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Methods of Representations Theory in Combinatorial Optimization Problems*

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An approach to combinatorial optimization problems is developed in this paper from the point of view of the theory of symmetric group representations. An assignment problem which generalizes the ordinary assignment problem is tied to each representation of a finite group. For a symmetric group, this problem includes also the travelling-salesman and other problems. It is proven that, for almost all representations of a symmetric group, the assignment problem is NP-complex. An approximate algorithm which yields a guaranteed relative error is constructed for solving these problems. The investigations are based on the analysis of the convex hull of the orbit of a point relative to the group action in the space of the representation operators.

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INTRODUCTION

Many problems in combinatorial optimization reduce to the problem of finding an extremum of a linear function on the convex hull of some set of points in Euclidean space. Sometimes, this set has the structure of the orbit of some point under the action of a finite group. This is precisely the situation in the assignment, the travelling-salesman, and other problems. The use of representations theory methods enables one to make considerable progress in the study of the combinatorial type of similar polyhedra and in the development of approximate algorithms with guaranteed estimates. Many authors [1] have examined the possibilities of applying group-theoretic methods to a number of discrete programming problems. Here, the problem has been formulated as that of optimizing a real function on a (symmetric) group. However, representations theory has not yet been applied to these questions. The basic idea of this paper is to study functions on a finite group with the aid of representations by which they can be defined. We say that a function $f\colon G \to \mathbf{R}$ passes through the representation $\pi\colon G \to GL(n, \mathbf{R})$ of the group G if there exists a linear functional $c \in (\mathrm{Hom}\, V_\pi)^*$ on the space V_π of the representation operators π such that $f(g) = \langle c, \pi(g)\rangle$. Methods of representations theory are applied in this paper systematically to combinatorial optimization problems.

The basic results consist of the following. First, one poses the general π-assignments problem which is a natural generalization of a number of combinatorial optimization problems such as the assignment, the travelling-salesman, and other problems; in it, π is an arbitrary representation of a symmetric group. Second, it is proven that almost all these problems are NP-complex. Third, approximate algorithms for solving the π-assignments problem with speed and error estimates are constructed on the basis of a more detailed analysis. We emphasize that, from an algebraic-geometric point of view, all these problems reduce to a description of all the senior-dimension faces (or to finding one of them) of the convex hull of the set from a given series of sets, that is, to the description of its polar. If the number of senior-dimension faces increases polynomially with the set dimension or if there is a polynomial description of the set of faces, then the corresponding optimization problem belongs to the P class. A close point of view is found in [2]. This approach will be developed in what follows for more general problems. We will describe briefly the contents of this paper. Notations and basic concepts are introduced in section 1. In section 2, the π-assignments problem is formulated and examples of combinatorial optimization

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problems that reduce to π-assignments problems are presented. Section 3 presents the proof of the mutual reducibility of function optimization problems on a symmetric group which are passed through various representations and, as a consequence, a theorem on the NP-complexity of π-assignments problems for some series of symmetric group representations is obtained. A series of approximate algorithms for π-assignments problems is constructed, and estimates of the arising errors are presented in section 4. Concluding remarks and unsolved questions are included in section 5. All the concepts, assertions, and definitions necessary for understanding the formulations of the theorems are explained in the text. The more special facts of the theory of representations of finite and especially symmetric groups that are used in the proofs can be found in [3, 4]. One can use [5, 6] as the sources which systematize in detail the information about the optimization problems of interest to us.

