Constructing the Triangle-Free Planar 3-Connected Graphs

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#1071

Abstract

We shall call a graph triangle-free if it does not contain any cycles of length three. We give a method of recursively constructing all of the triangle-free planar 3-connected graphs. Duality gives us a construction of the planar 3-connected 4-edge-connected graphs. A slight modification of our procedure gives a generating procedure for the planar 3-connected graphs without 3-valent vertices. The graphs are generated using four operations, and we show that all four operations are necessary.

1. Introduction. One of the most fundamental theorems of planar 3-connected graphs is that they can be generated from the complete graph on four vertices by face splitting (see [1] and [9]). There are similar theorems for all 3-connected graphs (see [5], [10] and [11]). There are also a number of recursive generation procedures for various classes of planar 3-connected graphs (see [2], [3], [4], [7], and [8]). In this paper we investigate the family $\mathcal{T}_1$ of triangle-free planar 3-connected graphs, that is, those without any cycles of length three. We shall also generate the planar 3-connected graphs without 3-valent vertices, and we shall see using duality that we will have a recursive construction of the planar 3-connected 4-edge connected graphs.

2. Definitions and notation.

The graphs in this paper are without loops or multiple edges.

If a graph G is embedded in the plane, then the closures of the connected components of the plane minus G are called the faces of G. The planar 3-connected graphs are characterized by the fact that the faces are bounded by cycles, and that two faces meet on a vertex, an edge, or not at all (a proof of this can be found in section 2, Ch.5 of [6]). When the faces meet in only these ways, we say that the faces meet properly.

An edge of a planar 3-connected graph G, is removable provided removing it and coalescing any pairs of edges meeting resulting 2-valent vertices into single edges produces a 3-connected graph. The inverse of removing edges is face splitting. Face splitting consists of adding a new edge across a face, with endpoints on edges or vertices of the face, but the endpoints are not both on the same edge of the face. An edge is *-removable provided it is removable, and when it is removed the resulting graph has no triangles.

If an edge e of a 3-connected graph is not removable, then removing e produces a pair of faces that do not meet properly. These two faces thus have a multiply connected union. One of these two faces will be split by the inverse of the edge removal, and thus we will have a set of three faces F_1, F_2, and F_3 such that each two meet but there is no vertex belonging to all three. Any set $C = \{F_1, F_2, F_3\}$ of faces such that each two meet, but no vertex belongs to all three, will be called a 3-chain. With the graph embedded in the plane such that none of these three faces is the unbounded face, the union of the three faces has a complement that consists of two open regions - one that is bounded and is enclosed by the union of the faces, and one unbounded region. These are called the inner and outer regions of the chain, respectively. The outer region is bounded by a closed curve consisting of three simple paths, one along each face of C, with each two paths intersecting on a vertex. This curve will be called the outer curve, and its constituent paths will be called the outer paths. There is a similar inner curve, bounding the inner region, consisting of three inner paths, one along each face of C and meeting just at their endpoints. An edge e belonging to two faces of C will be called a connecting edge of C. If two faces of C meet on just a vertex v, then v will be called a connecting vertex of C. A 3-chain C is minimal provided no 3-chain has an inner region that is a subset of the inner region of C.

A cycle is a simple closed curve consisting of edges of the graph. If a cycle has exactly n edges it is called an n-cycle. A 3-cycle will be called a triangle. In planar 3-connected graphs there are two types of cycles. A cycle is separating if and only if there is an edge inside and an edge outside the cycle, otherwise the cycle is called nonseparating. The faces of planar 3-connected graphs have been characterized as the nonseparating cycles [11]. A separating n-cycle will be called an ns-cycle. We shall be particularly concerned with 4s-cycles. A 4s-cycle X is minimal provided no 4s-cycle encloses a proper subset of the faces enclosed by X.

