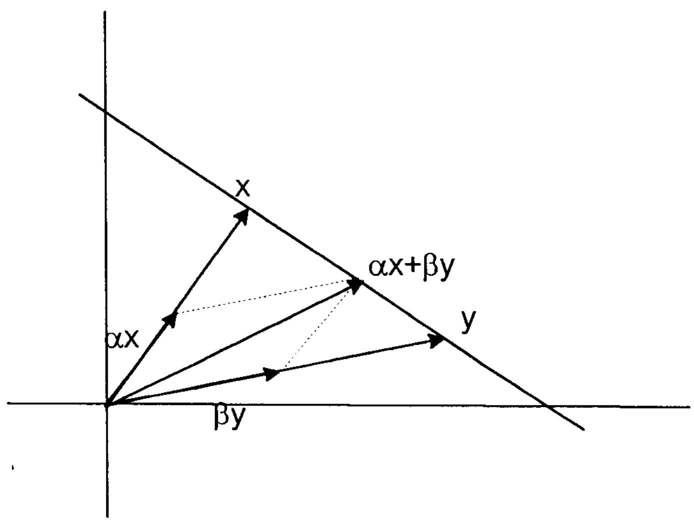
The Geometry of Convex Sets

1. The Algebraic Aspects of Convexity

Definition 1.1: A set S in E^n is <u>convex</u> iff given any two distinct points x and y in S, the segment joining x and y is in S.

Definition 1.2: A set S in E^n is <u>affine</u> iff given any two distinct points x and y in S, the line through x and y is in S.

There is a nice algebraic way of expressing segments and lines determined by two points x and y. Think of x and y as vectors and consider the sum $\alpha x + \beta y$, where $\alpha + \beta = 1$.



It is easy to see that this sum is a vector on the line through x and y by showing that $x - y$ and $x - (\alpha x + \beta y)$ are parallel vectors. To do this we simply show that one is a multiple of the other. This holds because $x - (\alpha x + \beta y) = (1-\alpha)x - \beta y = \beta x - \beta y = \beta(x-y)$. Note that if α and β are positive, as shown in the figure, we will be getting the points between a and y.

As a result of these observations we can reformulate our first two definitions:

Definition 1.3: A set S in E^n is affine (convex) iff for every x and y in S, and for all α and β such that $\alpha+\beta=1$ ($\alpha > 0$ and $\beta > 0$, with $\alpha+\beta=1$), the point $\alpha x+\beta y$ is in S.

If we take a combination $\alpha x+\beta y$ without any restrictions on α and β we have what is known as a linear combination of x and y, thus we shall make the following definition.

Definition 1.4: A set S in E^n is linear iff for each x and y in S and each α and β, the point $\alpha x+\beta y$ is in S.

There is one other type of set whose definition is similar to these.

Definition 1.5: A set S in E^n is positive iff for each x and y in S and for each $\alpha > 0$ and $\beta > 0$, the point $\alpha x+\beta y$ is in S.

Note that linear, positive and affine sets are convex.

We shall examine some of the basic algebraic properties of these sets.

Theorem 1.1: A set S is linear iff it is affine and contains **0** (**0** will be our symbol for the origin, or the zero vector.)

Proof: Clearly, from Definitions 1.3 and 1.4, a linear set is affine. Setting α and β equal to 0 in Definition 1.4 shows that **0** is in any linear set. To show the converse, suppose that S is affine and contains **0**. Let x and y be in S and let α and β be two real numbers. We shall show that $\alpha x+\beta y \in S$.

Case I. $\alpha+\beta \neq 0$. Let $z = (\alpha/(\alpha+\beta))x + (\beta(\alpha+\beta))y$. Since S is affine, $z \in S$. Now, $\alpha x+\beta y = (\alpha+\beta)z + (1-\alpha-\beta)\mathbf{0}$, thus $\alpha x+\beta y \in S$.

Case II. $\alpha+\beta = 0$. We shall assume without loss of generality that $\beta \neq -1$. Now let $z = (\beta/(1+\beta))y + (1/(1+\beta))\mathbf{0}$. The sum of the coefficients is 1, thus $z \in S$. Now $\alpha x+\beta y = \alpha x + (1+\beta)z$. The sum of the coefficients on the right hand side is 1, thus

$\alpha x + \beta y \in S$.

Theorem 1.2: A set is linear iff it is affine and positive.

Proof: It follows from the definition of linear that a linear set is affine and positive. Conversely, suppose that S is affine and positive. Let $x \in S$. Since S is positive, $2x \in S$. Now, $\mathbf{0} \in S$ because $-1(2x) + 2(x) = \mathbf{0}$ (note that the sum of the coefficients is 1). Now, S is affine and contains $\mathbf{0}$ thus S is linear.

Recall that a translate of a set S is a set $S + x$ which is defined to be $\{s+x \mid s \in S\}$

Theorem 1.3: A set is affine iff it is a translate of a linear set.

Theorem 1.4: A set S is positive iff it is convex and for each point in S the open ray from the origin through that point is a subset of S.

Exercises

1. Prove that the properties of being affine or convex are preserved by translations (ie. prove that a set is convex iff every translate of it is convex, and do the same for affine sets.) Does the same hold for linear and positive sets?
2. Prove Theorem 1.3.
3. Prove Theorem 1.4.
4. Prove that if two translates of an affine set intersect then the two translates are the same set.

Many of the following definitions and theorems about convex sets are also true for affine positive and linear sets. When this is so there will be a star on the word "convex" meaning that each starred occurrence of "convex" may be replaced by "affine", "positive", or "linear", and the theorem remains true.

Thm: the intersection of convex* sets is convex.

Proof: Let x and y be in the intersection of a family of convex sets. Then x and y are in each convex set. Thus for any $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta = 1$, $\alpha x + \beta y$ is in each convex set, and thus $\alpha x + \beta y$ is in the intersection. The intersection therefore satisfies the definition of convex.

Definition 1.6: The convex* hull of a set is the intersection of all convex* sets containing it.

The convex hull, affine hull, positive hull and linear hull of a set S are denoted by con S, aff S, pos S, and lin S, respectively.

Theorem 1.6: The convex hull of S is the smallest (with respect to containment) convex set containing S.

Proof: By Theorem 1.5 con S is convex. Suppose that there is a convex set C containing S and properly contained in con S. Then C is one of the convex sets in the family that is intersected to form con S, thus con $S \subset C$, which contradicts the strict containment of C in con S.

Definition 1.7: let $x_1,\ldots,x_n$ be points. Consider a sum of the form $\Sigma\, \alpha_i x_i$

(1) The sum is called a linear combination iff the α_i's are real numbers.

(2) The sum is an affine combination iff $\Sigma\, \alpha_i = 1$.

(3) The sum is a positive combination iff $\alpha_i \geq 0$ for all i and $\Sigma\, \alpha_i > 0$.

(4) The sum is a convex combination iff $\Sigma\, \alpha_i = 1$ and $\alpha_i \geq 0$ for all i.

Definition 1.8: Let S be a set of points. The set of all convex* combinations of points of S involving n points is called $con_n S$.

Theorem 1.7: The set of all convex* combinations of points of a set S is a convex* set.

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Proof: Let X be the set of all convex combinations of points in S, and suppose that x and y are in X. We can write

$$x = \Sigma\, \alpha_i x_i\,,\ \ x_i \in S,\ \ \Sigma\, \alpha_i = 1,\ \ \alpha_i \geq 0.$$

$$y = \Sigma\, \beta_i y_i\,,\ \ y_i \in S,\ \ \Sigma\, \beta_i = 1,\ \ \beta_i \geq 0.$$

Now consider the combination $\alpha x + \beta y$, $\alpha + \beta = 1, \alpha > 0, \beta > 0$. This is the combination $\Sigma\, \alpha\alpha_i x_i + \Sigma\, \beta\beta_i y_i$ which is a sum whose coefficients sum to 1 and are non-negative, thus this combination is in X. We have shown that X satisfies the definition of convex.

Theorem 1.8: For every n $\mathrm{con}^*{}_n S \subset \mathrm{con}^* S$.

Proof: The proof is by induction on n. For $n = 1$, the statement becomes $S \subset \mathrm{con}\, S$. For $n = 2$, the statement becomes, for each x and y in S the segment from x to y is in S. Suppose that the statement is true for $n = k\text{-}1$ and let $x \in \mathrm{con}_k\, S$. Then

$$x = \sum_{i=1}^{k} \alpha_i x_i\,,\ \ \alpha_i \geq 0,\ \ \Sigma\, \alpha_i = 1,\ \ x_i \in S.\ \text{We may assume that } \alpha_k \neq 1.$$

Let $y = \sum_{i=1}^{k-1} (\alpha_i/(1\text{-}\alpha_k))x_i$. the point y is now an element of $\mathrm{con}_{k-1}\, S$, and by induction is an element of con S. But x_k is an element of S, so $\alpha_k x_k + (1\text{-}\alpha_k)y = x$ is in con S. Thus $\mathrm{con}_k\, S \subset \mathrm{con}\, S$, and the inductive step is completed.

Theorem 1.9: The convex* hull of S equals the set of all convex* combinations of points in S.

Proof: Let X be the set of all convex combinations of points of S. Theorem 1.8 implies that $X \subset \mathrm{con}\, S$. Theorem 1.7 gives us that X is convex, but by Theorem 1.6, any convex set containing S and lying in con S must be con S.

In linear algebra the idea of linear independence was introduced. The following definitions and theorems show that there are analogous ideas for "affine", "positive" and "convex".

Definition 1.8: A point x is convexly* dependent on a set S iff it is in con S, otherwise it is called convexly* independent of S.

Definition 1.9: A set S is convexly* independent iff each point is convexly* independent of the other points of S. If S is not convexly* independent it is called convexly* dependent.

In linear algebra there were certain linearly independent sets that were called bases for the space. With sets that are affine, positive or convex we have the analogous ideas of bases.

Definition 1.10: A set B is a convex* basis for a set S iff B is convexly* independent and con* B = S.

Recall from linear algebra that every basis for n-space has exactly n points, and in fact, for any given space, every basis has the same number of points. (Here we would call these bases "linear bases"). For convex and positive, the behavior is somewhat different.

For any integer n, there are sets in the plane with a convex basis of n points. There are sets in the plane with infinite convex bases. Strangely, the plane itself has no convex basis. An affine basis for n-space always has n+1 points. There are positive bases for n-space with anywhere between n+1 and 2n points.

Affine bases share similar properties with linear bases. In particular:

Theorem 1.10: A set B in E^n is affinely independent iff every point has a unique representation as an affine combination of points of B.

The coefficients in the affine combination are called the affine coordinates of the point.

Theorem 1.11: A set of n+2 or more points in E^n is affinely dependent.

Since every point in E^n is a linear combination of basis elements, it follows that every point x in E^n is in $\text{lin}_n E^n$. In the plane there are sets with convex bases containing arbitrarily many points. Nonetheless, if x is in a convex set in the plane, x can be expressed as a convex combination of three or fewer convex basis elements. For example, if S were a convex 10-gon then the ten vertices would be a convex basis for S. If x is in S we can draw chords from one vertex to the others, breaking S up into triangles. The point x must be in one of the triangles, thus it is in the convex hull of three of the vertices. This property has the following generalization:

Theorem 1.12: (Caratheodory's Theorem) If $x \in \text{con}\, S \subset E^n$, then $x \in \text{con}_{n+1}\, S$.

Proof: First we show that x is in the convex hull of some finite subset of S and then show that if the finite subset has more than n+1 points we can reduce the number of points in the set and still have x in its convex hull.

By Theorem 1.9, con S is the set of convex combinations of points in S. thus x is a convex combination of some finite set K of points of S. But again by Theorem 1.9 this puts x in con K. Thus x is in the convex hull of a finite subset K of S. Suppose that K has more than n + 1 points. Call them $x_1,\ldots,x_k$. Since K has more than n+1 points it is affinely independent, thus we can write

$x_j = \sum_{i \neq j} \beta_i x_i$, $\Sigma\, \beta_i = 1$. Now let us define $\beta_j = -1$ (note that β_j has not appeared in the above sum). Now we have

$$\Sigma\, \beta_i x_i = \mathbf{0}. \quad S\, \beta_i = 0.$$

Since $x \in \{x_1,\ldots,x_k\}$ we can also write

$$x = \Sigma\, \alpha_i x_i\, ,\ \Sigma\, \alpha_i = 1\, ,\ \alpha_i > 0.$$

Now we shall look at all β_i' s that are negative and examine the numbers $-\alpha_i/\beta_i$. These are all positive numbers. We shall let τ be the smallest of these numbers. Now we look at the sum

$$\Sigma\,(\alpha_i + \tau\beta_i)x_i = \Sigma\, \alpha_i x_i + \tau\Sigma\, \beta_i x_i = \Sigma\, \alpha_i x_i + \mathbf{0} = x.$$

The coefficients are positive when β_i is positive, but when β_i is negative $\tau\beta_i \leq -\alpha_i$, so even in this case the coefficients are positive. The sum of the coefficients is

$\Sigma\, \alpha_i + \tau\Sigma\, \beta_i = 1 + 0$. Thus this sum expresses x as a convex combination of the points $x_1,\ldots,x_n$. However, note that one of the coefficients (the one for which $t = -\alpha_i/\beta_i$) is 0, thus x is now in the convex hull of a smaller set.

Theorem 1.13 (Radon's Theorem): If S is a set of n+2 points in E^n then S can be partitioned into two sets of points A and B such that con A $\cap$ con B $\neq \emptyset$.

Proof: Since S has n+2 points, it is affinely dependent and thus as in the previous theorem we can write

$$\mathbf{0} = \Sigma\, \alpha_i x_i\,,\ \ \Sigma\, \alpha_i = 0.$$

Now, let $A = \{x_i \mid \alpha_i > 0\}$ and $B = \{x_i \mid \alpha_i < 0\}$. Let $\alpha = \sum_A \alpha_i$.