1. Notations and Basic Concepts

A homomorphism of a group G into the set of linear mappings of the real space E into itself, $\pi : G \to \mathrm{Hom}\,(E)$ is called a representation of G in E. Thus, every $g \in G$ is interpreted as a linear operator in E and $\pi(gh) = \pi(g)\,\pi(h)$. A representation π is called irreducible if there exists no subspace $E_1 \subset E$; $E_1 \neq 0, E$ such that $\forall x \in E_1$: $\pi(g) \cdot x \in E_1$. A representation $\pi : G \to \mathrm{Hom}\,E$ is the sum of the representations $\rho : G \to \mathrm{Hom}\,E_1$, $\tau : G \to \mathrm{Hom}\,E_2$ if the space E is the direct sum of the spaces E_1 and E_2 and $\pi(g) = \rho(g) \oplus \tau(g)$ (denoted by $\pi = \tau \oplus \rho$). It is well known that any real representation of a finite group can be decomposed into the sum of irreducible representations (see, for example, [3]). A representation $\rho_n : (\rho_n(g)\,x)_i = x_{g(i)}$; $x = (x_1, \ldots, x_n)$; $g \in S_n$ is called the natural representation of a symmetric group S_n in $\mathbf{R}^n$. This representation is well known and arises in many combinatorial problems (see in what follows). We shall consider other representations of S_n too. It is known that irreducible real representations of S_n are parametrized by Young diagrams of λ or by partitions of the number n (written as $\lambda \vdash n$) (see also [4]). A partition of the number n into the addends $\lambda_1 \geq \ldots \geq \lambda_s$ will be denoted by $(\lambda_1, \ldots, \lambda_s)$. The irreducible representation of S_n which corresponds to partition λ will be denoted identically. The Young subgroup S_λ of those substitutions which preserve the partition λ corresponds to the diagram of λ. In other words, $S_\lambda = S_{(1,\ldots,\lambda_1)} \times \ldots \times S_{(n-\lambda_s+1,\ldots,n)}$. A representation induced from a unit representation of a Young subgroup will be denoted by $\pi(\lambda)$. It can be interpreted as an S_n action by multiplication on the left in a vector space in which the left-adjacent classes S_n / S_λ are the basis. If G is a finite group, then $\mathbf{R}\,[G]$ is its group algebra, that is, the space of formal linear combinations of elements from G equipped with ordinary multiplication; $\mathbf{R}\,[G]_+$ is the cone of nonnegative linear combinations, and $T(G)$ is the simplex: $\{\Sigma r(g)\,g : r(g) \geq 0;\ \Sigma r(g) = 1\}$. A scalar product $(\Sigma r(g)\,g, \Sigma s(g)\,g) = \Sigma r(g)\,s(g)$ exists in $\mathbf{R}\,[G]$. We will also use the following notations: conv and co are the convex and conic (convex) hull, respectively, Lin is the linear hull, ext is the set of extremal rays of the cone or the vertices of a polyhedron; $|M|$ is the cardinality of the set M. Let G be a finite group, and V_π be the $\mathbf{R}$-space of its rational (that is, all the representation operators are rational matrices; for all the cases considered in what follows this requirement is not a restriction) representation π. We denote by $P_\pi = \mathrm{conv}\,\{\pi(g) : g \in G\} \subset \mathrm{Hom}\,V_\pi = V_\pi \otimes V_\pi^*$ the convex, and by $K_\pi = \mathrm{co}\,\{\pi(g) : g \in G\}$ the conic hull of the representation operators. If π is a subrepresentation of a regular representation of the group G (that is, of a representation of G by leftward shifts in $\mathbf{R}\,[G]$), then the linear space $L_\pi = \mathrm{Lin}\,\{\pi(g) : g \in G\}$ is canonically isomorphic as a bimodulus to the corresponding ideal I_π of the group algebra $\mathbf{R}\,[G]$, and P_π and K_π are the orthogonal projections of the simplex T_G and the cone $\mathbf{R}\,[G]_+$ on L_π, respectively.

Lemma 1. Let π contain a unit representation. Then the conjugate, in the sense of the scalar product (,), cone K_π^* in I_π has the form $K_\pi^* = \mathbf{R}\,[G]_+ \cap I_\pi$. The points $\{\pi(g) : g \in \Gamma\}$ constitute a face of the polyhedron P_π if and only if there exist positive numbers $r(g) : g \in S_n \setminus \Gamma$ such that $\Sigma r(g)\,g \in I_\pi$.

Proof. Since $\mathbf{R}\,[G]_+^* = \mathbf{R}\,[G]_+$ and $I_\pi^* = I_\pi$, then $K_\pi^* = I_\pi \cap \mathbf{R}\,[G]_+$. Any $r \in I_\pi$ is uniquely representable in the form $r = \Sigma \langle c, \pi(g)\rangle\,g$, where $c \in L_\pi^* = I_\pi$. Since π contains a unit representation, there exists $c \in L_\pi^* : \langle c, \pi(g)\rangle = 0$; $g \in \Gamma$; $\langle c, \pi(g)\rangle > 0 : g \notin \Gamma$ for any face Γ of the polyhedron P_π. This proves the lemma.