We shall say that a vertex is inside a 3-chain C provided it is in the bounded open region bounded by the outer curve. A face is inside C provided it is in the bounded closed region bounded by the outer curve. An edge is inside C provided it is in this bounded closed region and it is not an edge of the outer curve. Faces, vertices, and edges that are inside C will also be called interior faces, vertices and edges, respectively. A vertex is strictly inside C provided it lies in the inner region. An edge is strictly inside C provided it is inside C and is not an edge of the inner curve. A Face is strictly inside C provided it is inside C and is not a face of C. Vertices, faces, and edges that are strictly inside C will also be called strictly interior vertices, faces, and edges respectively.

There are two types of configurations that can occur in a planar 3-connected graph that we shall deal with. These are shown in Figure 1. The first is a set of three 4-sided faces surrounding a vertex v. This configuration will be called a star, and v will be called the center of the star. The second, we shall call a box. It consists of a star with an additional 4-sided face meeting one of the vertices that is joined to the center of the star (see Figure 1).

3. Generating triangle-free graphs.

The planar 3-connected graphs without triangles will be generated from the graph of the cube using four operations. We shall show that the graphs can be generated by these operations by showing that the inverse of at least one of them can be performed on every triangle-free planar 3-connected graph, other than the graph of the cube, producing a planar 3-connected triangle-free graph. These inverse operations are:

1) Removing a *-removable edge.

2) Collapsing a star (See Figure 2A). Collapsing the star consists of contracting the three edges that meet the center v. When the star is collapsed it is possible that 2-valent vertices are created at a, b, or c . If this happens the two edges meeting a 2-valent vertex are coalesced into a single edge.

3) Collapsing a 4-sided face of a star. When we specify a collapsing we shall state which vertices are to be identified. One of them will always be the center of the star. (Figure 2B shows a collapsing of F, d to b.)

4) Simplifying a box. This can be thought of as removing two edges e and e' (see Figure 2C).

One might suspect that with so many operations that fewer might suffice. It is rather interesting that no subset of our set of four operations will suffice. The graphs in Figure 3 show the necessity of the four operations. Each graph admits exactly one of our inverse operations. Graph 1 admits only edge removal; graph 2 - only star collapsing; graph 3 - only face collapsing; and graph 4 - only box simplifying. (In Figure 3, graph 4, we have indicated a box and the two edges to be removed.) One might also conjecture that by adding the operation "edge contracting" (the dual of edge removal) we might be able to eliminate some of our operations and simplify things. Our four examples, however, show that this is not the case. In each of the examples, no edge can be contracted, thus even with this new operation at our disposal the other four are necessary

4. Minimal graphs.

We shall say that a planar 3-connected triangle-free graph G is minimal provided none of operations 1, 2, 3, and 4 may be performed on it to produce a planar 3-connected triangle-free graph. For the rest of this section G is always a minimal planar 3-connected graph.

Lemma 1: Every minimal 4s-cycle X of G admits a 3-chain consisting of faces F, F', F" inside X.

Proof: Let e be an edge of X. Since G is minimal, e is not *-removable. We shall begin by assuming that e is not removable. In this case it is a connecting edge of a 3-chain C = {F, F', F"}.

Case I. Only one face of C lies inside X. Let that face be F. The face F must contain two consecutive edges of X (recall that if two vertices of an edge lie on a face, the edge lies on the face, because faces meet properly). Let F_1 and F_2 be the two faces inside X that contain the two edges of X that are not on F. If $\{F, F_1, F_2\}$ is not a

Case II. Two faces of C lie inside X. Let these faces be F' and F". Now, F contains a vertex of e and must meet at least one other vertex v of X, thus F contains exactly two edges of X, namely e and an edge vw. The edges e and vw must be consecutive on X or else all edges of X lie on F, a contradiction. We may now assume that w is on e. Let F' be the face sharing e with F. The face F" cannot meet F on an edge because then {F, F', F"} surround w. Thus F" meets F only at v. Let H be the face of G lying inside X and containing vw. If H, F' and F" surround a vertex, then H is a triangular face. Thus {H, F', F"} is the desired 3-chain.