We can now write $\sum_A (\alpha_i/\alpha)x_i = \sum_B (-\alpha_i/\alpha)x_i$. One side of the equation is a convex combination is points in A while the other is a convex combination of points in B. Thus each sum represents a point in the convex hulls of both A and B.

Exercises

1. Prove that if X and Y are convex then

    con $(X \cup Y) = \{\, \alpha x + \beta y \mid x \in X,\ y \in Y,\ \alpha,\beta \geq 0, \text{ and } \alpha + \beta = 1\}$. (In other words, the convex hull of X union Y is the union of all segments from X to Y.) Is a similar statement true when we replace "convex" by "affine"?

2. Prove that convex and affine independence are invariant under translations. (That is, a set is convexly independent iff every translate is convexly independent, etc.) Is this also true for positive and linear independence?

3. Prove that every translate of a linearly independent set is affinely independent.

4. Prove that if X is affinely independent then for each $x \in X$, the set $(X - x) - \{\mathbf{0}\}$ is linearly independent.

5. Using your knowledge of linear algebra prove that any maximal affinely independent set in E^n has exactly $n+1$ points.

6. Prove $\text{lin } S = \text{aff}(\text{pos } S)$, while it is not necessarily true that $\text{lin } S = \text{pos}(\text{aff } S)$.

7. Prove that every affinely independent set in E^n can be extended to an affine basis for E^n.

8. Prove that $\text{aff}_n S \subset \text{aff } S$ and $\text{pos}_n S \subset \text{pos } S$.

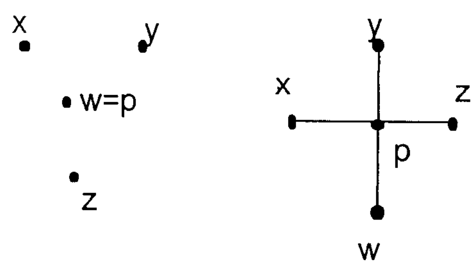
2. Helly's Theorem

In this chapter we look at a strange intersection property of convex sets. The property is given in what is known as Helly's theorem. We shall actually have two Helly Theorems, one for finite families of convex sets and one for infinite families. We shall also see some applications to geometric problems. We begin with a special case.

Theorem 2.1: <u>If A, B, C, and D are four convex sets in the plane, and if the intersection of each three of these sets is nonempty, then the intersection of all four is nonempty.</u>

<u>Proof:</u> Let $x \in B \cap C \cap D$, $y \in A \cap C \cap D$, $z \in A \cap B \cap D$, and $w \in A \cap B \cap C$.

By Radon's Theorem, the set $\{x,y,z,w\}$ can be partitioned into two nonempty sets X and Y such that con $X \cap$ con $Y \neq \emptyset$. Let $p \in$ con $X \cap$ con Y. We illustrate the possible cases:



case I case II

One case (shown as case I) is where one of X or Y is a single point and the other set forms the vertices of a triangle. In this case we know that w is in A, B, and C by the choice of w, but since x, y, and z are all in D, and D is convex, w is in D. Thus w is in the intersection of all four sets.

The other case (shown as case II) is where X and Y are both sets of two points. The convex hulls of these two sets are two segments which intersect at some point p. Now, since x and z are both in B and D, and since B and D are convex, we have that p is in B and D. Since y and w are both in A and C, and since A and C are both convex, p is in A and C. Now p is in all four sets, in other words it is in their intersection.

Theorem 2.2: If $A_1,\ldots,A_n$ is a family of convex sets in the plane such that the intersection of each three is nonempty then the intersection of all n sets is nonempty.

Proof: The proof is by induction on n. The previous theorem starts the induction when $n = 4$ (the theorem is trivial for $n < 4$). Suppose that the theorem is true when $n = k$ and consider convex sets $A_1,\ldots A_k, A_{k+1}$. We look at the following collection of sets: $A_1,\ldots A_{k-1}, (A_k \cap A_{k+1})$. This is a family of k convex sets in the plane. If each three sets in this family have nonempty intersection then we can apply our inductive hypothesis and conclude that there is a point lying in all of them. Such a point, however, would also lie in each of our original k+1 sets. Let us consider the intersection of any three sets in this family. If we take the three sets from $A_1,\ldots A_{k-1}$ then, of course, they have nonempty intersection by hypothesis. If we choose $(A_k \cap A_{k+1})$ and two other sets then their intersection is just the intersection of four sets of our original family. For these four sets we know that if each three have nonempty intersection then all four have nonempty intersection, by Theorem 2.1. Thus our new family has the required intersection property and by induction we have the point we want that lies in all of the sets.

Theorem 2.3 (Helly's Theorem, finite version): If for a finite family of convex sets in E^n, each n+1 sets have nonempty intersection then the entire family has nonempty intersection.

In order to understand the next version of Helly's Theorem we need some definitions from the field of topology.

Definition 2.1: A point x is a limit point of a set S in E^n iff each ball centered at x contains points of S other than x.

Definition 2.2: A set is closed iff it contains all of its limit points.

Definition 2.3: A point x of a set S in E^n is an interior point iff there exists a ball centered at x lying entirely in S.

Definition 2.4: A set S in E is open iff each point of S is an interior point of S.

Definition 2.5: A point x of a set S in E^n is a boundary point iff each ball centered at x contains points of S and points not in S.

Definition 2.6: The boundary of a set S in E^n is the set of its boundary points.

People often confuse the term "bounded" with the notion of a boundary of a set. One should note that these are entirely different ideas.

It should be noted that the interior and the boundary of a set depend not only on the set but also on the space that it is in. A disc in the plane has interior points and it's bopundary is a circle. A disc in E^3 has no interior points and every point is a boundary point. Note also that every point of a set is either an interior point or a boundary point.

Definition 2.7: A point x of a set S in E^n is a relative interior point of S iff it is an interior point of S in the affine hull of S.

For example, the center of a disc in E^3 is a relative interior point of the disc but not an interior point.

Definition 2.8: A set S in E^n is bounded iff there is a sphere enclosing it (ie. the set is contained in some ball).

We are now ready to state the second form of Helly's Theorem:

Theorem 2.4 (Helly's Theorem, bounded version): If $\mathcal{C}$ is a collection of closed bounded convex sets in E^n such that each $n+1$ sets have nonempty intersection then the intersection of all sets in $\mathcal{C}$ is nonempty.

Note that in this version of the theorem we can have infinitely many sets while in the first version only finitely many were allowed.

Many theorems in convexity are about closed bounded convex sets, thus these sets have a special name.

Definition 2.9: A closed bounded convex set is called a convex body.

We turn now to an application of Helly's Theorem dealing with diameters of sets.

Definition 2.10: The diameter of a set S in E^n is the maximum distance between points in S.

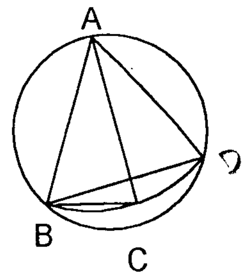
Note that a set may not have a diameter. A half plane has no diameter because distances between points can be arbitrarily large. An open disc (that is a disc without its bounding circle) Has no diameter because even though there is an upper bound on distances, no pair of points realizes a maximum distance.

We now consider the following problem: Suppose we have a set S of diameter 1 in the plane, and suppose that we don't know exactly which set of diameter it is. We would like to place a disc over the set so that the disc entirely covers the set S. Certainly we can do this with a disc of radius 1. all we would have to do is place the center of the disc over a point of the set and then it covers the set S. It would seem, however that one should be able to use a smaller disc. The question then becomes: What is the smallest radius of a disc that will cover every set in the plane of diameter 1?

Before we see how Helly's Theorem provides the answer we need a few geometric facts.

Lemma 2.1: If T is an isosceles triangle with two sides of length 1 and the other side of length at most 1, then T can be covered by a disc of radius $(1/3)\sqrt{3}$.

Proof: Let ΔABC be the triangle with sides AB and AC of length 1. We construct the equilateral triangle ΔABD as shown, and cover it with a disc of radius $(1/3)\sqrt{3}$.



Now we note that C lies on an arc centered at A of radius 1. Since this arc has radius greater than the radius of the circle, the portion joining B and C lies inside the circle, thus the disc covers ΔABC.

Lemma 2.2: A triangle all of whose sides are of length at most 1 can be covered by a disc of radius $(1/3)\sqrt{3}$.

Proof: We take the two side of a smallest angle of the triangle. Any of these two sides that has length less than one, we extend to produce a triangle with two sides of length 1. These sides have been chosen such that they lie on an angle of at most 60°, thus the triangle we have formed satisfies the hypotheses of Lemma 2.1. We can cover this triangle with a disc of radius $(1/3)\sqrt{3}$, and the same disc then covers the original triangle.

Theorem 2.5: Any set of diameter 1 in the plane can be covered by a disc of radius $(1/3)\sqrt{3}$.

Proof: Let S be a set a set of diameter 1 in the plane. Consider the collection $\mathcal{C}$ of all discs of radius $(1/3)\sqrt{3}$ centered at points of S. The centers of any three of them are the vertices of a triangle of maximum side length 1 (S had diameter 1). By Lemma 2.2, such a triangle can be covered by a disc D of radius $(1/3)\sqrt{3}$. This means that the center of D has distance at most $(1/3)\sqrt{3}$ form each of the centers of the original three discs. In other words, the center of D in each of the three original discs. The three discs are arbitrary, thus we have shown that each three discs in the collection $\mathcal{C}$ have a point in common. By Helly's Theorem, the collection of discs has nonempty intersection. Let x be a point in this intersection. The point x is thus of distance at most $(1/3)\sqrt{3}$ from every center, in other words, x is of distance at most $(1/3)\sqrt{3}$ from each point of S. Now the disc of radius $(1/3)\sqrt{3}$ centered at x covers S.

Here is an application dealing with the idea of fitting a line to data where there is a margin of error in the data. In such cases the data would be represented by segments rather than points when put into graphical form.

Theorem 2.6: Given a finite collection of vertical segments in the plane, there exists a line intersecting them all if for each three segments there is a line intersecting them.

Proof: The equation of a nonvertical line m in the plane is $y = ax + b$. Suppose that we consider a vertical segment with endpoints (x_1,y_1) and (x_1,y_2). To see if the line m intersects the segment we plug x_1 into the equation and see what value we get for y. If y lies between y_1 and y_2 the line intersects the segment. Assuming that $y_2 > y_1$, the line $y = ax + b$ intersects the segment iff $y_1 \le ax_1 + b \le y_2$. Now consider the ab-plane and the set S_1 of all pairs (a,b) such that $y = ax + b$ intersects the segment. The pairs (a,b) must satisfy the linear inequality $ax_1 + b \ge y_1$, and $ax_1 + b \le y_2$. The set of points in a plane satisfying a linear inequality is a half plane, thus S_1 is the intersection of two half planes and is thus a convex set.

Now, we have a convex set associated with each vertical segment. (The vertical segments are in the xy-plane while the associated convex sets are in a different plane, namely the ab-plane.) Let the convex sets be $S_1,\ldots,S_n$. Note that a point (a,b) being in S_i means that the line $y = ax + b$ intersects the i'th segment. Saying that a line intersects three segments is saying that there is a point (given by the coefficients in the equation of the line) in each of the corresponding convex sets. If each three segments can be intersected by a line then there is a point in each three of the convex sets. By Helly's Theorem, there is a point that lies in all of the convex sets. the coordinates of that point are the coefficients of a line intersecting every segment.

The next application deals with the idea of the existence of a point that serves as a "center" of a set, in the sense that every line through the point cuts the set into two pieces that come as close as possible to being the same size. Not every set admits a point such that every line through it cuts the set in half. In fact the best result that holds for all sets in the plane is the following:

Theorem 2.7: For any figure F in the plane, there exists a point p such that every line through p contains at least 1/3 of the area of F on each side.

Proof: Let $\mathcal{C}$ be the collection of closed half planes that contain more than 2/3 of the area of F. Consider the intersection of any three sets of $\mathcal{C}$. We want to show that the intersection is nonempty. The way to see this is to look at the complement of this intersection. By a theorem of set theory, the complement of the intersection is the union of the complements of the three sets. The complement of each of these sets contains less than 1/3 of the area of F, thus the union of these three complements does not contain all of F. The intersection of the three sets is therefore nonempty.

By Helly's Theorem, there is a point p that lies in a in all sets of $\mathcal{C}$. Consider any line m through p. Suppose that one side of m contains less than one third of the area of F. A very small translation of m will give another line n parallel to m still containing less than 1/3 the area of F on one side, and with p lying on that side. Now, the other side of n is a halfplane in $\mathcal{C}$, yet p does not lie in that halfplane, a contradiction.

Note: Neither of the Helly's theorems that we have had fits the conditions in the above theorem. What kind of modification in the problem is needed so that we may really apply one of these theorems?

For our next application we need to know about hyperplanes and halfspaces.

Definition 2.11: The dimension of an affine set is the number of points in an affine basis for the set, minus one.

Recall that each affine set is just a translate of a linear set so you can think of the dimension of the affine set as being the dimension of the corresponding linear set. Thus a line is of dimension one, a plane is of dimension 2, etc.

Definition 2.11: A hyperplane in E^n is an affine set of dimension $n-1$.

For example, in the line, the hyperplanes are points, in the plane the hyperplanes are lines, In 4-space the hyperplanes look like 3-space, and so on.

Just as lines in the plane are given by linear equations in two variables, hyperplanes in E^n are given by linear equations in n variables.

Theorem 2.8: H is a hyperplane in E^n iff H is the set of all points $(x_1,\ldots,x_n)$ satisfying the equation $a_1x_1+,\ldots,+a_nx_n = b$, for some real numbers $a_1,\ldots,a_n$, b.

Definition 2.12: If H is a hyperplane given by $\Sigma a_ix_i = b$, then the sets of points satisfying $\Sigma a_ix_i > b$, and $\Sigma a_ix_i < b$ are called the two open halfspaces of H. An open halfspace together with the points on the hyperplane form a closed halfspace.

