maximal lattice free body is also maximal lattice free. It will be shown, however, that A1. implies that the complex is __locally finite__ in the sense that every lattice point z is a vertex of a finite number of such simplices [White, 1983]. It can also be shown that each $(n-1)$ face $(z^1,...,z^{i-1},z^{i+1},...,z^n)$ is incident to precisely two n-simplices in the complex, with opposite derived orientations, so that the complex is an orientable pseudo-manifold [Scarf, 1981a].

The simplicial complex $K(A)$, being locally finite, can be provided with a topology in a natural fashion: Give each simplex in $K(A)$ its customary topology, and realize $K(A)$ as the disjoint union of its vertices and the relative interiors of its simplices and subsimplices. A set in the union is defined to be closed if its intersection with the closure of each simplex in $K(A)$ is a closed set. The associated topological space will be written as $|K(A)|$.

When $n = 2$, there is a basis, b^1,b^2, for the lattice Z^2, such that the simplices in $K(A)$ are the lattice translates of $(0,b^1,b^1+ b^2)$ and $(0,b^2,b^1+ b^2)$, thereby providing a simple geometric realization of the abstract

complex as R^2. When $n \geq 3$, however, the convex hulls of two distinct n-simplices may contain common interior points, and the global structure of the complex is no longer transparent. Some indication of this structure is suggested by the following result which also appears in [White, 1983].

Assume that the rows of A are permuted so that $\det[1',A] > 0$, and for each simplex $S = (z^0, z^1, \ldots, z^n)$ in K(A), define

$$\text{index}(S) = \text{sign } [\det(z^1 - z^0, \ldots, z^n - z^0)],$$

or zero if the determinant vanishes. Then for generic $x \in R^n$,

$$\Sigma(\text{index}(S) | x \in \text{convex}(S)) = 1.$$

The result suggests that the natural geometric realization of the complex K(A) by means of the convex hulls of its simplices may be an intricate folding of R^n into itself; and this is, in fact, our major conclusion.

Theorem 1. $|K(A)|$ is homeomorphic to R^n.

The group of integers Z^n acts on K(A), since the lattice translate of a simplex in K(A) is also a simplex in K(A). We may, therefore, obtain a new complex, KZ(A), by identifying those simplices or subsimplices of K(A), which are lattice translates of each other. Local finiteness of K(A) implies that the new complex has a finite number of simplices of each dimension $j = 0, \ldots, n$. In particular KZ(A) has a single vertex. KZ(A) is not a conventional simplicial complex, but is a more general object known as a CW complex [Massey, 1980].

KZ(A) is realized as the disjoint union of a single vertex and the relative interiors of the simplices and subsimplices of K(A), with lattice translates identified. As before, a set in the union is closed if its intersection with the closure of any simplex is a closed set. The

topological space obtained in this fashion is denoted by $|KZ(A)|$, and is clearly homeomorphic to $|K(A)|/Z^n$.

White [1983] contains an additional result, bearing on the structure of $KZ(A)$. Again assume that the rows of A are permuted so that $\det[1',A] > 0$. Then

$$\Sigma[\det(z^1 - z^0,\ldots,z^n - z^0)]/n! - 1,$$

with the sum taken over a set of representatives in $K(A)$ of the maximal simplices in $KZ(A)$. The result states that the sum of the signed volumes of the simplices in $KZ(A)$ is equal to unity, and is consistent with the following theorem, conjectured by Lex Schrijver.

Theorem 2. $|KZ(A)|$ is homeomorphic to the n-torus.

The demonstration of Theorem 1. involves the construction of a simplicial mapping of the vertices of $K(A)$ into R^{n+1}, such that the convex hulls of the images of the vertices of two distinct n-simplices no longer overlap. They form, in fact, the facets of a convex set in R^{n+1}.