We now have that every edge of X is removable or we have the desired 3-chain, thus we shall assume that every edge of X is removable. If an edge e of X is not *-removable then that edge meets a 3-valent vertex. Since this is true for every edge of X, there is a pair of nonconsecutive vertices x and y of X that are 3-valent. If the third edge at each of x and y both lie inside X, then G can be disconnected at the other two vertices of X, Thus for at least one vertex, say x, the third edge lies outside X.

Let F be the face inside X meeting x, and let F' and F" be the two faces inside X containing the two edges of X that don't meet x. If these three faces surround a vertex then F' and F" are triangular faces. If not, then they form a 3-chain with faces inside X. ■

Lemma 2: <u>If F is a face of G strictly inside a minimal 3-chain C, then F is not a face of a 3-chain</u>.

Proof: Suppose F is a face of a 3-chain C'. One of the regions of C contains exactly one face F' of C'.

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Case I. F' is in the outer region of C. In this case F' meets two connecting vertices x and y of C. Inside C are two faces F" and F"' of C', each meeting one of x and y. Let H be the face of C containing x and y. Now, either {H, F", F"'} is a 3-chain violating the minimality of C, or these three faces surround a vertex, and H is thus a triangular face.

Case II. F' is in the inner region. Again, F' meets two connecting vertices x and y of C. Let H be the face of C containing x and y. Now either F' is a triangular face, or else F' and the two faces of C, other than H, form a 3-chain that contradicts the minimality of C. ■

Lemma 3: <u>If e is an edge inside a minimal 3-chain C, and if e is not a connecting edge of C, then e is not a connecting edge of any 3-chain</u> (<u>and therefore e is removable</u>).

Proof: Such an edge would lie on a face that is strictly inside C, and is a face of a 3-chain. By Lemma 2 this is impossible. ■

In the following we shall assume that G is a minimal graph and

i) If G contains 3-chains, then C is a minimal 3-chain.

ii) If G contains 4s-cycles, then C lies inside a minimal 4s-cycle

(Lemma 1 allows us to do this.)

Lemma 4: <u>There is a vertex strictly inside C</u>.

Proof: If there were no edges strictly inside C, then C would surround a triangular face. If there were edges but no vertices strictly inside C, then every strictly interior edge would join a vertex of one interior path to a vertex of another interior path. Let P_1 and P_2 be two interior paths with vertices on each that are joined by a strictly interior edge. Let P_1 and P_2 meet at vertex z. If we choose an edge uw, with u on P_1 and w on P_2, such that the sum of the path lengths from u and w to z along P_1 and P_2 is minimal, then z, u, and w are the vertices of a triangular face. Thus there must be strictly interior vertices. ■

Lemma 5: If there are not two vertices joined by an edge strictly inside C, then C is the set of faces surrounding a star.

Proof: Suppose that v is the only strictly interior vertex of C. Each edge meeting v is joined to one of the interior paths of C. Furthermore there can't be two edges to the same interior path because two consecutive such edges would be edges of a triangular face. There must be at least three edges meeting v, and each must be joined to exactly one interior path. If there are strictly interior edges not joined to v, then the argument in Lemma 4 shows that there is a triangular face. If there is another strictly interior vertex not joined to v it cannot be joined to each of the three interior paths, because of planarity. Now v is the center of a star, and the chain is the set of faces surrounding it. ■

Lemma 6: There is a star with all of its faces inside C.

Proof: By Lemma 5 we may assume that there is an edge vw with v and w strictly inside C. By Lemma 3, vw is removable, and since G is minimal, one vertex (say v) of vw is 3-valent and belongs to a 4-cycle. Since vw is inside a minimal 4s-cycle, if one exists, the 4-cycle meeting vw must be a 4-sided face F_1, which is strictly inside C. Let the other vertices of F_1 be p, q, and r, as shown in Figure 4.

The edge pv is strictly inside C and thus it is removable, but not *-removable. Thus either p or v is 3-valent and belongs to a 4-sided face F_2 ($F_2 \neq F_1$).