Theorem 2.9: The two open halfspaces of a hyperplane H are convex and disjoint. Together with H they partition the space. Any segment joining a point in one halfspace with a point in the other halfspace will intersect H. The closed halfspaces are also convex.

Here's an application that shows when a hyperplane will separate two finite sets of points. (for this application we shall say that a hyperplane separates two sets provided the sets lie in different halfplanes of the hyperplane. In the next chapter we shall give a more formal definition and distinguish several different types of separation.)

Theorem 2.10: Let X_1 and X_2 be finite sets in E^n. If for each set of n+2 points $S \subset X_1 \cup X_2$, we have that a hyperplane exists that separates $S \cap X_1$ and $S \cap X_2$, then there exists a hyperplane that separates X_1 and X_2.

Proof: Note that the equation of a hyperplane is of the form $\Sigma a_1 x_1 = b$, and two points $z = (z_1,\ldots,z_n)$ and $y = (y_1,\ldots,y_n)$ will be separated by the hyperplane provided $\Sigma a_i z_i > b$ and $\Sigma a_i y_i < b$ (or the reverse inequalities) hold. For each such point z in X_1 we shall consider the set of all points $(a_1,\ldots a_n,b)$ such that $\Sigma a_i z_i > b$. This is the set of points in (n+1)-space satisfying a linear inequality, thus it is a halfspace and therefore is convex. For each point y in X_2 we associate the set of all $(a_1,\ldots,a_n,b)$ such that $\Sigma a_i y_i < b$. This is also a convex set in E^{n+1}. For each set S of n+2 points in $X_1 \cup X_2$ we have that a hyperplane separates the points of S in X_1 form the points of S in X_2. In other words there are coefficients $a_1,\ldots,a_n$, b such that the above inequalities hold for the points in S. This is the same as saying that the sets in E^{n+1} associated with the points in S have a point in common (namely the point giving these coefficients.) By Helly's Theorem, there is a point that lies in all of the sets that we have associated with the points of $X_1 \cup X_2$. The coordinates of that point will be the coefficients that make the required inequalities hold for all of the points in $X_1 \cup X_2$, thus they are the coefficients for the equation of a hyperplane separating X_1 and X_2.

And finally an application sometimes referred to as the "art gallery theorem".

Consider a polygon P in the plane. Suppose that a point p lies inside the polygon. We shall say that a point q on the polygon is visible from p if the segment from p to q meets the polygon only at q. (Note that we did not require the polygon to be convex. If P were convex then every point on P would be visible from every point inside P.)

Theorem 2.11: Let P be a polygon in the plane. If for each three points on P there is a point p inside the polygon from which the three points are visible, then there is a point inside the polygon form which every point of P is visible.

Exercises

1. Prove that for any positive integer n there exist n convex sets such that each two have a point in common but the intersection of the n sets is empty. (Hint: Think thin.) Prove the analogous theorem in E^3.

2. Prove that a set is closed iff it contains its boundary.

3. Prove that Theorem 2.4 is not true if we remove the requirement that the sets be closed.

4. Let S be a convex body in E^n and let X be a finite set of points. Suppose that each n+1 points of X can be covered by a translate of S (different translates for different sets of n+1 points). Prove that some translate of S covers X.

5. Prove that the intersection of closed sets is closed.

3. Separation, Support, and Extreme Points

The following can be thought of as a generalization of the notion of a tangent line.

Definition 3.1: A hyperplane H supports a set X provided X lies in one closed halfspace of H and $H \cap X \neq \emptyset$.

Definition 3.2: A hyperplane H separates sets X and Y iff X and Y lie in different closed halfspaces of H. If X and Y lie in different open halfspaces we say that H strictly separates X and Y.

Theorem 3.1: If S is a convex body with an interior point, then for any given direction there will be two supporting hyperplanes perpendicular to that direction.

We shall give a sketch of a proof. Since S is a convex body there is a sphere enclosing S. We can take two hyperplanes perpendicular to the given direction such that the sphere lies between them. Now imagine moving the hyperplanes toward S, stopping each hyperplane when it first touches S. This gives the two supporting hyperplanes.

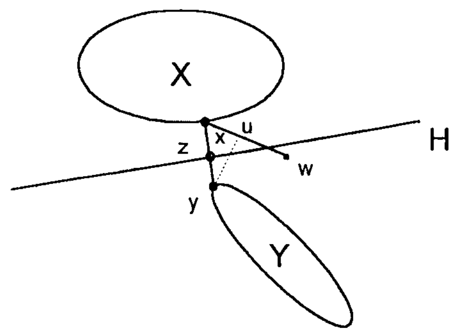
(Note: convexity is not used in this proof, but the set being closed must be used. Imagine what happens with this "proof" if the set S is an open ball, that is, S is the set of points inside a sphere.)

In order to get our main separation theorem we shall use the following lemma from topology. the proof will be omitted.

Lemma 3.1: Given two disjoint closed bounded sets X and Y, there are points $x \in X$ and $y \in Y$ such that the distance from x to y is the minimum distance between points of X and Y.

Theorem 3.2: If X and Y are disjoint convex bodies then there is a hyperplane that strictly separates X and Y.

Proof: Let $x \in X$ and $y \in Y$ be points realizing the minimum distance between points of X and Y. Let Z be the midpoint of the segment xy and let H be a hyperplane perpendicular to xy passing through z. Suppose that a point w of x lies on the same side of H as y.



Since the segment xw is not parallel to H, the angle formed by xy and xw is less than 90°. Now by taking a point u on xw close to x we get a triangle Δyxu where $\angle x$ is acute and $\angle u$ is obtuse. Now it follows that yu is shorter than yx. Now, since X is convex and x and w are in X, u must be in X. This, however, contradicts the fact that the distance from x to y was the minimum distance between points of X and Y. A similar argument works if there is a point of X lying on H.

Now we have all of X lying in the open halfspace of H that doesn't contain y. The same argument shows that all of Y lies in the other open halfspace. ■

Another separation theorem that we won't prove:

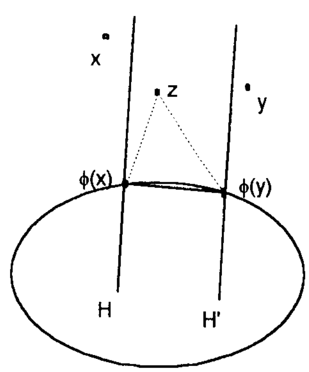
Theorem 3.3: Any two disjoint convex sets can be separated by a hyperplane.

The proof of our second support theorem will depend on the use of the nearest point function.

Definition 3.3: For any convex body X the <u>nearest point function</u> of X is the function ϕ that takes every point of X onto itself, and every point p not in X onto the point of X nearest p.

Lemma 3.2: <u>The nearest point function doesn't increase distances.</u>

Proof: Let X be a convex body, and let x and y be two points not in X. Let H and H' be hyperplanes perpendicular to the segment $\phi(x)\phi(y)$, with $\phi(x)\in H$ and $\phi(y)\in H'$.



We note that if we take any point z strictly between H and H', then z, $\phi(x)$, and $\phi(y)$ form a triangle where the angles at $\phi(x)$ and $\phi(y)$ are less than 90°, thus the foot of the perpendicular from z to the line through $\phi(x)$ and $\phi(y)$ lies between $\phi(x)$ and $\phi(y)$. This implies that the foot of the perpendicular (which is in the convex set X) is closer to z than $\phi(x)$ or $\phi(y)$. As a result, z cannot be x or y, so neither x nor y can be in the region between H and H'.

Suppose that x and y were on the opposite side of H from H'. Then y, $\phi(x)$, and $\phi(y)$ would form a triangle where the angle at $\phi(x)$ was obtuse and the angle at $\phi(y)$ was acute. It follows that $\phi(x)$ is closer to y than $\phi(y)$ is. (Remember the largest side of a triangle is opposite the largest angle.) Similarly we can show that x and y cannot be on the same side of H'. Now, however x and y are separated by a pair of hyperplanes whose distance apart as the length of $\phi(x)\phi(y)$. Thus the distance from x to y is as least as great.

The case where one of x and y is in X is handled similarly, and the case where x and y are both in X is trivial. ■

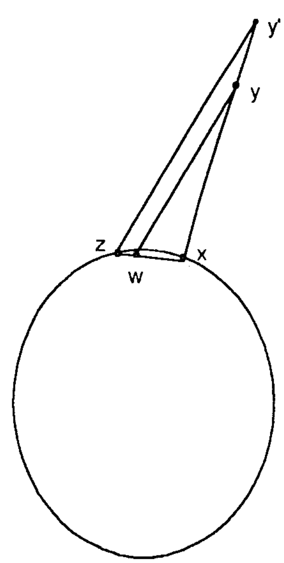
Lemma 3.3: The Nearest point function is continuous.

We won't get into $\varepsilon-\delta$ arguments here. We shall just mention that continuity will follow from Lemma 3.2.

Lemma 3.4: Let X be a convex body and $y \notin X$. If $\phi(y)$ is the nearest point of y then $\phi(y)$ is the nearest point of every point on the open ray from $\phi(y)$ through y.

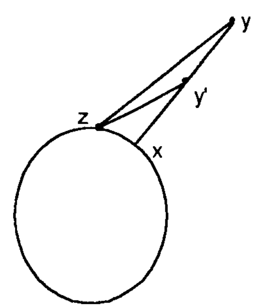
Proof: Let $x = f(y)$. Suppose y' is on the open ray from x through y, and that $z \neq x$ is the nearest point of y'.

Case I. y is between y' and x. Let the line parallel to line y'z through the point y intersect segment zx at w.



The triangle $\Delta y'zx$ is similar to Δywx. If the distance from y' to z is less than the distance from y' to x, then by similar triangles the distance from y to w is less than the distance from y to x, contradicting the fact that x is the nearest point for y (note that $w \in X$ by convexity.

Case II. y' is between y and x. If z is the closest point for y', then the distance from z to y' plus the distance from y' to y is greater than the distance from y to z, and less than the distance from y to x. Thus z is closer to y than x, a contradiction. □



Theorem 3.4: If p is a point on the boundary of a convex body X then p is a nearest point to some point not it in X.

Proof: Let S be a sphere enclosing the convex body X, (such that $S \cap X = \varnothing$). Let S' be a sphere of radius r centered at p. For any point a inside S and not in X there is a nearest point b in X. By Lemma 3.2, b lies in S'. The open ray from b through a will intersect S at a point c. Now we let the radius r of S' approach 0. As $r \to 0$ b must approach p, and c will approach some point d on S. Now by continuity of the nearest point function we have that since $\phi(c) \to p$ as $c \to d$, we must have $\phi(d) = p$. Thus p is a nearest point to d. ■

The most important consequence of all of this is the following theorem.

Theorem 3.5: If p is a boundary point of a convex body X, then there is a supporting hyperplane of X through p.

Proof: Let d be a point not in X such that p is the nearest point of X to d. Let H be a hyperplane through p perpendicular to the segment pd. Let A be one of the two halfspaces of H, with $X \subset A$. An argument similar to that given in Theorem 3.2 shows that no point of X lies on the other side of H from A. ■

Exercises

1. Let X be a convex body. Prove that the nearest point of any given point x is unique. (You may assume that a nearest point exists.)

2. Prove that a line through an interior point of a convex body X intersects the boundary of X in exactly two points.

3. Prove that if a closed bounded set X with an interior point has the property that every line through an interior point intersects X in a segment, then X is convex.

4. The <u>width</u> of a convex body is defined to be the maximum distance between parallel supporting hyperplanes. Prove that the width of a convex body equals its diameter.

5. Fill in the details of the proof of Theorem 3.5.

6. Let X be a closed set with interior points. Suppose that through every point on the boundary there is a supporting hyperplane. Prove that X is convex.

Extreme points

Definition 3.4: A point x of a set S is an extreme point of S iff it is not the midpoint of any segment of S. We shall denote the set of extreme points of S by ext S.

One of the most important theorems about extreme points is the following:

Theorem 3.6: If S is a convex body then S is the convex hull of its extreme points.
Proof: The proof is by (strong) induction on the dimension n of the convex body S (The dimension of a convex set is the dimension of its affine hull.) If S is 0-dimensional then S is a point and the theorem is clearly true. Suppose the theorem is true when $n \leq k$ and suppose that S is $(k+1)$-dimensional (We shall now assume that S is in E^{k+1}). We will take an arbitrary point x in S and show that x is in the convex hull of the extreme points of S.

If x is a boundary point, then we take a hyperplane H supporting S at x. The set $S \cap H$ is a convex body of dimension less than $k+1$. By induction, x is in the convex hull of the extreme points of $S \cap H$. Since the extreme points of $S \cap H$ are extreme points of S (see exercise 1), we have that x is in the convex hull of the extreme points of S.

If x is an interior point, then we can take a line through x which intersects the boundary of S in two points p and q. Now by the above argument we have:

$\{p,q\} \subset \text{con ext } S$, thus $\text{con}\{p,q\} \subset \text{con con ext } S$ (see exercise 2)

Thus we have $x \in \text{con}\{p,q\} \subset \text{con ext } S$.

Our argument about the arbitrary point x has shown that $S \subset \text{con ext } S$. On the other hand $\text{ext } S \subset S$, thus $\text{con ext } S \subset \text{con } S = S$, and therefore $S = \text{con ext } S$.

Exercises

1. Prove that if H supports a convex set S, then the extreme points of $H \cap S$ are extreme points of S.

2. Prove that if $S \subset T$ then con $S \subset$ con T.

3. Prove that a point x of a convex set S is an extreme point of S iff $S - \{x\}$ is convex.

4. We define a point of a convex set S to be an <u>exposed point</u> provided it is the intersection of S with a supporting hyperplane. Show that a convex body is not necessarily the convex hull of its exposed points (there are examples in E^2).

5. Show that neither the set of extreme points nor the set of exposed points of a convex body is necessarily a closed set. (There are examples in E^3 that take care of both exposed and extreme points.)