Remark. The set of functions $f : S_n \to \mathbf{R}$ which are passable through the π representation coincides with the ideal I_π.

Examples. (1) if ρ is a natural representation in $\mathbf{R}^n = \mathbf{R}\{e_i\}$, $i = 1, \ldots, n$; $\rho(g)\,e_i = e_{g(i)}$, then $L_\rho = \{a \in M_n R;\ a\mathbf{1} = a^*\mathbf{1} = c(a)\mathbf{1};\ \mathbf{1} = (1, \ldots, 1)\}$ and $\dim L_\rho = (n-1)^2 + 1$. P_ρ is a polyhedron of bistochastic matrices. One can easily see that K_ρ^* is a conic hull of n^2 elements, namely the indicators of the two-sided adjacency classes $h_1 S_{n-1} h_2$. Their explicit form is $x_{ij} = e_{ij} + (1/(n-1))\,E^{ij}$, where e_{ij} is a unit of the matrix, and $(E^{ij})_{ks} = 1 - \delta_{ik}\,\delta_{is}$.

(2) Let τ_n be the representation of S_n in $\mathbf{R}^{n(n-1)} = \mathbf{R}\,\{e_{ij} : 1 \leq i \neq j \leq n\}$ that is induced from a unit representation of the

Young subgroup $S_{n-2,1,1} = S_{n-2}$. $\tau_n(g)\, e_{ij} = e_{g(i)\, g(j)}$, $1 \le i \ne j \le n$. In this case, $\dim L_\tau = ((n^2 - 3n)/2 + 1)^2 + ((n^2 - 3n)/2)^2 + (n-1)^2 + 1$. The extremal rays of the cone K_τ^* are by no means exhausted by rays of the form $\{\alpha \Sigma g : g \in a S_{n-2} b;\ \alpha \ge 0;\ a, b \in S_n\}$. We will construct a family of extremal rays. Let $I \cup J$ be a partition of the set $[1 : n-2]$. Consider the element

$$e_{I,J} = \sum_{\substack{g(n)=n \\ g(1)\in I}} g + \sum_{\substack{g(n-1)=n-1 \\ g(1)\in J}} g - \sum_{\substack{g(n)=n \\ g(n-1)=n-1}} g.$$

One can easily establish that $\alpha e_{I,J} : \alpha \ge 0$ is an extremal ray of the cone K_τ^*. The equation of the corresponding senior-dimension face P_τ in the coordinates $x_{j_1 j_2}^{i_1 i_2}$ of the space Hom $(\mathbf{R}^{n(n-1)})$ is

$$\sum_{i\in I} x_{ni}^{n1} + \sum_{j\in J} x_{n-1\, j}^{n-1\, 1} - x_{n\, n-1}^{n\, n-1} \geqslant 0.$$

Remark. Analogously, one can construct for any representation $\pi(\Lambda_n)$, where $\Lambda_n = (n-k, \lambda_2, \ldots, \lambda_s)$; $k > 1$, $O(2^n)$ senior-dimension faces of the polyhedron $P_\pi(\Lambda_n)$.

2. The π-Assignments Problem

Examples. Let V_π be the space of the representation π of a finite group G and $c \in (\mathrm{Hom}_Q\, V)^*$. The following question will be called the π-assignments problem (for short, Problem 1): Find $\min_{g\in G} \langle c, \pi(g)\rangle = \min_{x \in P_\pi} \langle c, x\rangle$.

The question: Does there exist $g \in G$: $\langle c, \pi(g)\rangle \le a$ ($a \in \mathbf{Q}$) will be called the ASSIGNMENT SUBSTANTIATION problem (for short, Problem 2) (for terminology, see [2]). As the following examples demonstrate, many familiar combinatorial optimization problems reduce to π-ASSIGNMENTS PROBLEMS.

Examples.

(1) Let ρ_n be a natural representation of S_n in $\mathbf{R}^n$. Problem 1 is the ordinary ASSIGNMENTS PROBLEM [5].