Case I. F_2 meets p. We note that if p is 3-valent, it lies inside C, so F_2 lies inside C. The edge pq cannot be a connecting edge of C because then v would not lie strictly inside C. Thus we may apply the above argument to edge pq and get that it meets a 4-sided face F_3 on a 3-valent vertex. If that 3-valent vertex is p then p is the center of the desired star. If that 3-valent vertex is q, we note that if q is 3-valent then it cannot be a connecting vertex of C, and thus F_3 lies inside C, and q is the center of the desired star.

Case II. F_2 meets v. We apply an argument similar to that in Case I, and get a star centered at either v or r. ■

Lemma 7: <u>There is either a box with all faces inside C, or a star with all faces strictly inside C</u>.

Proof: By Lemma 6 there is a star S inside C. Let the faces of S be F, F', and F". Let the vertices of S be labeled as shown in Figure 5.

Case I. Two faces, F and F' of S, are faces of C. Since separating vertices of C must lie on F or F', x is not a separating vertex, and since it lies on the strictly interior face F", x is strictly interior. Now, the edge e (see figure) is not a connecting edge of a 3-chain, thus it is removable. Since G in minimal, either b is 3-valent; or x is 3-valent, and e meets a 4-sided face K at x. If K contains a and f, then we have a box inside C. If K does not contain f, then a is not 3-valent and e' is *-removable unless e' meets a 4-sided face J at x. If J and K exist, then they are inside C because they meet x, and it is easily seen that we can collapse F", x to y. Thus b is 3-valent, and by symmetry, a is 3-valent.

The other would have to meet c. In G the two faces would have to meet on a vertex that is a separating vertex of S, but such a vertex is a vertex of F or F', in which case the collapsing would not make them meet improperly.

Case II. One face F, of S is in C. The edges e and e' are again removable, and by the above argument a and b are 3-valent. Now, by symmetry, c is 3-valent, and S is clearly collapsible.

The only remaining case is where the faces of S are strictly inside C. ■

Lemma 8: There is a box inside C.

Proof: Suppose not. Then by Lemma 7 there is a star S with faces strictly inside C. Let the faces and vertices of S be labeled as in Figure 6. If collapsing S destroys 3-connectivity, then the resulting graph will have two faces that meet improperly. Thus these faces belong to a 3-chain in G, one of whose faces is destroyed when the star is collapsed. The only faces, however, that are destroyed are faces of the star, and they do not belong to any 3-chains. It follows that we may collapse S unless a triangle is formed. This can happen in three ways.

Case I. Vertex a is 3-valent, d is not, and a and d belong to a triangle after the collapsing. Then F meets a 4-sided face H on edge da. Now, edge da is removable, and is *-removable unless F and H meet a third 4-sided face K meeting a. If K meets c, then K and the three faces of S form a box. If K does not meet c, then g is at least 4-valent, and we may collapse F, h to a.

Case II. Vertices a and d are 3-valent, vertex b is not, and a, d, and b lie on a triangle after the collapsing. Then these three vertices belong to a 4-sided face F' that forms a box with the faces of F.

triangle after the collapsing. Then these three vertices belong to a 4-sided face F' that forms a box with the faces of F.

Case III. Vertices a, d, and b, are 3-valent and belong to a 3-sided face after collapsing. Now these three vertices belong to a 5-sided face vadbu. The edge uv is removable because no face of S is in a 3-chain. If uv is not *-removable then uv meets a 4-sided face H on a 3-valent vertex. Let that vertex be v. By symmetry we may now assume that g is 3-valent. Now F, F', F" and H form a box inside C. ■

Lemma 9: There is a box B inside C with at most one face of B as a face of C.

Proof: From the previous lemmas we know that there is a box B inside C. Let the faces be labeled as in Figure 7. Suppose that two faces of B are faces of C. They cannot be F_1 and F_3 because then one of F_2 and F_4 would lie outside C. By symmetry we may assume that F_1 and F_2 are faces of C. Let F be the third face of C. Figure 7 shows the six ways that F can meet B.