6. Let $\{x_1,...,x_k\}$ be a finite set in E^n. Prove that the extreme points of con $\{x_1,...,x_k\}$ are a subset of $\{x_1,...,x_k\}$. Hint: Use induction and exercise 1, page 8.

7. Prove Caratheodory's Theorem by induction on the dimension of the set. Use the intersection of the set with hyperplanes to reduce the dimension. You may assume that the convex hull of a finite set is always closed and bounded.

8. Prove that every nonempty convex body has an extreme point.

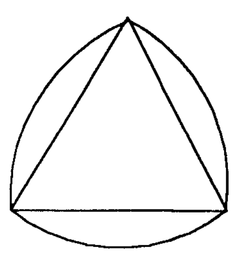
4. Sets of Constant Width

Some of the most interesting convex sets are the so-called sets of constant width.

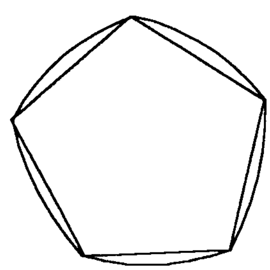
Definition 4.1: A convex body is a set of constant width iff it has the same width in every direction.

The disc is obviously a set of constant width. These sets become interesting once one realizes that besides the disc there are many other such sets in the plane, such as the Reuleaux polygons.

Definition 4.2: A Reuleaux polygon is a set constructed in the following way: We start with a regular (2n+1)-gon. Using each vertex as a center we join the end points of the edge opposite the vertex by an arc. The Reuleaux polygon is the simple closed curve formed by these arcs together with all points inside this curve.



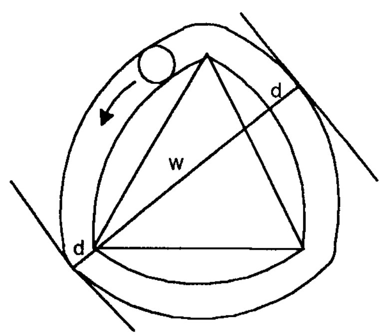
Reauleaux triangle



Reuleaux pentagon

We can see that the Reuleaux polygons are sets of constant width because any pair of parallel supporting lines will pass through a vertex of the original polygon and the opposite arc. Since the arcs all have the same diameter the distances between the pairs of supporting lines is the same in all directions.

If you imagine placing a disc so that it touches the boundary, but otherwise lies outside of a Reuleaux polygon P, and then rolling the disc around the outside of P, the disc will sweep out a region that surrounds P. This region together with P will form a set of constant width. If the width of P is w and the diameter of the disc is d, then we will have created a set of constant width w + 2d. Such a set is not another Reuleaux polygon. This can be seen because a Reuleaux polygon has "corners" while the set we have just produces will have a completely smooth boundary.



Definition 4.3: A convex ~~set~~ S of diameter d is as set of constant diameter iff for every point x on the boundary of S there is a point y in S such that the distance from x to y is d.

Definition 4.4: A set S of diameter d is complete iff for every point $x \notin S$, the set $S \cup \{x\}$ has diameter greater than d.

We shall see that there is a close connection between constant width, constant diameter, and completeness. We start with the two easiest connections.

Theorem 4.1: Every set of constant width is a set of constant diameter.

Theorem 4.2: Every set of constant width is complete.

We shall state a very nontrivial theorem about completeness:

Theorem 4.3: Every bounded set with a diameter can be completed. (That is, given any bounded set S of diameter d, there exists a complete set of diameter d containing S.)

Definition 4.5: If S is a set of diameter d and T is a complete set of diameter d containing S, we call T a completion of S.

The Reuleaux triangle is a completion of an equilateral triangle, (in fact the only completion). A disc centered at the center of a square is a completion of the square although there are other completions as well.

The most difficult connection between the above three properties is that complete sets are sets of constant width. In order to establish this we shall need a series of lemmas.

Lemma 4.1: If S is a complete set of diameter d, then S is the intersection of all balls of radius d centered at points on the boundary of S.

Proof: Let $T = \cap B_\alpha$ be the intersection of all such balls. Let $x \in S$. If $x \notin T$, then there is some ball B_α centered at a point $y \in S$ for which y is not in that ball. Thus the distance from y to x is greater than d, contradicting the fact that d is the diameter of S.

Suppose that $x \in T$. If $x \notin S$ then since S is complete, $S \cup \{x\}$ has diameter greater than d. Thus there is a point $y \in S$ such that the distance from y to x is greater than d. It follows that there is a boundary point z of S such that the distance from z to x is greater than d (See exercise 10). Now, however, x is not in the ball of radius d centered at z, a contradiction. We have shown that $S \subset T$ and $T \subset S$, thus $S = T$, and we are done. ■

Definition 4.6: An arc of a circle is a minor arc iff it subtends an angle of at most 180°.

Lemma 4.2: If x and y are two points in a disc D of radius d, and if Z is a minor arc of radius d joining x and y, then $Z \subset D$.

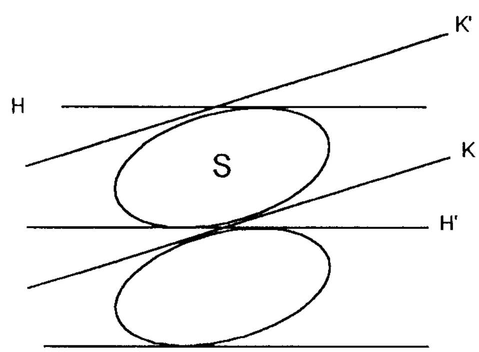
Proof: If any portion of Z lies outside D then there are two points u and v of Z where Z intersects the boundary of D. The portion of Z joining u and v, however, either lies on the boundary of D (because the curves have the same radius) or lies in D. ■

Corollary 4.1: If x and y are two points in a complete set S of diameter d, then any minor arc Z of radius d joining x and y will lie in S.

Proof: $S = \cap B_\alpha$ where the B_α's are balls of radius d centered at points on the boundary of S. The points x and y lie in each of these balls, thus by Lemma 4.2, Z lies in each of these balls. Thus Z lies in the intersection of these balls which is S. ■

Lemma 4.3: If S is a convex body and H and H' are supporting hyperplanes for which the width w of S is a minimum, then there exist a points $x \in H \cap S$ and $y \in H' \cap S$ such that the segment xy is perpendicular to H and H'.

Proof: We take a translation in the direction perpendicular to H in the direction from H to H'. We apply this translation to H, H', and S.



If S', the image of S under the translation meets S at some point x, then x and the image under the inverse translation (the translation perpendicular to H the same distance int he opposite direction) are the two points we are looking for. If S' misses S, then there is a hyperplane K separating S and S'. The image K' of K under the inverse translation will miss S (because K misses S'). Also, the set S will be between K and K'. The distance d between K and K' is less than or equal to w, while the distance between supporting hyperplanes parallel to K will be less than d. This implies that the width in the direction perpendicular to K is less than w, contradicting the fact that w was the minimum width. ■

Lemma 4.4: If S is a complete set then S is a set of constant diameter.

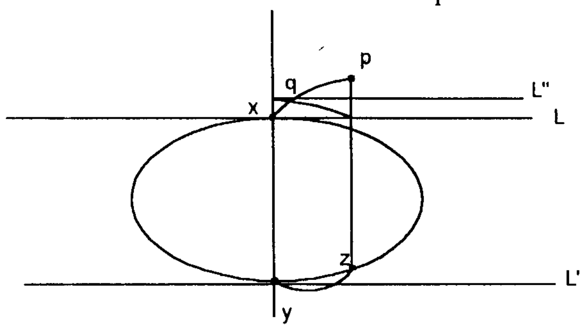
Proof: The proof really belongs in topology or analysis rather than geometry, but we shall give a sketch. First, using continuity arguments (similar to those we are about to give) one proves that a complete set is closed. Now let x be a boundary point of S. For each positive integer n we let B_n be a ball of radius 1/n centered at x. We choose a point x_n in each B_n where x_n is not is S. Since we cannot add x_n to S without increasing the diameter , there is a point y_n for each x_n such that the distance from y_n to x_n is greater than d. The sequence of points $\{x_n\}$ converges to x. The sequence $\{y_n\}$, since it lies in a bounded set, has a convergent subsequence, converging to a point y, which is in the set S because S is closed. The subsequence of the x_n's corresponding to the convergent subsequence of the y_n's will converge to x. Since distance is a continuous function (of two variables) the distance from x to y is greater than or equal to d. Thus the distance from x to y is d, and we have satisfied the definition of constant diameter. ■

Now we are ready to prove our theorem.

Theorem 4.4: If S is a complete set then S is a set of constant width.

Proof: Let H and H' be parallel supporting hyperplanes of S of minimum distance apart, and let the diameter of S be d. Let $x \in H \cap S$ and $y \in H' \cap S$. By Lemma 4.3, x and y can be chosen such that segment xy is perpendicular to H and H'. Since the diameter is the maximum width, we are done if we can show that the distance between H and H' is d. We shall assume that the distance from x to y is less than d and reach a contradiction.

By Lemma 4.4 S is of constant diameter, thus there is a point z in S such that the distance from z to x is d. The remainder of our argument now takes place in the intersection of S with the 2-dimensional plane P determined by x, y, and z.



For convenience we shall say that the line $L = H \cap P$ lies <u>above</u> the line $L' = H' \cap P$. Let L" be the line parallel to L, of distance d from y and lying above L. The circle C of radius d centered at y lies on or below L". The circle of radius d centered at z passes through a point p above L" on a line through z parallel to xy. Thus the arc of this circle from x to p of radius d intersects C to the right of line xy at a point q. This point of intersection is a center for a minor arc arc Z of radius d joining z and y.

Now we look at where the circle C' of radius d centered at q intersects L'. One point of intersection is x. Since the center is not on line xy and xy is perpendicular to L', Z is not tangent at x. Thus C' intersects L' at another point u. If u is on the opposite side of line xy from z then Δqxu is an isosceles triangle with equal sides px and pu, but with y as an obtuse angle (recall $xy \perp L'$). This is impossible, thus u lies on the same side of xy as z. Now u is on the arc Z. This implies that Z passes below L', and thus Z does not lie entirely in S. Corollary 4.1, however, tells us that Z must lie in S. ■

So far we have the following: complete $\Leftrightarrow$ constant width

$\Downarrow$ $\Downarrow$

constant diameter

To complete the equivalences of these three concepts we have:

Theorem 4.5: Any set S of constant diameter is complete.

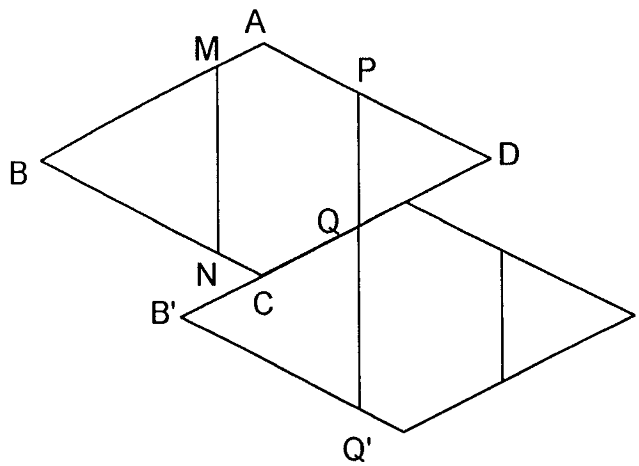
Proof: Let $x \notin S$, and let y be the closest of S to x. Let d be the diameter of S and let z be a point in S of distnace d from y. It is easily shown (Exercise) that the distance from x to z is greater than d, thus adding x to S increases the diameter, and thus S is complete. ■

The Circumference of a Set of Constant width.

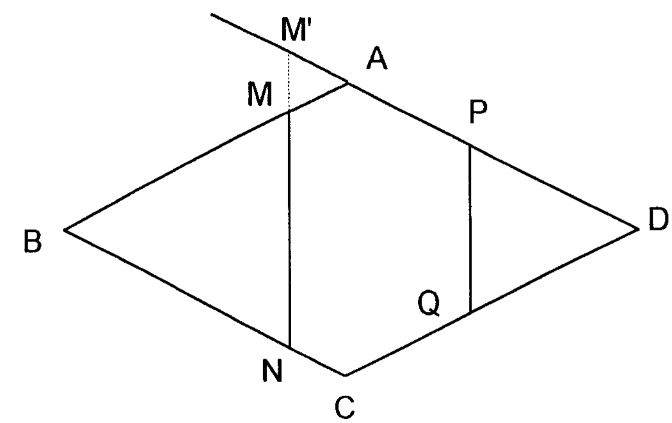
If you compute the circumference of the disc of width 1 and the Reuleaux triangle of width 1 you will get the same number. This makes one wonder about the circumferences of other sets of constant width. To answer this we shall need the following lemma.

Lemma 4.5: Let ABCD be a rhombus, and let MN and PQ be two line segments perpendicular to the diagonal BD, such that $M \in AB$, $N \in BC$, $P \in AD$, and $Q \in CD$, and such that the distance between NM and PQ is a constant value h. Then the perimeter of the hexagon AMNCQP is independent of the placement of the two segments.

Proof: We shall represent the length of segments in the following way: If PQ is the segment then $\overline{PQ}$ will be its length. First we shall show that $\overline{MN} + \overline{PQ}$ is a constant. To see this we take a translate of the rhombus and place it next to the original rhombus as shown:



Now we have that $\overline{MN} + \overline{PQ} = \overline{PQ'}$ and this will hold for any choice of the position of PQ, because this distance is just the length cut off by the parallel lines AD and B'Q'. Next we show that $\overline{AM} + \overline{AP}$ is a constant. We extend the side DA and the segment MN as shown.