Before presenting the details of this construction it may provide motivation for the study of $K(A)$ to say a few words about the relationship between maximal lattice free bodies and integer programming. Consider the integer program

(1) $\text{Min } \Sigma a_{0,j} \cdot z_j$, subject to

$\Sigma a_{i,j} \cdot z_j \le b_i$, $i - 1,\ldots,n$, and $z \in Z^n$.

Let $N(0)$ be an arbitrary set of lattice points which is symmetric about the origin. For a particular right-hand side b, a lattice point z is said to be a local minimum with respect to $N(0)$ if z is a feasible solution to (1), and if, for every $h \in N(0)$, $z + h$ is either infeasible or yields a higher value of the objective function $\Sigma a_{0,j} \cdot (z_j + h_j)$.

(2) $y_i - \exp(ta_i \cdot x)$ for $i - 0, \ldots, n$,

with t a fixed positive real number. Clearly the image of R^n under this

mapping is contained in the set $M - \{y - (y_0, \ldots, y_n) | y_i > 0, \Pi y_i - 1\}$ and,

as we shall see, the image is this entire hyperboloid. (Scarf, 1973

contains an earlier but less useful version of this mapping.)

Let $V - \{v\}$ be the image of the lattice of integers Z^n. We can define

a multiplication in R^{n+1} by taking the coordinate-wise product of two

vectors. With this multiplication, M is an Abelian group with identity $1 -$

$(1, \ldots, 1)$, and V a discrete subgroup. Finally, let C be the convex hull of

V. (For simplicity of notation, we omit explicit reference to the

dependence of V, C and other constructions on the parameter t.)

C is not the convex hull of finitely many points and so it need not be,

and is not, a convex polytope. We can, nevertheless, define faces of C in

the following way. L is a face of C if there is a half-space $H \subset R^{n+1}$ with

bounding hyperplane H^0 such that $C \subset H$ and $C \cap H^0 - L$. The vertices, or

zero dimensional faces of C, are precisely the vectors $v \in V$, since each

such vector is on the boundary of the strictly convex body $\{y | y > 0, \Pi y_i \geq$

$1\}$. The line segment $[v,u]$ will be an edge of C (a one dimensional face) if

and only if the segment $[1, vu^{-1}]$ is also an edge. More generally the facial

structure of C is identical at each vertex; the convex hull of $(u, v^1, \ldots, v^k)$

will be a face of C if and only if the convex hull of $(1, v^1 u^{-1}, \ldots, v^k u^{-1})$ is

also a face of C. (If $\{y : \Sigma h_i y_i \geq h\}$ is the half-space which verifies that

the first convex hull is a face, then $\{y : \Sigma h_i u_i y_i \geq h\}$ verifies that the

second convex hull is a face, and conversely). As we shall see (Theorem 3),

there is a t_0 such that for $t > t_0$, the facets - or faces of dimension n -

of C are simplices and the number of facets containing any particular vertex

is finite. As a consequence, C will have the facial structure of a polytope at each vertex.

The intuition behind the proof of Theorem 1. is that, for t tending to infinity, the convex hulls of the images of two distinct simplices in K(A) tend to have no interior points in common. To appreciate this fact we simply notice that if $(z^0, z^1, \ldots, z^n)$ is a simplex in K(A), and if $\alpha_j > 0$, for $j = 0, \ldots, n$, then

$$\text{limit}_{t \to \infty} \ (1/t)\log(\Sigma_j \alpha_j \exp(ta_i . z^j)) = \max_j (a_i . z^j),$$

so that all of the interior points in the convex hull of the images of $(z^0, z^1, \ldots, z^n)$ are clustered around the point $y = (y_0, \ldots, y_n)$ with

$$y_i = \exp(t\max_j (a_i . z^j)).$$

The vectors (b_i) with $b_i = \max_j (a_i . z^j)$ are different for different maximal lattice free bodies, so that these cluster points are far from each other for large t. In order to make this intuition precise, we shall require some preliminary arguments. We start with three lemmas concerned with the facial structure of C.

Lemma 1. The image of R^n under the mapping (2) is the entire set $M = (y | y > 0, \Pi y_i = 1)$.