Configuration 6 is not possible because $\{F, F_3, F_4\}$ is a 3-chain that contradicts minimality. In 1, 3, and 4, vertices x and y are at least 4-valent. Thus if we remove edges e and e' (and both can be removed because the edges are not connecting edges of 3-chains) we do not create any triangles. Removing these two edges is essentially operation 4. In 2 and 5 we may collapse F_4, v to w, unless z is 3-valent and lies on a 4-sided face F_5 that contains w, z, and y. In this case, however, we have a box with faces F_1, F_3, F_4, and F_5 that has only one face as a face of C. ■

Lemma 10: There is a box B with all faces strictly inside C.

Proof: By Lemma 9 there is a box B inside C with at most one face as a face of C. Let the edges faces and vertices be labeled as in Figure 7.

Case II. The face is not F_1 or F_3. By symmetry we may assume that the face is F_4. If x and z are both at least 4-valent, we may collapse F_4, v to w. We shall suppose therefore that z is 3-valent. This forces one of the faces H of C to meet F_4 at vertex w, and the other face H' of C to meet F_4 only at x. The vertex y must be 3-valent or we can remove e and e'. The edge wx is *-removable unless w is 3-valent and wzyq is a 4-sided face. Now we have that H' meets q. But now $\{H', F_1, F_2\}$ is a 3-chain contradicting minimality, or these three faces surround the vertex u. If they surround u, then either H' is a triangular face or vertices of H' can be separated from other vertices of the graph by removing x and q. ■

Theorem 1: The only minimal graph is the graph of the cube.

Proof: The above arguments give the existence of a box B in G. If there is a 4s-cycle X in G, the above arguments give a box with faces strictly inside a minimal 3-chain inside X. Similarly if there are 3-chains we have a box strictly inside a minimal 3-chain. Let the faces and vertices of B be as shown in Figure 8 . We note that each of the edges ab, bc, cd, df, fg, and ga are removable, thus each is *-removable unless it has a 3-valent vertex. We also note that each of F_2 and F_4 is collapsible h to g and i to c, respectively, unless one of a, f and one of b, d is 3-valent. One of a and d must be 3-valent or we can remove e and e', and similarly one of b and f must be 3-valent. Figure 8 shows the two possible arrangements (up to symmetry) of the 3-valent vertices (the vertices that must be 3-valent are indicated by large dots).

In 1, if the vertex a is 3-valent, then g and c disconnect G, or else g is joined to c, and G is the graph of the cube. If the valence of a is greater than 3 then edge e'' is *-removable unless g, f, d, and c lie on a 4-sided face. Thus g is joined to c, and unless G is the graph of the cube, a and c disconnect G.

In 2, vertices g and c disconnect the graph unless g is joined to c, and G is the graph of the cube. ■

Corollary: The triangle-free planar 3-connected graphs can be generated from the graph of the cube using the following four operations:

i) Splitting a face.

ii) Expanding a 3-valent vertex to a star.

iii) Expanding to a 4-sided face, a vertex belonging to two 4-sided faces.

iv) Replacing a pair of adjacent 4-sided faces with a box.

(These are the inverses of the operations 1,2,3 and 4.)

It is an easy exercise to show that these operations preserve 3-connectedness. Clearly they preserve planarity. The only restriction on their application is that they must not be applied in such a way that a triangle is ever formed. It is easily seen that the only operation that could form such a triangle is face splitting.

We note here that in our arguments we only used the fact that G had no triangular faces. Thus the above arguments also show that the family $\mathcal{J}_2$ of planar 3-connected graphs without triangular faces may be generated by these four operations. (Such an argument can be simplified somewhat for these graphs since we would not have to include any lemmas dealing with 4s-cycles.) Our four examples show the necessity of the four operations in generating $\mathcal{J}_2$ also. With this family of graphs, the difference in the generating procedure is that when any of the above operations is applied, it is admissible to apply it in such a way that a 3s-cycle is created. In other words, we may do face splittings that create triangles that are not faces. Applying duality we have:

Corollary: The planar 3-connected graphs without 3-valent vertices can be generated from the graph of the octahedron by the duals of operations i, ii, iii, and iv.

Applying duality to the family $\mathcal{T}_1$ also is interesting.