The triangle ΔAM'M can be shown to be isosceles, because the angles at M' and M are congruent. Now, however we see that $\overline{AP} + \overline{AM} = \overline{AP} + \overline{AM'}$ which is the length of line AP cut off by the parallel lines MN and PQ, and depends only on h. Similarly, $\overline{CN} + \overline{CQ}$ is a constant, and thus the circumference of the hexagon is constant. ■

Theorem 4.6: <u>Every set of constant width w in the plane has perimeter πw.</u>

Proof: We will let S be a set of constant width w and D be a disc of diameter w. We begin by circumscribing about S and also about D, a rhombus with angles of 120° and

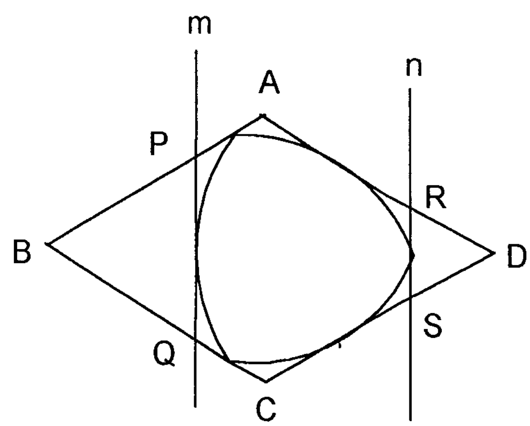
60°. This is easily done because these are sets of constant width. The two rhombuses are congruent and thus have the same perimeter. Next we cut off the two 60° vertices with a pair of supporting lines. This circumscribes an equiangular hexagon about both S and D. By the previous lemma, these two hexagons have the same perimeter. Next we cut off each of the six vertices using pairs of parallel supporting hyperplanes, chosen in directions that make the resulting 12-gons equiangular. We note that when each cut is made, if we restrict our attention to the four supporting lines that are being cut, we have the same result as when we cut off the vertices of the rhombus determined by these four supporting lines. As the previous lemma shows, the perimeter of the polygon circumscribing the disc will change by the same amount as the perimeter of the polygon circumscribing the set S. As we repeat this process we create a sequence of circumscribing equiangular $3 \cdot 2^n$-gons for the disc and for S, and for each n the polygons have the same perimeter. These polygons will converge to the boundary of the disc ans the boundary of S (We aren't going to go into the details here, but convergence depends on the polygons being equiangular.), and the perimeters will converge to the perimeters of the two sets. Since the disc has perimeter πw, so does S. ■

Borsuk's Theorem

We turn now to the idea of covering a set of diameter 1 with sets of diameter less than 1. Obviously, it will take more than one set of diameter less than one to cover a set of diameter 1. It is easy to see that there exist sets of diameter 1 in the plane that cannot be covered by any two sets of diameter less than one (the equilateral triangle, for example). If, however, we are allowed to use three sets of diameter less than one to cover a set of diameter 1, it's not clear that there are sets that can't be covered. (The ground rules of this "game" are that we are shown the set of diameter 1, then we are free to choose any three sets of diameter less than one to cover it.). The following theorem, called Borsuk's theorem shows that in fact we can always succeed, and it is proved using properties of sets of constant width.

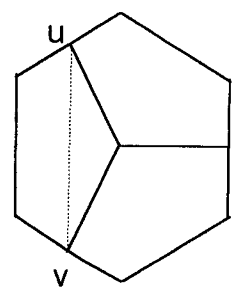
Theorem 4.6: <u>(Borsuk's Theorem) Any set S of diameter 1 in the plane can be covered by three sets of diameter less than 1</u>.

Proof: Let T be a completion of S. By Theorem 4.4, T is a set of constant width. We now circumscribe a 60° - 120° rhombus ABCD about T. Next we cut off the 60° vertices with parallel supporting lines, m and n, parallel to line AC.



Let the intersection of these lines with the rhombus be PQ and RS. What we are trying to do is to get a regular hexagon circumscribed about T. The hexagon APQCSR is equiangular, but it won't be regular unless $\overline{PQ} = \overline{RS}$. To see that we can get a regular hexagon, imagine what happens if we do the same construction but we start with the line AB rotated a small amount. The lengths of PQ and RS will be changed a small amount. Now imagine that we look at all constructions obtained for all rotations of AB through 180° (The set T is not rotated but the supporting lines have different positions for different amounts or rotation). After rotating 180°, the length of PQ has gone from its original length to the original length of RS, while the length of RS has gone from its original length to the original length of PQ. If the two lengths were not the same then somewhere during the rotation they must have the same length. (For students of calculus, recall the intermediate value theorem. The differences of the lengths is positive in one position and negative in the other position, thus somewhere in between, the difference is zero.)

Now we have a regular hexagon circumscribed about T. The distance between opposite sides of the hexagon is 1. We now cut this hexagon into three congruent pieces as shown:



The diameter of each of the three pieces is the length of the segment uv, which can be shown to be $(1/2)\sqrt{3}$. Thus we have covered T, and therefore S, with three sets of diameter less than 1. ■

Exercises

1. We define <u>a diameter</u> of a convex body to be a segment in the body whose length is the diameter of the set. Prove that any two diameters of a set of constant width in E^2 will intersect.

2. Prove that if two diameters of a set of constant width in the plane intersect at an end point then the end point lies on the boundary, and that boundary point has more than one supporting line through it. (Such a point is called a <u>corner</u> of the set.)

Characterize the centrally symmetric sets of constant width in the plane. (ie. Prove a statement of the form: A set is a centrally symmetric set of constant width in the plane iff...)

4. Prove that if a set of constant width in the plane has a corner, then there is an arc of a circle on the boundary of the set.

5. Let x be a corner of a set S of constant width in the plane. Suppose that m and n are supporting lines through x, such that the angle in which S lies is as small as possible. This angle is called the <u>angle of the corner</u>. Prove that corner angles for sets of constant width in the plane are always at least $120°$.

6.Prove that the only set of constant width in the plane with a corner angle of $120°$ is the Reuleaux triangle.

7. Prove that complete sets are closed and bounded.

8. Prove Theorem 4.1

9. Prove Theorem 4.2

10. Fill in the missing detail in the proof of Lemma 4.1, p 30.

11. Prove that every set of constant diameter is complete (Therorm 4.5).

5. Convex Polytopes

Definition 5.1: A <u>convex polytope</u> is a set that is the convex hull of a finite set of points in E^n. (From here on we shall usually say "polytope" rather than "convex polytope". Sometimes we shall say "<u>d-polytope</u>" when we wish to indicate that it has dimension d.) It seems obvious that a polytope has finitely many extreme points, but this does need proof.

Theorem 5.1: <u>If U is a finite set in E^n, and if $P = \text{con } U$, then the extreme points of P are points in U</u>.

Proof: See Exercise 6, Ch. 3.

Polytopes in E^1 are just the closed segments. The 2-polytopes are the convex polygons. In E^3 their structure becomes more complex. A few 3-polytopes are familiar to most who have had high school geometry - The tetrahedron, octahedron, cube, icosahedron, and dodecahedron. As can be seen by the definition, however, these are but a few of the infinitely many different kinds of 3-polytopes.

There is an alternate way to define a polytope.

Theorem 5.2: <u>If P is a bounded d-dimensional set in E^n, $n \geq 1$, that is the intersection of a finite collection of closed halfspaces, then P is a polytope</u>.

Proof: Our proof is by induction on the dimension of P. If the dimension of P is 0 or 1, the theorem is clearly true.

Let P be $\cap H_i^+$ where each H_i^+ is a closed halfspace bounded by a hyperplane H_i, for $1 \leq i \leq k$. Let p be an arbitrary extreme point of P. Since p is in the intersection of the halfspaces, it is either in the interior of all of the halfspaces or is on one of the bounding hyperplanes. If it is in the interior of each halfspace then it is in the interior of P and is not an extreme point, thus p is on a hyperplane H_i. Now we consider the set

$$P' = H_i \cap (\cap H_j^+) \text{ where } H_j \neq H_i.$$

The intersection of the hyperplane H_i with any one of the closed halfspaces H_j^+ is a closed halfspace in H_i, thus P' is a bounded set that is the intersection of finitely many closed halfspaces in H_i. By induction, P' is a polytope. We also have that p is in P'. Since p is an extreme point of P it is an extreme point of P'. By Definition 5.1, P' is a convex hull of a finite set, and by Theorem 5.1 the extreme points of P' are a subset of this finite set, thus P' has finitely many extreme points.

We now have that any extreme points of P must lie in one of finitely many finite sets of extreme points of sets of the form $H_i \cap (\cap H_j^+)$, thus there are only finitely many extreme points of P. Finally we recall from Theorem 3.6 that a convex body is the convex hull of its extreme points, and this gives us that P is the convex hull of a finite set and is thus a polytope. ■

The converse we shall present without proof:

Theorem 5.2: <u>If P is a d-polytope in E^d then P is the intersection of a finite collection of closed halfspaces</u>.

From these basic properties of polytopes the following can be proved.

Theorem 5.3: <u>The intersection of a d-polytope P with a hyperplane is a polytope. If the hyperplane passes through an interior point of P then the intersection is a (d-1)-polytope.</u>

Theorem 5.4: <u>Every projection of a polytope is a polytope</u>.

Definition 5.2: A <u>face</u> F of a d-polytope P is the intersection of P with a supporting hyperplane. If F is of dimension d-1, then it is called a <u>facet</u> of P. If it is of dimension d-2 it is called a <u>subfacet</u> of P. If it is of dimension 1, it is called an <u>edge</u> and if it is of dimension 0 it is called a <u>vertex</u>.

For example, for 3-polytopes the facet are polygons and the subfacets are edges. We present a list of theorems that give most of the basic properties of faces. We shall omit the proofs as some of them tend to get rather technical.

Theorem 5.5: Every face of a polytope is a polytope.

Theorem 5.6: If F is a face of P and G is a face of F, then G is a face of P.

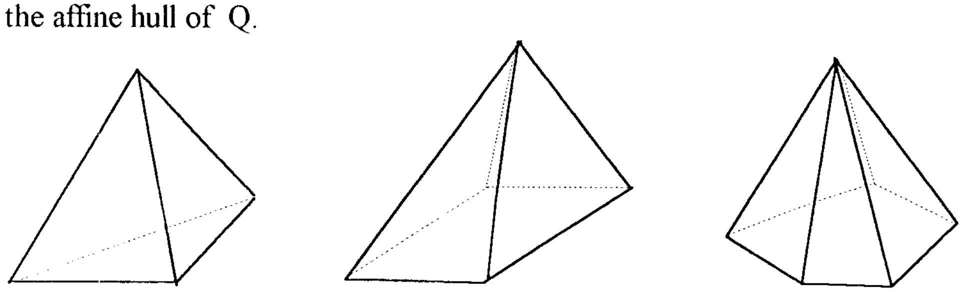
Theorem 5.7: Every point on the boundary of a polytope P lies in a facet of P.

Theorem 5.8: Each subfacet of a polytope P lies in exactly two facets of P.

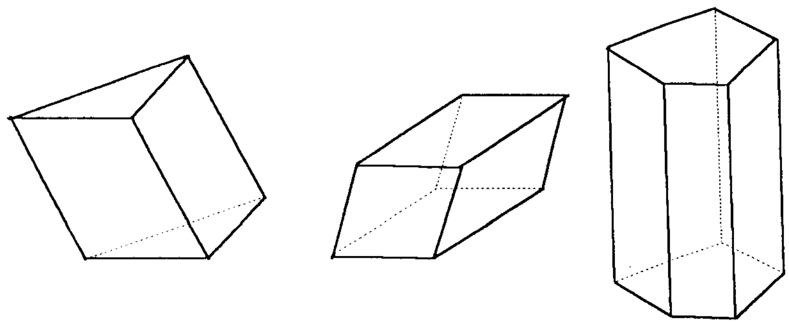
Theorem 5.9: Each vertex of a d-polytope P lies in at least d facets of P and at least d edges of P.

We shall now give some examples of polytopes.

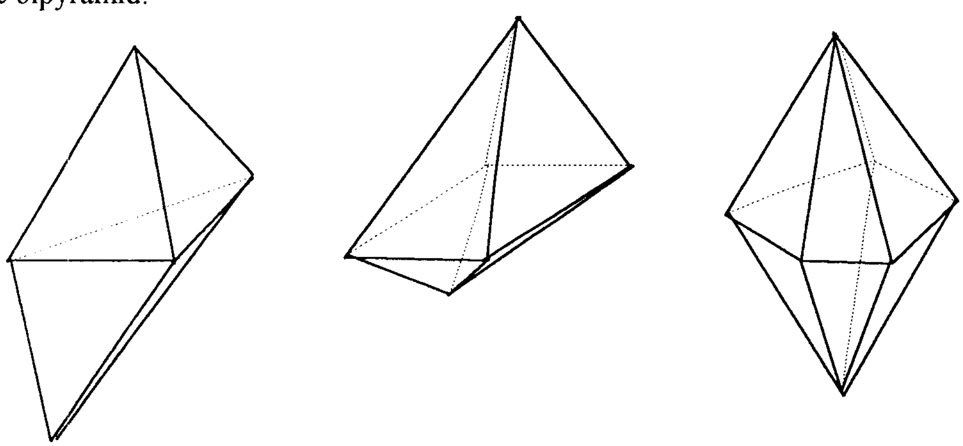
Definition 5.3: A d-pyramid is the convex hull of a (d-1)-polytope Q and a point not in the affine hull of Q.



Definition 5.4: A d-prism is the convex hull of two (d-1)- polytopes P and P' where P' is a translate of P that does not lie in the affine hull of P.



Definition 5.5: A d-bipyramid is the convex hull of a (d-1)-polytope P and a segment that intersects the interior of P, with one end point on one side of the affine hull of P (in E^d) and the other end point on the other side. The polytope P is called the equator of the bipyramid.