Proof: Let y be an arbitrary point in M. The n x n minors of A are non-singular so that there is a solution, x, to the system of equations

$$a_i . x = (1/t)\log y_i, \text{ for } i = 1, \ldots, n.$$

Since $\Pi y_i = 1$, the same x satisfies the corresponding equation for $i = 0$. ⊗

Lemma 2. Let $\varepsilon > 0$ be given. Then for each i, there is a $v \in V$ with $v_j \leq \varepsilon$ for all $j \neq i$.

Proof: Take $i = 0$. From A1., the set $(x | a_j . x \leq (1/t)\log\varepsilon, \ j = 1, \ldots, n)$ is a cone with non-empty interior and therefore contains a lattice point.

The argument for other i is similar. ⊗

Lemma 3. Let y^0 be on the boundary of C and let $h = (h_0, \ldots, h_n) \neq 0$ be such that $h.y \geq h.y^0$ for all $y \in C$. Then $h_i > 0$ for all i.

Proof: y^0 is in the set $(y|y > 0, \Pi y_i \geq 1)$, so that all of its coordinates are strictly positive. If one of the coordinates of h, say h_i, were < 0, then, as ε tends to zero, the vectors v described in Lemma 2. will lie in C and violate the inequality $h.v \geq h.y^0$. It follows that $h_i \geq 0$, and that $h.y^0$ is strictly positive since the coordinates of h are ≥ 0, but not all zero, and $y^0 > 0$. But then if $h_i = 0$, these same vectors in V will violate $h.v \geq h.y^0$. ⊗

Lemma 3. implies that the boundary of C consists of polytopes. We now turn to three lemmas concerning the structure of K(A).

Lemma 4. Each lattice point $z \in Z^n$ is contained in a finite number of simplices in K(A).

Proof: Let the lattice point be $z = 0$, with simplex $(0, z^1, \ldots, z^n)$, associated with the maximal lattice free body given by $(x| Ax \leq b)$, where $b_i = \max_j a_i.z^j \geq 0$. Any particular set of this form contains a finite number of lattice points; if there were an infinite sequence of such maximal lattice free bodies, all differing from each other, then in this sequence one of the coordinates, say b_0, would tend to infinity. But this is impossible since the cone defined by $a_i.x \leq 0$, for $i = 1, \ldots, n$, has a non-empty interior which contains infinitely many lattice points. ⊗

N(0), the set of neighbors of the origin, is identical with the set of $(z^i - z^j)$, with z^i and z^j vertices of a common simplex in K(A). We note that Lemma 4. implies that N(0) is finite. Lemma 4. has another useful

implication.

Lemma 5. Define $\delta_1 = \min\{|a_i.h| : h \in N(0), i = 0,1,\ldots,n\}$. Let S be a finite set of lattice points and let $K = \{x | Ax < b\}$, where $b_i = \max\{a_i.z : z \in S\}$. Then if K contains a lattice point, it will contain a lattice point ξ with $a_i.\xi \le b_i - \delta_1$ for all i.

Proof: Let $\xi \in K$ and let i be an index such that $b_i - \delta_1 < a_i.\xi < b_i = a_i.z^i$ with z^i a particular lattice point in S. Then ξ and z^i are not neighbors. It follows that the smallest body of the form $\{x | Ax \le c\}$ containing ξ and z^i, will also contain an interior lattice point. If the conclusion of Lemma 5 is not valid for this interior lattice point in this smaller body - which is strictly contained in K - we repeat the argument. But the argument must terminate after a finite number of repetitions, since each repetition eliminates at least one interior lattice point. $\otimes$

We have one final Lemma before proceeding to the proof of Theorem 1.

Lemma 6. There is a $\delta_2 > 0$, such that if $(z^0, z^1, \ldots, z^n)$ is a simplex in K(A), ordered so that for each i, $\max_j a_i.z^j$ is assumed at z^i, and if z is a lattice point different from $z^0, z^1, \ldots, z^n$, then $a_i.z \ge a_i.z^i + \delta_2$ for some i.