Lemma 11: A graph G is in $\mathcal{T}_1$ if and only if its dual is a planar 3-connected 4-edge-connected graph.

proof: Let G be in $\mathcal{T}_1$. Clearly G*, the dual of G, is planar and 3-connected. Suppose that we may disconnect G* by cutting three edges. Each pair of these edges belongs to a common face. The vertices that correspond to these faces under duality thus lie on an triangle, a contradiction. Thus G* is 4-edge-connected. The argument is easily reversed to get the reverse implication. ■

Corollary: The planar 3-connected 4-edge connected graphs can be generated from the graph of the octahedron by the duals of operations i, ii, iii, and iv.

Here, when applying the operations one must not apply vertex splittiing (the dual of face splitting) in such a way that a set of three faces is formed such that each two share a common edge.

5. remarks. The family of 3-connected graphs without triangles is an interesting family. It would be of interest to find a generating procedure for this set. Could our four operations, or perhaps our four plus vertex splitting, generate this set of graphs from the graph of the cube and the complete bipartite graph $K_{3,3}$?

Without duality at our disposal there doesn't appear to be any connection between the set of 3-connected graphs without triangles and those without 3-valent vertices. It would be of interest to find a recursive construction of the 3-connected graphs without 3-valent vertices.

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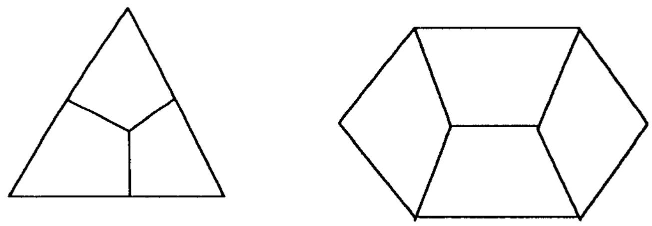
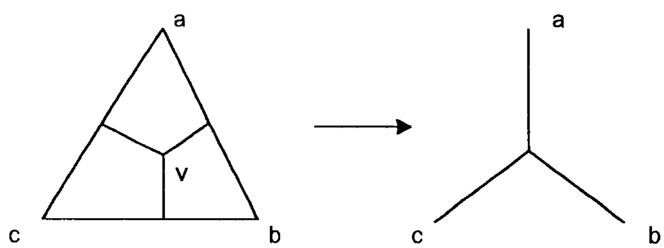


Figure 1



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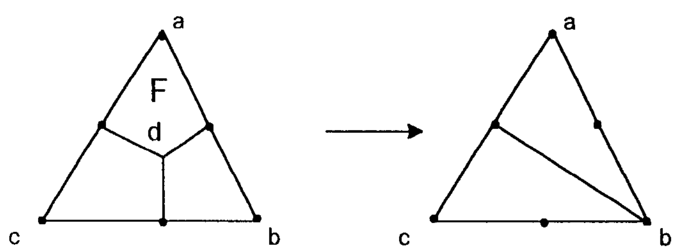

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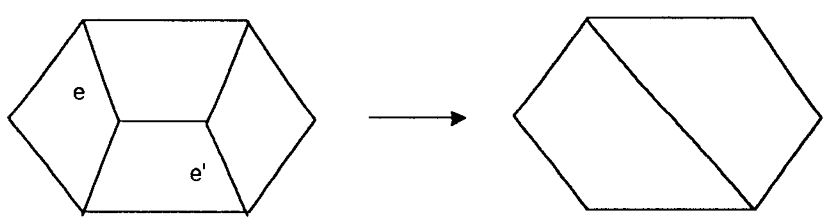

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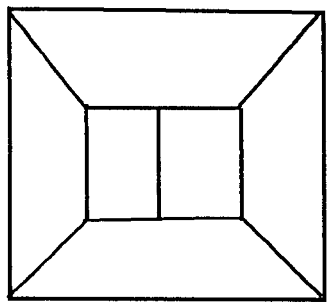
Figure 2



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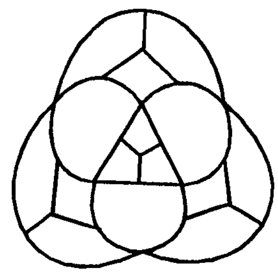