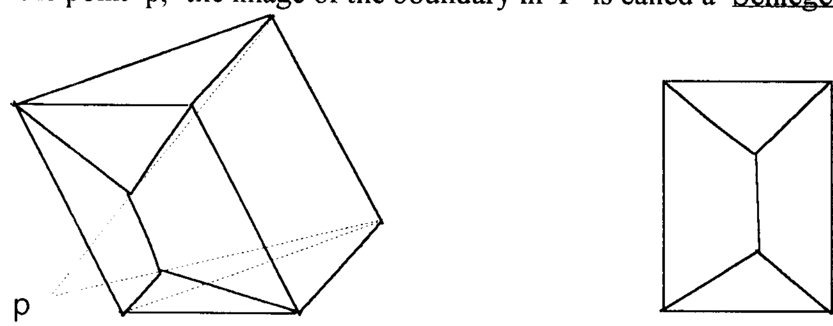
Definition 5.6: A 0-simplex is a point. A d-simplex is a pyramid over a (d-1)simplex.

Definition 5.7: A 0-cube is a point. A d-cube is a prism over a (d-1)-cube.

Definition 5.8: A 1-octahedron is a segment. A d-octahedron is a bipyramid with a (d-1)-octahedron as its equator.

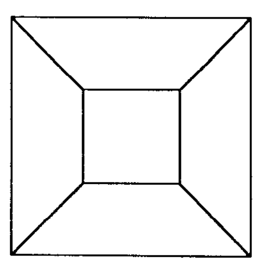
All but the simplest of the 3-polytopes are difficult to draw, and 4-polytopes are impossible to draw. There is, however a simple method of showing the facial structure of 3- and 4-polytopes. It involves the use of Schlegel diagrams.

Definition5.9: Let P be a d-polytope, and let p be a point not in P but close to the centroid of a facet F of P. If we project the boundary of P onto F by projecting through the point p, the image of the boundary in F is called a Schlegel diagram of P.



A Schlegel diagram of a triangular prism

The Schlegel diagram shows the combinatorial structure of the polytope, in other words it shows how the various faces "fit together". The Schlegel diagram does not accurately represent any metric properties of the polytope, that is, it doesn't show properties that are measured, such as area, volume, edge length, angle size, etc. So, for example from the Schlegel diagram of the cube



we see that there are eight vertices twelve edges and six facets. We see that every facet is a quadrilateral and that each vertex has exactly three edges meeting it. We cannot, however, say what the volume of the original cube was.

The Schlegel Diagrams of 3-polytopes are examples of a large class of structures called graphs.

Definition 5.10: A graph is a configuration consisting of a finite set of points (called the vertices of the graph) with various pairs of of the points joined by arcs (called the edges of the graph). The configuration consisting of just the vertices and edges of a polytope (or of its Schlegel diagram) is often called the graph of the polytope.

Definition 5.11: If v is a vertex of a graph or of a polytope, the number of edges meeting v is called the valence of v. If v has valence i we say that v is i-valent.

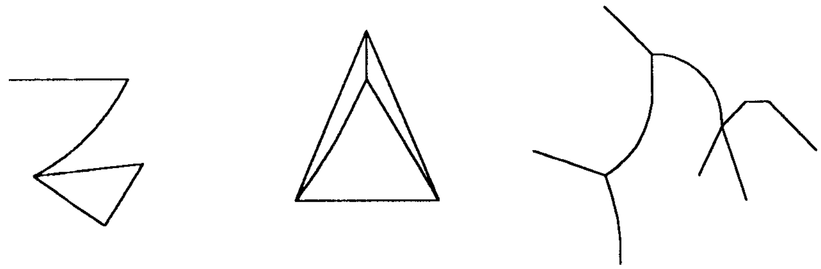
We shall prove one of the most important properties of 3-polytopes by proving a similar theorem about a certain class of graphs.

Definition 5.12: A graph is planar provided it can be drawn in the plane without any two edges crossing each other.

Definition 5.13: A graph is connected iff between any two vertices there is a path of edges joining them.

Definition 5.14: If a graph is drawn in the plane (without edges crossing) then the graph breaks the plane into regions. These regions are called the faces of the graph.

For example the first graph below has two faces (one is an unbounded region) the second has four and the third has just one.



The following is a fundamental theorem in graph theory and will establish an important combinatorial property of 3-polytopes.

Theorem 5.10: If G is a planar connected graph drawn in the plane, and if V, E, and F are the numbers of vertices edges and faces, respectively, of G then V-E+F = 2.

Proof: The proof is by induction on the number of edges of G. If G has no edges, then since it is connected it must have exactly one vertex, thus V-E+F = 1-0+1 = 2. Suppose that the theorem is true for planar connected graphs with k edges and suppose that G has k+1 edges.

We are going to remove an edge from G and apply induction. We must be careful, however, because removing an edge could leave the remaining graph disconnected and the induction hypothesis would not apply.

Suppose that we start at a vertex and begin traveling along edges, never backtracking along an edge. Since there are a finite number of edges, one of two things will happen. Either we will arrive at a vertex from which we can't leave (and thus the vertex meets just one edge of the graph) or we return to some vertex that we have visited before, in which case part of our journey is a simple closed curve.

If we arrive at a vertex v that meets just one edge e of the graph, we remove v and e producing a graph G'. Clearly the remaining graph is still connected. By induction, V-E+F = 2 for the graph G'. But G has one more vertex and one more edge than G' and thus going from G' to G gives a net change of 0 in V-E+F.

If our journey has a simple closed curve C in it, we remove an edge e of C to produce G'. The graph G' is still connected because given any two vertices in G there is a path connecting them. If this path used e we can substitute a path along the remainder of C in place of e. By induction V-E+F = 2 for G'. Now G' has one less face than G because a face inside C was merged with a face outside C when e was removed. Thus G has one more face and one more edge than G'. The net change for V-E+F is again 0. ■

Theorem 5.11(Euler's equation): If V, E, and F are the numbers of vertices, edges, and facets, respectively of a 3-polytope P, then V-E+F = 2.

Proof: The theorem follows immediately from the previous theorem since the Schlegel diagram of P is a planar connected graph. ■

The number of vertices of a 3-polytope does not uniquely determine the number of facets as is shown by the pyramid over a rectangel and a bypyramid with a triangular equator. In face no one of V, E, or F uniquely determines either of the other two. This leads to questions about the maximum and minimum numbers of one given the number of another.

Lemma 5.1: For any 3-polytope P, $2E \geq 3V$ and $2E \geq 3F$.

Proof: Imagine that we place a mark on each edge near each vertex. The number of marks will then be 2E because we have placed two on each edge, one near each of its two vertices. Since each vertex meets at least three edges, the number of marks is at least 3V, thus $2E \geq 3V$. A similar marking argument yields the second inequality (See Exercise). ■

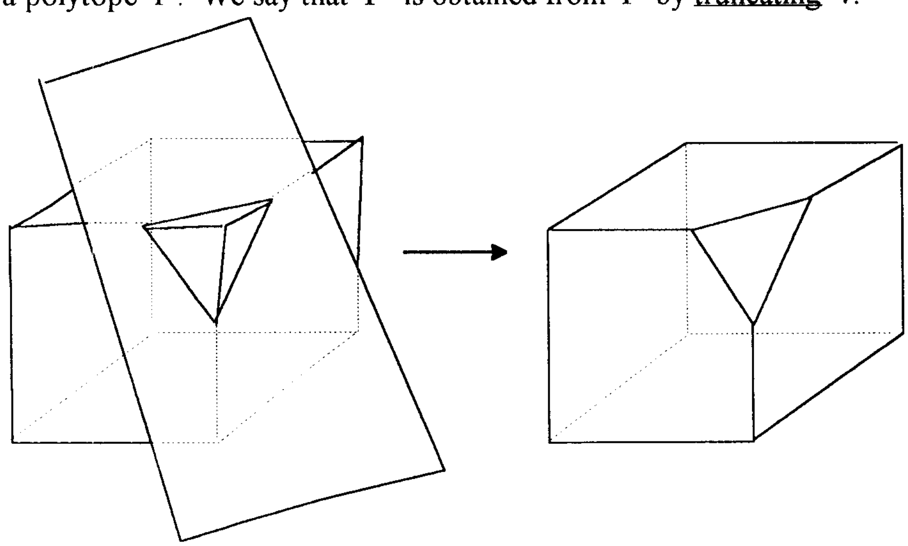
Theorem 5.12: For any 3-polytope $F \geq (v+4)/2$.

Proof: By Lemma 5.1 we have $E \geq 3V/2$. Substituting for E in Euler's equation gives the inequality. ■

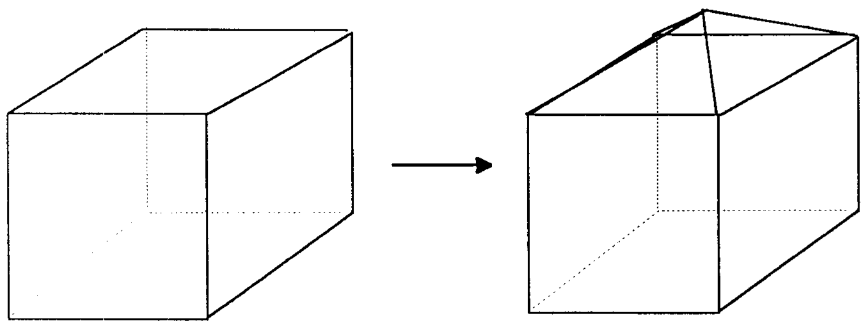
Theorem 5.12 only answers half the question about the minimum number of facets for a given number of vertices. There is still the question of whether the inequality is sharp. That is, for a given V is there a 3-polytope with (V+4)/2 facets. Unless we phrase this question more carefully, the obvious answer is no. There is no such 3-polytope for V = 9, for example. (Do you see the simple reason why?)

We shall introduce some more construction techniques sos that we will have a much richer family of examples that we can draw from.

Definition 5.15: Suppose v is a vertex of a d-polytope P, suppose that H is a hyperplane strictly separating v from the other vertices of P, and suppose that H^+ is the halfspace of H containing the other vertices of P. The intersection of P with H^+ is a polytope P'. We say that P' is obtained from P by truncating v.



Definition 5.16: Let F be a facet of a polytope P and let v be a point very close to the centroid of F. If v is close enough to F then taking the convex hull of v and P consists of erecting a pyramid with F as a base and v as the apex and "gluing" the pyramid to P. We say that the polytope $P' = (\text{con } P \cup \{v\})$ is obtained from P by capping the facet F.



Theorem 5.13: For every even integer $v \geq 4$, there exists a 3-polytope with v vertices and $(v+4)/2$ facets.

Proof: The proof of this theorem is a good example of the technique of strengthening the induction hypothesis to make induction work. We shall see that we may more easily prove the following statement:

> For every even integer $v \geq 4$, there is a 3-polytope with $(v+4)/2$ facets and a 3-valent vertex.

Our proof is by induction on n. For $n = 4$ the tetrahedron is an example (and the only example) with four facets and four 3-valent vertices. Suppose that the theorem is true for $n = k$. Since we are proving this for even n, our induction step is to show that the theorem holds for $k+2$. Let P be a 3-polytope with k vertices, $(k+4)/2$ facets and a 3-valent vertex v. Let P' be obtained from P by truncation vertex v. Then P' has $k + 2$ vertices and $(k+6)/2$ facets. Furthermore, each of the three vertices created by truncating is 3-valent. ■

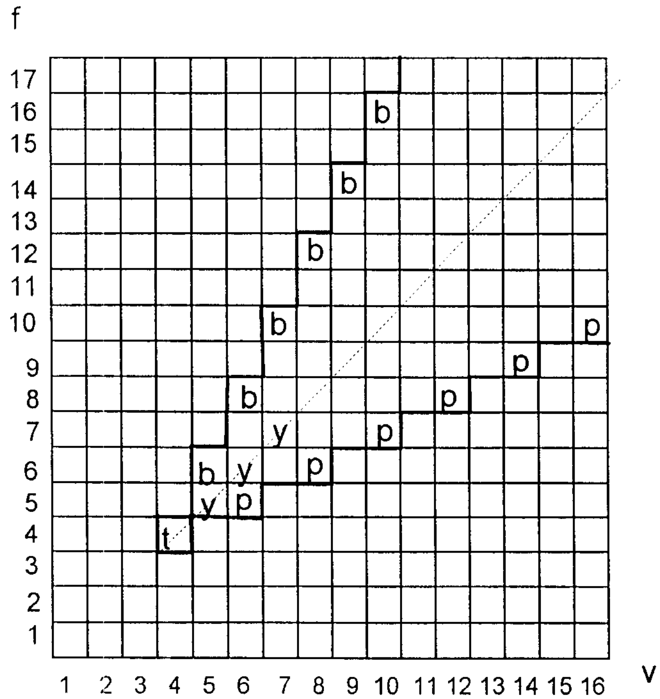
Theorem 5.14: For every even $v \geq 4$, there is a 3-polytope with v vertices and $2v - 4$ facets. (See Exercise)

We are now ready to look at a more general question: Given positive integers v, e, and f, when will there exist a 3-polytope with v vertices, e edges, and f facets? We have seen that the three integers must satisfy Euler's equation, thus if we have two of these numbers the third is uniquely determined. It therefore suffices to ask the question: Given positive integers v and f, when are there exist polytopes with v vertices and f facets?

We have seen two inequalities that v and f must satisfy. We shall now show that as long as v and f satisfy these two inequalities the polytopes will exist.

The following figure shows the first quadrant of the vf-plane. The region that satisfies our two inequalities is outlined. We shall show that there is a polytope corresponding to each square in the region. The square corresponding to the tetrahedron has a "t" in it. We have placed "p's" and "b's" in the squares that correspond to prisms and bipyramids, respectively. Several squares corresponding to pyramids have been filled with "y's", and we note that all squares on the diagonal will correspond to pyramids.

Now, note that if polytope P occupies a square and if P' is obtained from P by truncating a 3-valent vertex, then P' occupies a square one row up and two columns over from P. If P' is obtained from P by capping a triangular face then P' occupies a square two rows up and one column over from P.