Proof: It is sufficient to demonstrate Lemma 6 for those simplices in K(A) with $z^0 = 0$, since the general simplex may be reduced to this case by subtracting z^0 from each z^i. The number of such simplices is finite and it is therefore sufficient to find a δ_2 for each simplex and take the smallest of these. But, of course, there is such a δ_2 for any particular simplex $(0, z^1, \ldots, z^n)$ in K(A), since otherwise there would be an infinite set of lattice points, $z(\varepsilon)$, satisfying $a_i.z(\varepsilon) < a_i.z^i + \varepsilon$, for $i = 0,\ldots,n$, and ε tending to zero. $\otimes$

III. The Proof of Theorem 1.

Theorem 1. will be demonstrated by means of the following result, which shows that for large t, the convex hulls of the images of the vertices of simplices in K(A) provide a geometric realization of the complex.

Theorem 3. There is a t_0 such that for each $t > t_0$, the n-facets of C are simplices. Moreover, for such t, the simplex $(z^0, z^1, \ldots, z^n) \varepsilon K(A)$ if and only if the images of its vertices $(v^0, v^1, \ldots, v^n)$ are the vertices of an n-facet of C.

Proof: The proof consists of two parts. We begin by letting $(z^0, z^1, \ldots, z^n)$ be an arbitrary set of n+1 lattice points whose images $(v^0, v^1, \ldots, v^n)$ lie on an n-facet of C, for some particular $t > t_1 = (\log(n+1))/\delta_1$, and demonstrate that these n+1 lattice points form a simplex in K(A).

Let h be the strictly positive normal to the hyperplane supporting the facet containing $(v^0, v^1, \ldots, v^n)$, for the particular t in question. We have

(3) $\Sigma h_i \exp(t a_i . z) \geq 1,$

for any lattice point z, with equality for $z = z^0, \ldots, z^n$.

Consider first $z = z^j$. It follows from the above <u>equality</u> that for each index i,

$$h_i \exp(t a_i . z) \leq 1, \text{ so that}$$
$$a_i . z^j \leq -(1/t)\log(h_i), \text{ and therefore}$$
$$\max_j [a_i . z^j] \leq -(1/t)\log(h_i).$$

We wish to show that there are no lattice points other than $z^0, z^1, \ldots, z^n$ in the body $(x | Ax \leq b)$, with $b_i = \max_j [a_i . z^j]$, so that $(z^0, z^1, \ldots, z^n)$ is a simplex in K(A). Let z be such a lattice point so that

$a_i.z \le \max_j[a_i.z^j] - \delta_1$, for all i, using Lemma 5. But then inequality (3) implies that there is an index i with

$$h_i \exp(ta_i.z) \ge 1/(n+1), \text{ and}$$

$$a_i.z \ge -(1/t)\log(h_i) -(1/t)\log(n+1)$$

$$\ge \max_j[a_i.z^j] -(1/t)\log(n+1), \text{ a}$$

contradiction if $t > t_1 = (\log(n+1))/\delta_1$.

We now turn to the second part of the argument and demonstrate that the convex hull of the images of the vertices of a simplex $(z^0, z^1, \ldots, z^n)$ in K(A) form a facet of the boundary of C for large t. Let $(v^0, v^1, \ldots, v^n)$ be such a set of images and let h satisfy $h.v^j = 1$, for $j = 0, 1, \ldots, n$. We will demonstrate the existence of a t_2, for this simplex, such that $h.v \ge 2$, for all $t \ge t_2$, and for all v which are images of lattice points other than z^j. Assume that the vertices have been permuted so that $a_i.z^i = \max_j[a_i.z^j]$.

