COLORING POLYHEDRAL MANIFOLDS[a]

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1. INTRODUCTION

It is well known that the four color conjecture for maps on the sphere is equivalent to the four color conjecture for convex three-dimensional polyhedra (see Section 4). It is also well known that every map on the torus can be colored with seven or fewer colors, and that there are toroidal maps requiring seven colors [3]. One might guess that there exist toroidal polytopes that require seven colors. Surprisingly, this is not the case. We shall prove that every toroidal polytope can be colored with six or fewer colors.

2. DEFINITIONS

If a graph is embedded in a surface, the connected components of the complement of the graph are called its *faces*. If every face is a cell, we shall call the graph a *map*. The *dual* of a graph in the torus is constructed the same way as the dual of a graph in the plane. We place a vertex in each face and join vertices when the corresponding faces meet on an edge. The vertices are joined by one edge for each edge that the two faces have in common. Generally, the edges of the dual are drawn so that they cross the corresponding edges of the original graph.

A *toroidal polytope* is a topological torus consisting of convex polygons such that:

(i) The intersection of two polygons is either an edge of both, a vertex of both, or empty.

(ii) No two polygons that meet lie on the same plane.

The polygons will be called the *faces* of the polytope.

A *polyhedral immersion* of a torus is the continuous image in E^3 of a toroidal map such that:

(i) The image of each n-sided country of the map is a convex n-gon with the vertices of the country taken one-to-one onto the vertices of the n-gon.

(ii) No two n-gons in the immersion that meet on an edge are coplanar.

By a *vertex*, *edge*, or *face* of the immersion we shall mean the images of a vertex edge or country of the map. The *valence* of a vertex of the immersion is the valence of its pre-image in the map.

A toroidal polytope is *simple* provided each vertex has valence 3. A polyhedral immersion is *simple* provided it is the image of a map on the torus all of whose vertices are 3-valent.

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Since the vertices and edges of a toroidal polytope form a graph on the torus, we can speak of the dual graph of a toroidal polytope. Since two polygons can meet only on a vertex or an edge or not at all in a polytope, we see that the dual graph has no multiple edges or loops.

We shall need an inequality that follows from Euler's equation for graphs embedded in the torus. For any graph embedded in the torus, let V be the number of vertices, E the number of edges, and F the number of faces of the graph.

LEMMA 1. *For any graph in the torus we have*

$$V - E + F \geq 0,$$

with equality if each face is a cell.

(The reader is referred to [1] where a good treatment of Euler's equation for orientable surfaces is given.) We shall call this *Euler's inequality*. In the case where we have equality it is known as *Euler's equation* for the torus.

3. THE MAIN RESULT

LEMMA 2. *If v_i is the number of i-valent vertices in a graph without loops and multiple edges on the torus, then*

$$\sum (6 - i)v_i = 0$$

if and only if the graph is a triangulation of the torus. Otherwise, the sum on the left is greater than 0.

Proof. Since the graph is without loops and multiple edges, every face has at least three edges, and we have

$$3F \leq 2E$$

with equality if and only if every face is a triangle. We also have Euler's inequality, $V - E + F \geq 0$, with equality if each face is a cell.

The sum $\sum (6 - i)v_i$ equals $6V - 2E$. Combining our two inequalities, we get $6V - 2E \geq 0$, with equality if and only if every face is a triangle. From this the desired result follows. ▮

LEMMA 3. *There are no simple polyhedral immersions of the torus.*

Proof. We add the sum of the two-dimensional angles of the faces of such an immersion in two different ways and get a contradiction. Since there are exactly three faces meeting at each vertex, the sum of the angles is less than 2π at each vertex. Thus the sum of the angles is less than $2\pi V$. The sum of the angles of an n-sided face is $\pi(n - 2)$. Summing these quantities over all faces gives us

$$\sum \pi(n - 2) = 2\pi E - 2\pi F.$$

We conclude that $2\pi V > 2\pi E - 2\pi F$, which contradicts Euler's equation for the torus. ▮

THEOREM. *Toroidal polytopes are 6-colorable.*

Proof. Let T be a toroidal polytope and let G be its dual graph. We shall show that we can remove vertices from G, one at a time, such that at each step the vertex to be removed will have valence at most 5 in the graph that we have at that step.

Once this has been shown, the result is immediate because we can remove vertices one at a time until at most six remain, color them, and then return them one at a time assigning a color to each as it is returned. Since the returned vertex is at most 5-valent, there is always a color available for it.

We can choose a vertex of valence at most 5 in G because, by LEMMA 3, T is not simple; thus, G is not a triangulation of the torus; thus, $\sum (6 - i)v_i > 0$, for G, and so such a vertex exists in G. What we must show is that as we proceed, we will always be able to choose such a vertex at later steps.

By LEMMA 2, we know that $\sum (6 - i)v_i > 0$, provided there is a face with at least four edges. Suppose that at some step we have produced a proper subgraph H of G such that H has no faces with at least four edges. In this case, H is a graph whose faces are all triangles. If H is planar, then it is well known that there will be a vertex of valence at most 5 (see, for example, [1, Section 4.1]). If the graph is not planar, then H is a triangulation of the torus which is a proper subgraph of G.

We shall show that this is impossible.

We do this by showing that it implies the existence of an immersion of a simple toroidal polytope. Suppose that u, v, and w are three vertices of G that determine a face of H, but do not determine a face of G. Corresponding to these three vertices will be three faces U, V, and W, of the polytope T. These three faces will form a region that is a topological annulus. This annulus encloses a topological cell C consisting of faces of T that correspond to the vertices of G enclosed by the circuit determined by u, v, and w. We now modify the polytope. The faces U, V, and W meet pairwise on three edges. Each of these three edges has one vertex in C. We take a plane through these three vertices. This plane may cut some of the faces U, V, and W into two pieces. In each case where a face is cut into two pieces we throw away the piece that has an edge in common with C. We also throw away all of C, and fill in the "hole" with the convex hull of the three vertices.

We carry out this procedure for each triangle in G that is a face of H but not of G. When we are done we will have faces corresponding to each of the vertices of H and no other faces. The graph H will be the dual graph of the immersion of the toroidal polytope that we have constructed. Since H is a triangulation, we have created a simple polyhedral immersion of the torus, a contradiction. ∎

4. REMARKS

If we substitute "2-manifold" for "torus" in the definition of toroidal polytope we get the definition of a *polyhedral 2-manifold*. The author knows of no polyhedral manifolds of any genus requiring more than four colors. Recently, McMullen *et al.* [4] developed some new techniques of construction that produce polyhedral 2-manifolds all of whose faces are hexagons or all octagons. Perhaps among these is an example requiring five or six colors. The few that the author has tried can be colored with four colors. The author conjectures that all toroidal polytopes are 4-colorable.

In view of the results in [4], there exist polyhedral 2-manifolds whose average face size is at least 8. Do there exist polyhedral 2-manifolds with arbitrarily large average face sizes?

LEMMA 2 is a special case of a theorem of Grünbaum [2, Chapter 11, exercise 7] that every simple cell complex is the boundary complex of a convex polytope.

To prove the equivalence of the four color problem for maps in the plane and for convex polytopes, one first reduces the four color conjecture to the case of 3-connected graphs (a simple argument will do this), then one uses the theorem of Steinitz [5] which states that the planar 3-connected graphs are the graphs of the convex three-dimensional polytopes.

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