Next note that every unfilled square below the diagonal is one row up and two columns over from a square in the region that is on or below the diagonal. Every polytope indicated in the figure that is on or below the diagonal has a 3-valent vertex, furthermore truncating a 3-valent vertex creates a new polytope with a 3-valent vertex.

Every unfilled square above the diagonal is one column over and two rows up from a square in the region that is on or above the diagonal. All polytopes indicated in the figure that are on or above the diagonal have triangular facets, and capping a facet creates a new polytope with a triangular facet.

Thus, with these two operations we can fill out the rest of the region, and we have:

Theorem 5.15: For every v and f greater than or equal to 4, there is a 3-polytope with v vertices and f facets provided $2v-4 \leq f \leq (v+4)/2$.

Exercises

1. Suppose that every vertex of a 3-polytope P is 4-valent. Find an equation for v in terms of e.

2. Suppose that each facet of a 3-polytope P has at least four sides. Find a sharp inequality for f in terms of e.

3. Describe an infinite family of 3-polytopes all of whose facets are 4-sided.

4. Can a 3-polytope have all vertices 4-valent and all facets 4-sided ?

5. Prove that no 3-polytope has exactly seven edges.

6. Prove that for any $n \geq 6$ and $n \neq 7$ there exists a 3-polytope with exactly n edges.

7. Prove Theorem 5.3.

8. Prove Theorem 5.8.

9. Prove theorem 5.9.

10. Prove Theorem 5.10.

11. The triangular prism has two types of Schlegel diagrams. One type is shown in this chapter Draw the other type.

12. Draw a schlegel diagram for a 3-dimensional octahedron.

13. Draw a picture of the Schlegel diagram of a bypyramid over a tetrahedron.

14. Draw the Schlegel diagram of a the polytope obtained by truncating each vertex of a tetrahedron.

15. There are simpler examples than given in the text to show that the inequalities $F \leq (V+4)/2$, and $F \geq 2V-4$ are sharp. What are they?

16. Let p_i be the number of i-sided facets of a 3-polytope P. For example, for the cube $p_3 = 0$, $p_4 = 6$, and $p_i = 0$ for $i \geq 5$. For a triangular prism $p_3 = 2$, $p_4 = 3$, and $p_i = 0$ for $i \geq 5$. Consider a Schlegel diagram for P. This diagram consists of f-1 polygons (called the bounded faces of the diagram) filling out a polygon F (The polygon F is the facet that the boundary was projected into, and is called the unbounded face of the diagram.) Suppose we take a bounded face and take the sum of the angles of that face. In terms of the number of edges of that face, what sum do we get? Suppose we do this for each bounded face and add the results. In terms of the p_i's what sum do we get?

Suppose now that instead of adding these angles one face at a time, we add the angles by taking a sum at each vertex and adding the results. What sum do we get? Obtain Euler's equation by equating the two sums obtained by adding the angles in these two different ways.

The following useful facts can be established using linear algebra, and are useful in some of the above problems. You may use them here without proof.

a) If a hyperplane H contains a relative interior point of a k-polytope P in E^n and does not contain P then $P \cap H$ has dimension k-1.

b) If a hyperplane H contains a relative interior point p of a k-face, $k \leq d-1$, F of a polytope P, and if H contains F, then an arbitrarily small movement of H can be made so that the resulting hyperplane contains p and does not contain F.

c) If a hyperplane H intersects a k-polytope P in E^n and misses its vertices, then H intersects a relative interior point of P.

6. Eberhard's Theorem

In this chapter we shall look at problems such as these:

1. Is there a 3-polytope with three triangles, one quadrilateral and one hexagon (as its only facets)?
2. is there a 3-polytope with five pentagons eight hexagons and 17 7-gons?
3. Is there a 3-polytope with 127 triangles 44 quadrilaterals and 11 octagons?

The first question is easily answered without any new theorems (see Exercise 1). After proving our main theorem of this chapter we shall see that the answer to the second question is no. As for the third question, I don't know the answer.

The general question would be the following: Let a set of numbers p_i be given. Is there a 3-polytope with p_i i-sided facets? For example the third question would be: Is there a 3-polyotpe for which $p_3 = 127$, $p_4 = 44$, and $p_8 = 11$ (and, of course, $p_i = 0$ for all other i)?

Although this is similar to a question that we answered in Chapter 5, a general method for answering this question has never been found. We shall examine partial solutions.

We begin with the most basic inequality known for the numbers p_i.

Definition 6.1: Let p_i be the number if i-sided facets of a 3-polytope P. then the vector $(p_3,p_4,\ldots,p_i,\ldots)$ is called the p-vector of P.

For example the p-vector for a pentagonal prism is (0,5,2). The p-vector for a pyramid over a 9-gon would be (9,0,0,0,0,0,1)

The above question is the same as asking: Given a vector, when is it the p-vector of some 3-polytope?

Definition 6.2: A 3-polytope is <u>simple</u> iff each vertex has valence 3.

The tetrahedron, cube, and prisms are examples of simple 3-polytopes.

Theorem 6.1: <u>If $(p_3, \dots, p_n)$ is the p-vector of a 3-polytope, then $\Sigma\,(6\text{-}i)p_i \geq 12$, with equality iff the polytope is simple</u>.

Proof: Let V, E, and F be the numbers of vertices, edges and facets, respectively, for P. Suppose in each facet we place a mark near the middle of each edge. Then the number of marks will be $\Sigma i p_i$. Since two marks were placed near each edge, the sum is also 2E. The sum $\Sigma 6p_i$ will be six times the number of facets, thus $\Sigma(6\text{-}i)p_i = 6F - 2E$.

Now recall from Chapter 5 that we always have $2E \geq 3V$, and observe that in our derivation of that inequality, we actually had equality if each vertex had valence three. Recall also Euler's equation $V\text{-}E\text{+}F = 2$, from which we get $6V\text{-}6E\text{+}6F = 12$. now if we substitute 4E for 6V, the inequality $2E \geq 3V$ tells us that we will get $6F - 2E \geq 12$, with equality if the polytope is simple, thus we have:

$$\Sigma(6\text{-}i)p_i \geq 12, \quad \text{with equality if the polytope is simple.} \ \blacksquare$$

We can now see that the answer to question 2 is no, because plugging in to our inequality we would get $1\cdot5 + 0\cdot8 - 1\cdot17 \geq 12$, which is not true, thus there could not be such a 3-polytope.

Corollary 6.1: <u>Every 3-polytope has a facet with fewer than six edges</u>.

Corollary 6.2: <u>There is no polytope all of whose facets are hexagons</u>.

Suppose we use our inequality to tackle question 3. Plugging in we get $3\cdot127 + 1\cdot44 - 2\cdot11 \geq 12$, which is a true statement. Unfortunately, this tells us nothing. Theorem 6.1 does not tell us that a polytope will exist when the inequality is satisfied, in fact such a polytope won't always exist (See Exercise 8). There are numerous results covering special cases, the most famous is Eberhards's Theorem (Incredibly, Eberhard really was the one who discovered and proved it!).

Theorem 6.2: (Eberhard's Theorem) If $\Sigma(6-i)p_i = 12$, then there exists a value for p_6 such that $(p_3, \ldots, p_n)$ is the p-vector of some 3-polytope.

Note that the coefficient of p_6 is 0 in the sum, thus the sum is not effected by the number of hexagons. Eberhard's Theorem tells us that if someone specifies how many facets there are of all sizes except 6, and if these numbers satisfy the equation, then one can find a number of hexagons so that the polytope exists.

We regret being unable to furnish the proof of Eberhard's Theorem here. The original proof filled an entire book. In the 1960's Grünbaum, making use of a powerful theorem about Schlegel diagrams, which was unavailable to Eberhard, was able to reduce the proof to about 15 typewritten pages. Suffice it to say that this is a deep result about 3-polytopes.

Exercises

1. Prove without using the inequality developed in this chapter, that (3,1,0,1) is not a p-vector for a 3-polytope.

2a. Let v_i be the number if i-valent vertices of a 3-polytope. Use a marking process to evaluate Σiv_i.

b. What can be said about $\Sigma(6-i)v_i$?

3. A 3-polytope is called simplicial iff each facet is a triangle. Show that the inequality that you obtained in problem 2b is an equation when P is simplicial.

4. Prove that for any 3-polytope, $\Sigma(4\text{-}i)(v_i + p_i) = 8$. Hint: use your knowledge about $\Sigma 4v_i$ and $\Sigma 4p_i$.

5. Does there exist a 3-polytope whose vertices are all 4-valent and whose facets are all quadrilaterals?

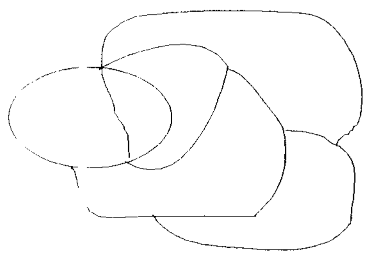
6. Does there exist a 3-polytope for which each two facets have a different number of edges?

7. Suppose that every vertex of a 3-polytope is 4-valent. What can be said about $\Sigma(4\text{-}i)p_i$?

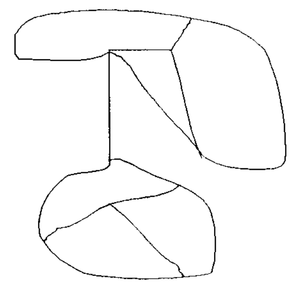
8. Find a vector, $(p_3, \ldots, p_n)$ such that $\Sigma(6\text{-}i)p_i = 12$, but there is no 3-polytope with that p-vector.

Chapter 7. Polytopes and Map Coloring

A map is a planar graph (drawn in the plane) such that each vertex has valence at least three and each region is bounded by a simple closed curve.



a map



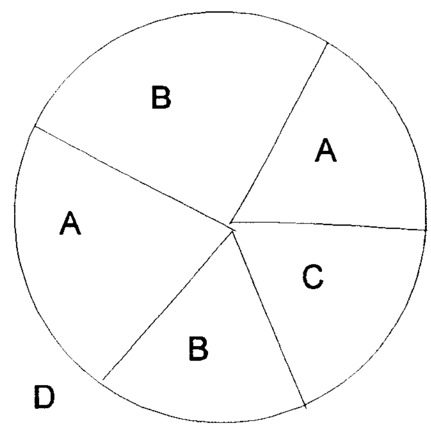
not a map

Note that Schlegel diagrams of 3-polytopes are all maps.

Definition 7.1: The regions of a map are called countries.

Definition7.2: A map is said to be colored with n colors iff colors are assigned to the countries such that no two countries meeting on an edge have the same color and at most n colors have been used. A map colored with n colors is said to be n-colored.

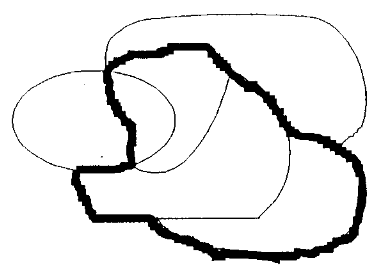
The following is an example of a map that is colored with four colors.



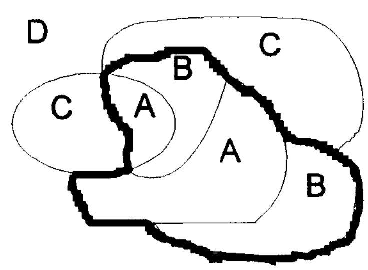
In 1852 it was conjectured that every map could be colored with four or fewer colors. In 1879 an English barrister named A. B. Kempe published a "proof" of the conjecture. It was not until eleven years later that an error in Kempe's work was found. Modifying Kempe's methods, Heawood proved that every map could be colored with five colors. There were, however no maps known that actually required five colors, thus the original conjecture was still unproved. As the years passed and no one could prove it, it became one of the most famous unsolved problems in mathematics. It was not until 1976 that Kenneth Apel and Wolfgang Haken at the University of Illinois finally produced a proof of the conjecture. It has remained a very controversial proof for two reasons. First, one part of the proof depends on generating a set of approximately 2000 maps (a set which we shall call M) according to certain rules. The set M was generated by hand and required nearly two years to accomplish. (To my knowledge no one has ever checked that part of the proof.) Second, another part involves using a computer to verify that the maps in M have certain properties. This was done using over 1000 hours of computer time. This part has been independently verified by others who have used their own programs to check the maps in M, however it has remained a source of discomfort to many mathematicians that an essential part of the proof is the work done by a computer. As for the first part of the proof, Apel and Haken have given a probabilistic argument that shows that if there were any mistakes in generating M, the probability that it will effect the validity of their proof is virtually zero!

In this chapter we shall see an unsuccessful attempt at a proof of the 4-color conjecture,and some of its interesting consequences.

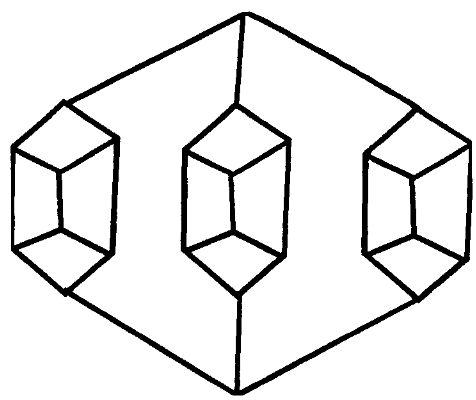
Definition 7.3: A Hamiltonian circuit in a map, or in a polytope, is a simple closed curve, consisting of edges, passing exactly once through each vertex. The following is an example of a Hamiltonian circuit in the above map:



Observe that if we have a map with a Hamiltonian circuit, we can color the countries inside the circuit with colors A and B and the outside countries alternately C and D, and we have 4-colored the map.