By Cramer's rule we have

$$h_0 = \det N/\det[\exp(ta_i.z^j)]$$

with N the matrix obtained by replacing row 0 by $(1, \ldots, 1)$ in the matrix appearing in the denominator. Let us estimate the denominator first. The determinant can be written as the sum of (n+1)! terms, each one based on a permutation of $(0, 1, \ldots, n)$. But for each permutation, σ, other than the identity, the corresponding term will be $[\Pi\exp(a_i.z^{\sigma(i)})]^t$, which is strictly less than

$$[\Pi\exp(a_i.z^i)]^t,$$

so that for large t this single term will be the asymptotic value of the determinant in the denominator. A similar analysis tells us that the numerator is asymptotically equal to the same product with index ranging from 1 to n, so that $h_0 \sim \exp(-ta_0.z^0)$, in the sense that

$h_0 \sim (1 + \varepsilon_0(t))\exp(-ta_0.z^0)$, with $\varepsilon_0(t) \to 0$ as $t \to \infty$.

By a similar argument we obtain

$h_i \sim (1 + \varepsilon_i(t))\exp(-ta_i.z^i)$, with $\varepsilon_i(t) \to 0$ as $t \to \infty$, for all i.

In particular there is a t_2 so that for all $t \geq t_2$, we have

(4) $h_i \geq 2\exp(-ta_i.z^i)\exp(-t\delta_2)$ for all i,

with δ_2 the constant referred to in Lemma 6.

In order to demonstrate that $con(v^0,v^1,\ldots,v^n)$ is a facet of the boundary of C, we shall show that $h.v \geq 2$, for $t \geq t_2$, and for any image, v, of a lattice point other than (z^j). From (4), we have

$h.v \geq 2\Sigma\exp(t(a_i.z - a_i.z^i - \delta_2)) \geq 2,$

for any integral z other than (z^j), since, for that z, Lemma 6 states that there must be at least one index i with $a_i.z \geq a_i.z^i + \delta_2$.

In this argument, the value of t_2 depends on the particular simplex $(z^0,z^1,\ldots,z^n)$; in order to complete the proof of Theorem 1 we must show that a single value suffices for all simplices. But for any t, the image of $(z^0,z^1,\ldots,z^n)$ is a facet of C if the image of $(0,z^1-z^0,\ldots,z^n-z^0)$ is a facet. There is a finite number of simplices in K(A) for which $z^0 = 0$, and we may select the largest value of t_2 associated with any of these.

The value of t_0 in the statement of Theorem 3 is the larger of t_1 and t_2. It should be remarked that we have demonstrated that for $t > t_0$, the n-facets of C are simplices. For if the convex hull of $(v^0,v^1,\ldots,v^n)$ is contained in an n-facet of C, we know that its preimage $(z^0,z^1,\ldots,z^n)$ is a simplex in K(A), and consequently the convex hull of $(v^0,v^1,\ldots,v^n)$ is, itself, an n-facet. ⊗

Theorem 3 yields a geometric realization of the complex K(A) as the boundary of the convex set C, for $t > t_0$ in the following fashion: given a

particular simplex in K(A) with vertices $(z^0, z^1, \ldots, z^n)$, associate the

point $x = \Sigma \alpha_j v^j$ with the formal convex combination $\Sigma \alpha_j z^j$, for each $\alpha =$

$(\alpha_0, \ldots, \alpha_n)$ with $\alpha_j \geq 0$, $\Sigma \alpha_j = 1$. With this mapping, $F: |K(A)| \rightarrow \partial C$,

vectors arising from different simplices in K(A), or from the same simplex

but with different formal convex combinations, are mapped to different

points on the boundary of C. Thus F provides a homeomorphism between $|K(A)|$

and ∂C, and Theorem 1 follows from the observation that ∂C is homeomorphic

to R^n.

