THE MINIMUM NUMBER OF VERTICES OF A SIMPLE POLYTOPE

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ABSTRACT

A d-polytope is a d-dimensional set that is the convex hull of a finite number of points. A d-polytope is simple provided each vertex meets exactly d edges. It has been conjectured that for simple polytopes

(i) $f_0 \geqq (d-1)f_{d-1} - (d+1)(d-2)$

and

(ii) $f_k \geqq \binom{d}{k+1} f_{d-1} - \binom{d+1}{k-1} k$ for $1 \leqq k \leqq d-2$

where f_i is the number of i-dimensional faces of the polytope. In this paper we show that inequality (i) holds for all simple polytopes.

Let P be a d-dimensional *convex polytope*, that is, a d-dimensional set that is the convex hull of a finite number of points. If each vertex of P has valence d then P is called a *simple* d-polytope; if each facet of P is a simplex then P is called a *simplicial* d-polytope. The term d-dimensional convex polytope shall hereafter be abbreviated *d-polytope*. Let f_i be the number of i-dimensional faces (abbreviated, *i-face*) of P. For many years lower bounds on f_i in terms of f_{d-1} for simple polytopes have been sought. (An interesting history of the problem can be found in Grünbaum [2, p. 188].) The problem is known as the Lower Bound Conjecture. The conjectured bounds are

(i) $$f_0 \geqq (d-1)f_{d-1} - (d+1)(d-2)$$

(ii) $$f_k \geqq \binom{d}{k+1} f_{d-1} - \binom{d+1}{k-1} k \text{ for } 1 \leqq k \leqq d-2.$$

Let P be a d-polytope obtained from the d-simplex by repeated truncations of vertices. We shall call such a polytope a *truncation polytope*. It is well known

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that equality holds in (i) and (ii) for any truncation polytope. It has been conjectured that these are the only polytopes for which equality holds except when $d = 3$ (for $d = 3$ the cube is a counterexample). We shall prove that (i) holds and that, for $d \neq 3$, equality holds only for truncation polytopes. By duality we will have lower bounds for the number of facets of a *simplical* d-polytope in terms of the number of vertices. We shall also observe that the same lower bounds on f_{d-1} in terms of f_0 may be obtained for triangulated $(d-1)$-manifolds.

We begin with some definitions and basic facts about simple polytopes and cell complexes. A *d-cell complex* is a collection $\mathscr{C}$ of k-polytopes $-1 \leqq k \leqq d$ such that

a) if $F \in \mathscr{C}$ then every face of F is in $\mathscr{C}$;
b) if F_1 and F_2 are in $\mathscr{C}$ then $F_1 \cap F_2$ is a face of both (possibly the empty face)
c) each k-cell is a face of a d-cell in $\mathscr{C}$.

A d-cell complex $\mathscr{C}$ is *strongly* connected if

d) given any two d-cells, F_1 and F_n, there is a sequence $F_1, F_2, \cdots, F_n$ such that $F_i \cap F_{i-1}$ is a $(d-1)$-cell. We shall use the following theorem of Sallee [1, p. 470]:

The graph of a strongly connected d-cell complex is d-connected. By the *graph* of a d-cell complex $\mathscr{C}$ we mean the graph formed by the 1-skeleton of $\mathscr{C}$.

If F is a face of a d-cell complex $\mathscr{C}$, then we define $\mathrm{Star}(F,\mathscr{C})$ to be the complex consisting of all faces of $\mathscr{C}$ meeting F and all of the faces of these faces. We define $\mathrm{Ast}(F,\mathscr{C})$ to be the complex consisting of all faces of $\mathscr{C}$ that miss F. We define $\mathrm{Link}(F,\mathscr{C}) = \mathrm{Ast}(F,\mathscr{C}) \cap \mathrm{Star}(F,\mathscr{C})$. If $\mathscr{C}$ is a cell complex consisting of facets of a simple d-polytope P and their faces, we define a vertex of $\mathscr{C}$ to be an *exterior* vertex of $\mathscr{C}$ if it meets an edge of P not in $\mathscr{C}$. In other words the exterior vertices are those which are $(d-1)$-valent in $\mathscr{C}$.