This suggests that one could prove the 4-color conjecture by proving that every map has a Hamiltonian circuit. The following map, however, shows that such a statement is not true (See Exercise 1):



It's a rather unusual coincidence that in the early work on the 4-color problem one of the theorems that was proved (actually it is a consequence of several theorems) was the following:

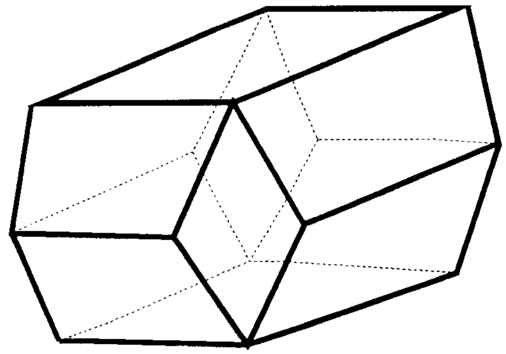
Theorem 7.1: <u>If the four color conjecture is true for Schlegel diagrams of 3-polytopes then it is true for all maps</u>.

This suggests that one could prove the 4-color theorem by proving:

Conjecture 7.1: <u>The Schlegel diagrams of all 3-polytopes have Hamiltonian circuits</u>.

(Or equivalently: <u>Every 3-polytope has a Hamiltonian circuit</u>.)

Unfortunately, this conjecture is not true as the following polytope shows.

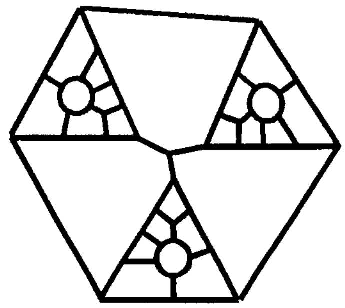


This polytope is called the Rhombic Dodecahedron. Note that each 3-valent vertex is surrounded by 4-valent vertices, and each 4-valent vertex is surrounded by 3-valent vertices, thus any circuit will alternate 3- and 4-valent vertices. But, there are six 4-valent and eight 3-valent vertices, thus no circuit can contain all vertices.

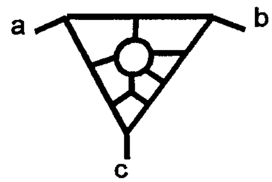
This counter example, however, did not end this line of investigation, for the following had also been proved:

Theorem 7.2: If all the Schlegel diagrams of all simple 3-polytopes are 4-colorable, then all maps are 4-colorable.

The Rhombic Dodecahedron is not a simple polytope, and thus is not a counter example to the conjecture that the simple ones all have Hamiltonian circuits. For years mathematicians searched for a simple 3-polytope with no Hamiltonian circuit, with no success. They also tried to prove that they all had such circuits, also without success. In 1932 a biologist named Jules Chuard published a "proof" that all simple 3-polytopes had Hamiltonian circuits (and thus claimed to have solved the 4-color problem). This proof was shown to be incorrect when in 1946 William Tutte found a simple 3-polytope with no Hamiltonian circuit. The following is a Schlegel diagram of his polytope.



The proof that there is no Hamiltonian circuit uses a property of a part of the graph called a Tutte triangle:



As one can check with a few minutes work exhausting cases, no path can enter and leave this portion of the graph at a and b and pass through every vertex of the Tutte triangle. This means that if there is a Hamiltonian circuit then the portion passing through each Tutte triangle must use the edge labeled c. Since there are three of these edges and they meet at a vertex we have that the circuit uses all three edges at that vertex. This is a contradiction because a circuit uses exactly two edges at a vertex.

Tutte's discovery did not end this way of trying to prove the 4-color conjecture. Note that if we take Tutte's graph and cut three of the edges meeting a Tutte triangle we will have disconnected the graph. In fact we will have disconnected it in a way that there will be two separate pieces each containing a country. When the graph of a simple 3-polytope has the property that we cannot separate two countries by cutting just three edges we say that the graph is cyclically 4-connected. Tutte's graph is not cyclically 4-connected. Examples that are cyclically 4-connected include the graphs of the prisms over polygons of at least four sides, and the dodecahedron. The following is another theorem that was known to researchers working on this problem:

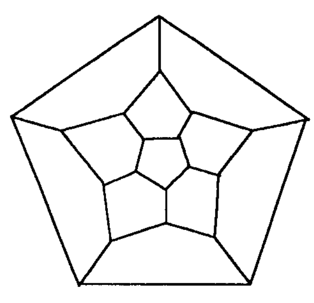
Theorem 7.3: If every cyclically 4-connected simple Schlegel diagram is 4-colorable then every map is 4-colorable.

In 1960 Tutte found a cyclically 4-connected simple Schlegel diagram without a Hamiltonian circuit. HOWEVER....

It was also known that if all cyclically 5-connected simple Schlegel diagrams were4-colorable then all maps were 4-colorable.

Definition 7.4: A simple Schlegel diagram of a 3-polytope is cyclically n-connected iff one can separate two ccountries by cutting edges, but one cannot separate two countries by cutting fewer than n edges.

For example the graph of the cube is not cyclically 5-connected but the graph of the dodecahedron is.



Schlegel diagram of the dodecahedron

Unfortunately, in 1965 Walter found a cyclically 5-connected simple Schlegel diagram with no Hamiltonian circuit. His example was quite complex, having 82 countries and 160 vertices.

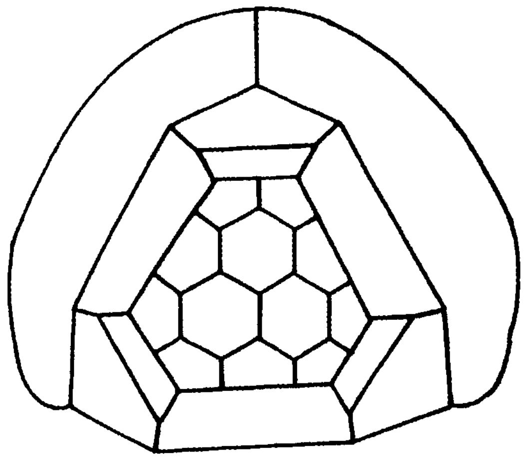
This however was not the end of the line. There is something called strong cyclic connectivity

Definition 7.5: A simple Schlegel diagram of a 3-polytope is strongly cyclically n-connected iff is cyclically n-connected, and the only way to separate two countries by cutting n edges is by cutting the edges meeting an n-sided country.

Theorem 7.4: If the strongly cyclically 5-connected simple Schlegel diagrams of 3-polytopes are all 4-colorable then all maps are 4-colorable.

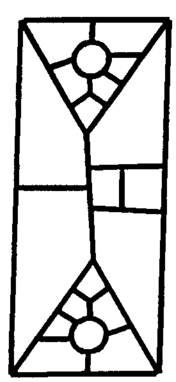
There was still hope. Perhaps all of these had Hamiltonian circuits. This would imply that they were all four colorable and thus all maps would be 4-colorable.

About 1970 Kozyrev, Grinberg and Tutte found a strongly cyclically 5-connected simple Schlegel diagram without a Hamiltonian circuit. To everyone's surprise, it had fewer vertices than Tutte's first example.



This finally ends this line of attack on the 4-color problem because there are no planar cyclically 6-connected graphs (Exercise 10).

Although one can't prove the 4-color conjecture with Hamiltonian circuits, the circuits themselves have become interesting. One problem of interest is to determine the smallest simple 3-polytope with no Hamiltonian circuit (here, smallest, means lest number of vertices.) In the middle 1960's three mathematicians independently found the following non-Hamiltonian example:



This graph has 38 vertices, and no smaller example has ever been found. Since it's discovery mathematicians have been trying to prove that 38 is the minimum number of vertices of a simple 3-polytope with no Hamiltonian circuit. The most recent result is that all simple 3-polytopes with 34 or fewer vertices have Hamiltonian circuits. Since simple 3-polytopes always have an even number of vertices, the question becomes: Is the minimum number of vertices for non-Hamiltonian simple 3-polytopes 36 or 38?

Exercises

1. Prove that the map given at the top of page 60 has no Hamiltonian circuit.

2. If the path does not return to it's starting point but passes through every vertex, it is called a Hamiltonian path. Does the rhombic dodecahedron have a Hamiltonian path? If not, what is the maximum length of any path (that doesn't intersect itself)?

3. Prove that if every map in which every vertex is 3-valent is 4-colorable then every map is 4-colorable. Hint: Suppose you have a map that has vertices of valence greater than three. What happens if you replace such vertices with small countries?

4. Prove that every simple 3-polytope has an even number of vertices.

5. Prove that for every integer n there is a map with at least n countries that is colorable with 3 colors.

6. Prove that for every n there is a map with at least n countries that can be colored with 2 colors.

7. Let P be the polytope obtained by capping each facet of a tetrahedron. Prove that P has a Hamiltonian circuit.

8. Let Q be obtained by capping every facet of an octahedron. Prove that Q does not have a Hamiltonian circuit.

9. If P is a polytope obtained from polytope Q by capping each facet, then P is called the <u>Kleetope</u> over Q (named after the great geometer Victor L. Klee). In problem 8 you showed that the Kleetope over an octahedron has no Hamiltonian circuit. Prove that the Kleetope over any simplicial 3-polytope, other than the tetrahedron, has no Hamiltonian circuit.

10. Prove that there are no planar cyclically 6-connected 3-polytopes. Hint recall that there must be a facet with five or fewer edges.

11. Prove that the 38 vertex example on page 63 has no Hamiltonian circuit. Hint: suppose that there is such a circuit. If the two Tutte triangels are shrunk to vertices what does this do to the circuit? What edges must the resulting circuit use?

Duality

Definition: If $X \subset E^n$, we define the polar dual X^* of X by

$$X^* = \{ x \in E^n \mid < x, y> \leq 1, \text{ for all } y \in X\}$$

where $<,>$ is the dot product in Euclidean n-space.

Lemma 1: For any $x \in E^n$, x^* is a closed halfspace whose bounding hyperplane is perpendicular to the vector x and which intersects the segment $\mathbf{0}x$ at a point p such that $||\mathbf{0}p||\ ||x|| = 1$.

Lemma 2: $X^* = \cap\ x^*$ over all $x \in X$.

the proof is an immediate consequence of the definition of polar dual.

Lemma 3: For any set X, X^* is closed, convex, and contains $\mathbf{0}$.

Proof: By Lemmas 1 and 2, X^* is the intersection of closed convex sets and thus is closed and convex. It is immediate from the definition of dual that $\mathbf{0}$ is in every dual.

Lemma 4: If $X \subset Y$, then $Y^* \subset X^*$.

This follows immediately from the definition or from Lemma 2.

Theorem 1. If $X = \text{con } S$, then $X^* = S^*$.

Proof: By Lemma 4, $X^* \subset S^*$. Suppose that $x \in S^*$. We need to show that $<x,z> \leq 1$ for all $z \in X$. We have $z = \Sigma\ \alpha_i x_i$, $x_i \in S$, $S\ \alpha_i = 1$, $\alpha_i \geq 0$. Now, by linearity of the inner product, $<z,x> = \Sigma\ <\alpha_i x_i, x> = \Sigma\ \alpha_i <x_i, x> \leq \Sigma\ \alpha_i = 1$. Thus $x \in X^*$ and $X^* \subset S^*$, and we have $X^* = S^*$.

the dual of the set of extreme points of P. This, however, is the intersection of a finite collection of closed halfspaces. Since the origin is in the interior of P the dual is bounded and hence is a polytope.

Corollary 8.5 A polytope is the intersection of a finite collection of halfspaces.

Proof: Let P be the polytope. We translate P to the origin and observe that $P = P^{**}$. Since P^* is a polytope, we have that P is the intersection of a finite collection of halfspaces.

Definition 8.6 A face of a polytope is the intersection of the polytope with a supporting hyperplane. A 0-dimensional face is called a vertex. A 1-dimensional face is an edge, a (d-1)-dimensional face of a d-polytope is a facet, and a (d-2)-dimensional face is a subfacet.

Theorem 8.7 Every point in the boundary of a polytope lies in a facet of the polytope.

Definition 8.8 A d-pyramid is the convex hull of a (d-1)-polytope P and a point p not in the affine hull of P. The polytope P is called the base of the pyramid and p is called the apex.

Definition 8.9 A d-prism is the convex hull of a (d-1)-polytope P and its image under a translation that takes P to a polytope not in the affine hull of P. The polytope P and its image under the translation are called the bases of the prism.

Definition 8.10 A d-bipyrimid is the convex hull of a (d-1)-polytope P with a segment xy with x on one side of the affine hull of P and y on the other, and xy intersecting the interior of P. The polytope P is called the equator of the bipyramid.

Corollary 1: *If X is a convex body, then X^* is the intersection of the duals of the extreme points.*

Theorem 2: *If X is closed, convex, and contains* 0, *then* $X^{**} = X$.

Proof: If $p \in X$ then $\langle p,y\rangle \le 1$ for all $y \in X^*$. Thus $p \in X^{**}$. If $p \notin X$, then there is a hyperplane H given by $\langle \mathbf{a},x\rangle = b$, where $\mathbf{a} \in E^n$, strictly separating p and X. Since $0 \in X$ we have $\langle \mathbf{a},y\rangle < b$ for all $y \in X$, and thus $\langle \mathbf{a},p\rangle > b$. Now $\langle \mathbf{a}/b,x\rangle < 1$, for all $x \in X$, thus $\mathbf{a}/b \in X^*$. But $\langle \mathbf{a}/b, p\rangle > 1$, thus $p \notin X^{**}$. Thus $X^{**} = X$.

Exercises

1. Prove that for any set $X \subset E^n$, $X^{**} = \text{cl con}(X \cup \{\mathbf{0}\})$, where cl stands for the *closure* of a set.

2. Can the dual of a convex set be affine, linear, or positive? Is the dual of a convex set always linear, affine, positive? Is the dual of a positive always positive? is the dual of an affine set always affine?

3a. Prove that for any family of sets $\{A_x | \ x \in \alpha\}$,

$$(\cup A_x,\ x \in \alpha)^* = (\cap A_x^*,\ x \in \alpha).$$

b. Prove that $(\cap A_x,\ x \in \alpha)^* = (\cup A_x^*,\ x \in \alpha)$ does not always hold.