For the proof of Theorem 2, we need an argument from equivariant

topology which we shall state in a somewhat informal way. Consider the

composite mapping obtained by first mapping the formal convex combination

$\Sigma \alpha_j z^j$ to x on the boundary of C, as above, and then mapping x to y in M, by

$y_i = x_i / [\Pi x_j]^{1/n}$. This composite mapping, $T: |K(A)| \rightarrow M$, is a realization

of the complex K(A) as the hyperboloid M. Furthermore the mapping T is

compatible with (equivariant for) the actions of Z^n on $|K(A)|$ (by

translation) and on M (by coordinatewise multiplication by points in V),

i.e., if (y_i) is the coordinate vector associated with the formal convex

combination $\Sigma \alpha_j z^j$, then the coordinate vector associated with its lattice

translate $\Sigma \alpha_j (z^j + z)$ is $(y_i v_i)$, with v the image of z under the mapping

(2). Therefore, the set of lattice translates of a given point in $|K(A)|$

corresponds to a single point in M/V, so that T induces a homeomorphism

between $|KZ(A)| = |K(A)|/Z^n$ and the quotient space M/V. Theorem 2 follows

from the observation that M/V is homeomorphic to the n-torus R^n/Z^n.

IV. Some Remarks

A major limitation of the present analysis is that the number of rows

of the matrix A exceeds the number of variables only by unity, so that the corresponding integer programming problems possess the same number of constraints as variables. The simplicial complex of maximal lattice free bodies can be readily defined for a general matrix A with m > n columns and n rows, but the complex will contain maximal lattice free bodies with k vertices for every integer k between (n+1) and $\min[m, 2^n]$. The complex is no longer given by R^n, and the complex mod (Z^n) is not the torus. It would be a matter of considerable interest to characterize the global structure of these simplicial complexes.

Some results are available for dimensions n = 2,3. If n is a 4 x 2 matrix, the maximal lattice free bodies consist of a sequence of parallelograms of area one, and four triangles; two associated with the first parallelogram in the sequence and two with the last [Scarf, 1981b]. When A is a 4 x 3 matrix, all of the maximal lattice free bodies lie between two parallel, adjacent lattice hyperplanes [Scarf, 1985].

The proof of this last result is extremely tedious, and it could conceivably be simplified by the structural results of the present paper. We know that the simplicial complex of maximal lattice free bodies, with lattice translates identified, is the 3-torus. Its second homology group, $H_2(K)$, will therefore be generated by three 2-tori. One example of such a 2-torus is obtained from a pair of triangular faces in KZ(A) whose union forms a planar parallelogram of unit area. The theorem would be quite easy to prove if the existence of such a pair of triangular faces could be established.

A possible line of attack is by means of the Fundamental Group, a free Abelian group with three generators. The Fundamental Group can be presented

in the following way: orient each of the edges (the neighbors of the origin) of the complex. Each 2-face of the complex will then define a relationship among its three edges. The homotopy group is then given by the set of words composed of edges and their inverses, with multiplication given by concatenation, all modulo this set of relations. But this group is commutative. An instance of commutativity is provided by a pair of triangular faces whose union is a parallelogram; is it possible that commutativity of the homotopy group requires at least one such example?

Knowledge of the lattice hyperplane associated with such a pair of triangular faces permits a considerable simplification in the 3-variable integer programming problem defined by the matrix and a particular right hand side. Let the family of lattice hyperplanes be given by $h.z = h_0$, for integral h_0, and let

$$f(h_0) = \text{Min } \Sigma a_{0,j} . z_j, \text{ subject to}$$
$$\Sigma a_{i,j} . z_j \le b_i, \; i = 1,2,3,$$
$$h.z = h_0, \text{ and } z \in Z^3.$$

This family of 2-variable integer programs (there is a unimodular transformation so that $h.z$ is one of the new variables) has the property that $f(h_0)$ is unimodal: it has a single local minimum. A global minimum for the original 3-variable problem is obtained by finding a particular h_0, such that $h_0 \pm 1$ yield higher values of the objective function.

Is it possible that knowledge of a generator of the n-1 dimensional homology group $H_{n-1}(K)$ permits a corresponding reduction in dimension for the n-variable problem? More generally, can the combinatorial structure of the simplicial complex K(A) be used effectively in the development of algorithms for integer programming?

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