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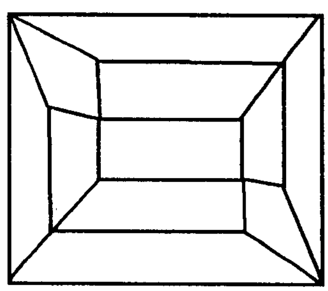

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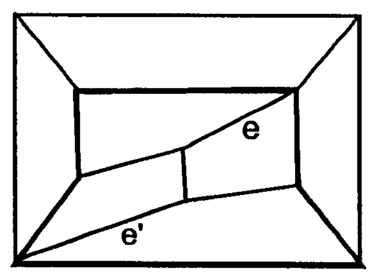

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Figure 3

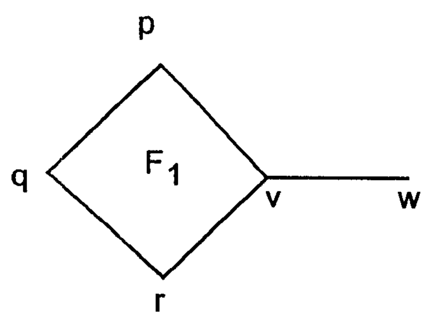

Figure 4

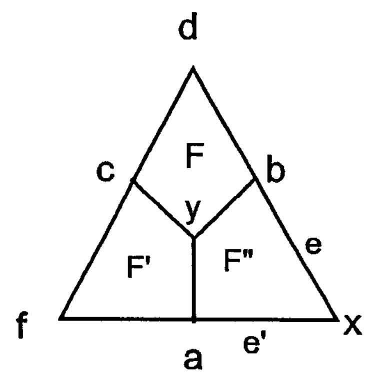

Figure 5

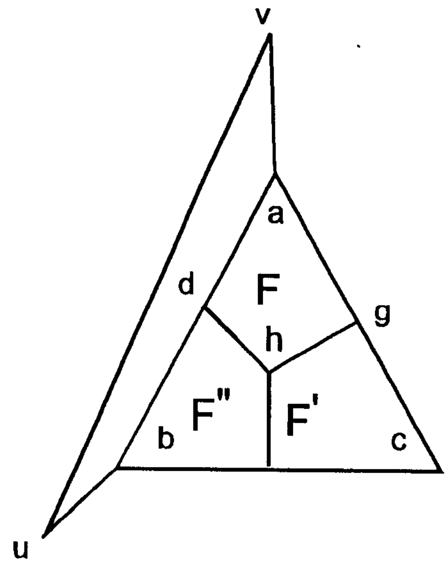

Figure 6

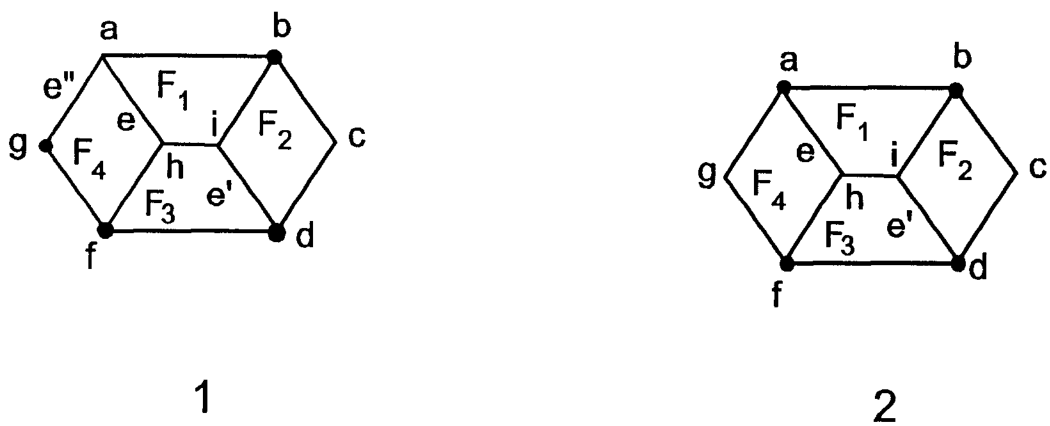

Figure 8

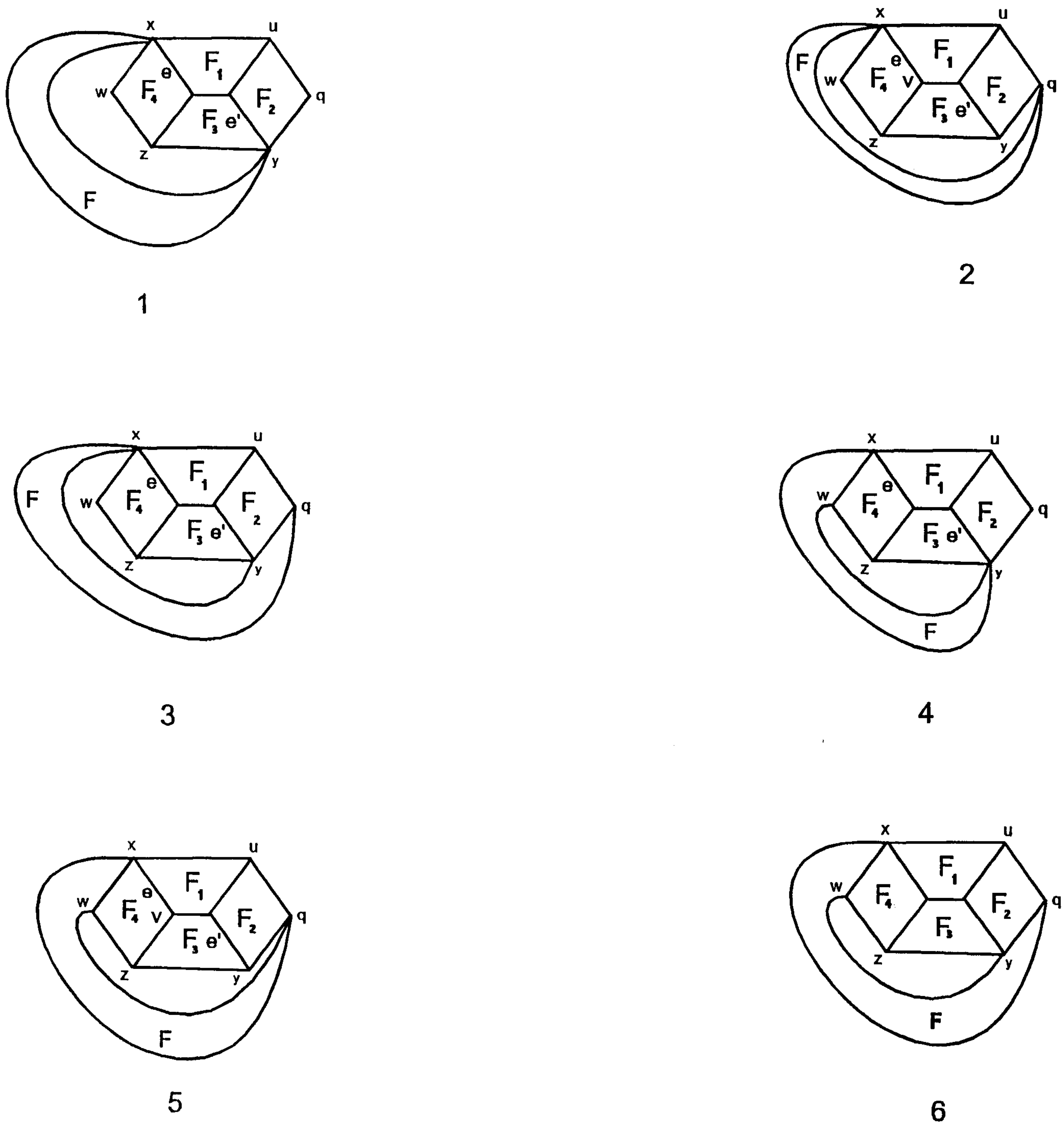

Figure 7