THEOREM 1. *If P is a simple d-polytope then*

$$f_0 \geqq (d-1)f_{d-1} - (d+1)(d-2).$$

PROOF. Let P be a simple d-polytope with f_0 vertices and let v be any vertex of P. Let $\mathscr{C}_0 = \mathrm{Ast}(v,P)$. Then $\mathscr{C}_0$ is a strongly connected $(d-1)$-cell complex and each of the d vertices of P adjacent to v is an exterior vertex of $\mathscr{C}_0$. We may suppose that $\mathscr{C}_0$ contains at least two $(d-1)$-cells, since otherwise P would be a d-simplex for which the theorem is trivial. The desired inequality (i) is equivalent to $f_0 \geqq (d+1) + (d-1)(f_{d-1} - d - 1)$, which follows easily from repeated application of the following lemma.

LEMMA 1. *Suppose $\mathscr{C}$ is a strongly connected $(d-1)$-cell complex which is a subcomplex of the boundary complex of a simple d-polytope. Suppose further that $\mathscr{C}$ contains at least two $(d-1)$-cells and at least one exterior vertex. Then there exist distinct vertices $v_1, \cdots, v_{d-1}$ and two nonempty strongly connected $(d-1)$-cell subcomplexes $\mathscr{C}_1$ and $\mathscr{C}_2$ of $\mathscr{C}$ such that:*

a) *$\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ and $\mathscr{C}_1 \cap \mathscr{C}_2$ is of dimension $d-2$.*

b) *Each v_i is an exterior vertex of both $\mathscr{C}_1$ and $\mathscr{C}_2$.*

c) *No v_i is an exterior vertex of $\mathscr{C}$ or any $(d-1)$-cell complex containing $\mathscr{C}$.*

PROOF OF LEMMA 1. Note that (c) will follow immediately from (b). Let v' be an exterior vertex of $\mathscr{C}$ and let F be the unique facet that contains v' and is in $\mathscr{C}$. We shall use a dual graph, G, defined as follows: The vertices of G will correspond to facets in $\mathscr{C}$ with two vertices being joined if and only if the corresponding facets meet on a subfacet. Let V_F be the vertex of G corresponding to F and let E be a minimal set of edges of G with endpoint V_F separating G. It is clear that we have separated G into two components and these components determine two strongly connected $(d-1)$-cell complexes $\mathscr{C}_1$ and $\mathscr{C}_2$ of $\mathscr{C}$ (we shall assume $F \in \mathscr{C}_1$). Similarly E determines a $(d-2)$-cell subcomplex E' of Ast (V, F). Since in a simple polytope two facets meet only if they meet on a subfacet, we see that $\mathscr{C}_1 \cap \mathscr{C}_2 = E'$ and E' separates $\mathscr{C}$ topologically.

We let F' be a facet of $\mathscr{C}_2$ that meets F on a subfacet and let v'' be a vertex of F that is not on E'. By Sallee's theorem we may choose $d-1$ independent paths from v' to v''. Let v_i be the first vertex on the ith path from v' to v'' which is also a vertex of E' and let e_i be the preceding edge. The edge e_i demonstrates that v_i is an exterior vertex of $\mathscr{C}_2$. It remains to be shown that v_i is an exterior vertex of $\mathscr{C}_1$. By the definition of E', v_i is incident on a subfacet in E', which in turn is incident on F and some facet F''' of $\mathscr{C}_2$. Let e_i' be the unique edge of F''' meeting v_i' which is not in F. If e_i' is not an edge of any facet in $\mathscr{C}_1$, then v_i is in an exterior vertex of $\mathscr{C}_1$. Otherwise, there is some facet F'' in $\mathscr{C}_1$ with e_i' as edge. Since P is simple and both F'' and F''' have v_i as vertex, they must meet in a common subfacet, which cannot be in E' because $E' \subset \mathrm{Ast}(v, F)$. This implies that E' does not separate $\mathscr{C}_1$ and $\mathscr{C}_2$, which is a contradiction. Thus v_i is also an exterior vertex of $\mathscr{C}_1$, and the proofs of the Lemma and Theorem 1 are complete.

By duality we have

(iii) $\qquad f_{d-1} \geqq (d-1)f_0 - (d+1)(d-2)$ for all simplicial polytopes.

polytopes. The theorem is now completed by observing the following which can be easily proved by induction.

LEMMA 2. *If d-polytopes P_1 and P_2 are isomorphic to stacked polytopes and if $P_1 \cap P_2$ is a facet of each, then $P_1 \cup P_2$ is isomorphic to a stacked polytope.*

REMARKS.

1) Theorem 2 can be modified to prove that, if equality (i) holds for a triangulated manifold $\mathscr{M}$, then $\mathscr{M}$ is isomorphic to the boundary of a stacked polytope.

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REFERENCES

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We could have dualized our entire proof giving a direct proof of (iii). This would have had one advantage, namely that the same proof would then work for all triangulated manifolds, again giving inequality (iii). This was not done for two reasons. First it loses its intuitive appeal and second, no dual form of Sallee's theorem has been published. The dual form of Sallee's theorem can be proved using theorems by D. W. Walkup (unpublished) on lattices of manifolds.

Let P be a d-polytope and let v be a point chosen close enough to some relative interior point of a facet F of P so that $\mathrm{con}(\{v\} \cup P) = \mathrm{con}(\{v\} \cup F) \cup P$. We say that the polytope $\mathrm{con}(\{v\} \cup P)$ is obtained from P by capping. A *stacked polytope* is one that is obtained from the simplex by repeated capping.

Using (iii) for triangulated manifolds we can prove

THEOREM 2. *If, for a simplical d-polytope P,*

(iv) $$f_{d-1} = (d-1)f_0 - (d+1)(d-2)$$

and if $d > 3$ then P is isomorphic to a stacked polytope.

PROOF. Suppose that for a simplical d-polytope P, (iv) holds. We apply the proof of Theorem 1 to the dual P^* of P. In the proof we apply Lemma 1 f_{d-1}—$d-1$ times to the antistar of some vertex of P^*. The last application of Lemma 1 will be to write a cell complex $\mathscr{C}$ as the union of two facets F_1 and F_2 of P^*. Each vertex of the subfacet $F_1 \cap F_2$ will be exterior to F_1 and F_2 but not to $\mathscr{C}$, thus if (iv) holds there are only $d-1$ vertices on $F_1 \cap F_2$, and $F_1 \cap F_2$ is a $(d-2)$-simplex. Corresponding to this subfacet will be an edge e in P whose linked complex, L, is the boundary of a $(d-2)$-simplex. Unless L is spanned by a $(d-2)$-face of P in the antistar of e we may create a new $(d-1)$-sphere from the boundary of P by replacing the star of e by two topological simplices whose intersection is a $(d-2)$-cell spanning L. If $d > 3$ then we have a manifold for which (iii) does not hold, which is a contradiction.

We may now assume that L is spanned by a $(d-2)$-cell S in the antistar of e. Let v be a vertex of e. Consider the set consisting of S and all $(d-2)$-faces of the star of e that meet v. This set is the boundary of a $(d-1)$-simplex S_1. Similarly, we can get a simplex S_2 by using the other vertex of e. If both of these simplices are facets of P, then P is a simplex and we are done. If not, then the affine hull of one of them, say S_1, separates P into two polytopes P_1 and P_2. If equation (iv) does not hold for P_1, then we may replace P_1 by a stacked polytope P_3 and obtain a polytope $P_3 \cup P_2$ that contradicts (iii). The same reasoning applies to P_2, thus P_1 and P_2 satisfy (i) and by induction they are isomorphic to stacked