(2) Let τ_n be a representation of S_n in $\mathbf{R}^{n(n-1)}$, $\tau_n(g)\, e_{ij} = e_{g(i)\, g(j)} : 1 \le i \ne j \le n$. Problem 1 is a quadratic ASSIGNMENTS PROBLEM [6].

(3) We will consider Problem 1 with a special functional c. If $c = v \otimes w^*$; $v \in V$; $w \in V^*$, then Problem 1 is a linear programming problem with the functional w^* on the polyhedron $P_\pi(v) = \mathrm{conv}\, \{v\pi(g) : g \in G\} \subset V$.

In particular, we have the following:

(a) If $\pi = \tau_n$ is the representation of Example 2, then Problem 1 with the functional $v \otimes w^*$ is the so-called quadratic deployment problem [6].

(b) Let ρ be a natural representation (Example 1). We will set $A \in M_n\, \mathbf{R}$ as the matrix of the Hamiltonian loop, that is, $A_{ij} = \delta_{i+1,\, j\, (\mathrm{mod}\ n)}$; $c \in (M_n\, \mathbf{R})^*$. The problem

$$\min\{\langle c, \rho(g) A \rho(g^{-1})\rangle, g \in S_n\} = \min\{\langle c \otimes A, x\rangle, x \in P_\tau = \mathrm{conv}\, \{\rho(g) \otimes \rho(g), g \in S_n\}\}$$

is the travelling-salesman problem [5, p. 12] since $\rho(g) A \rho(g^{-1})$ passes over a set of Hamiltonian loop matrices as g passes over S_n. Because $\tau_n = (n-2, 1, 1) \oplus (n-2, 2) \oplus 2(n-1, 1) \oplus (n)$ and A lies in the space of the representation $(n-2, 1, 1) \oplus (n-2, 2) \oplus n$, the travelling-salesman problem is a π-assignments problem, where $\pi = (n-2, 1, 1) \oplus (n-2, 2)$ with the special functional c; $\dim P_\tau(A) = n^2 - 3n + 1$.

(c) If one sets $\tilde{A} = A + A^{-1}$, where A is the matrix of example (3b), then the τ_n-assignments problem with the functional $c \otimes A$ is a symmetric travelling-salesman problem (see [5] p. 359). Analogously to example (3b), it turns out to be a π-ASSIGNMENTS problem for $\pi = (n-2, 2)$. $\dim P_\tau(\tilde{A}) = (n^2 - 3n)/2$.

(d) The problem of search for the Hamiltonian loop in an oriented graph can be reduced to the SUBSTANTIATION problem of an $(n-2, 1, 1)$-assignment with a special functional.

Let Γ be an oriented graph with n vertices. We will set $c_{ij} = 1$ if ij is an arc of Γ and $c_{ij} = 0$ otherwise; $\tilde{A} = A - A^{-1}$, where A is the matrix of example (3b). A $g \in S_n$ such that $\langle c \otimes \bar{A}, \tau(g)\rangle \geq n$ exists if and only if there is a Hamiltonian loop in Γ. Since $\bar{A}$ is in the space of the representation $(n-2, 1, 1)$, we have the special case of the SUBSTANTIATION problem for an $(n-2, 1, 1)$-assignment.

(e) We will set $B \in \mathbf{M}_n R$, n even, $B_{ij} = 1$ if $\{i, j\} = \{2k, 2k-1\}$, $k = 1, \ldots, n/2$, and $B_{ij} = 0$ otherwise. The problem

$$\max\{\langle c, x\rangle, x \in \operatorname{conv}\{\rho(g) B \rho(g^{-1})\}\} = \max\{\langle c \otimes B, x\rangle, x \in P_\tau = \operatorname{conv}\{\tau(g)\}\}$$

is the problem of maximum weighted pairwise matching (see [5, p. 254]), since $\rho(g) B \rho(g^{-1})$ passes over the set of symmetric permutation matrices as g passes over S_n. Because $B \in V_{(n-2,2)} \oplus V_{(n)}$, we have a special-form $(n-2, 2)$-ASSIGNMENTS PROBLEM.

(4) Let a canonical simple matroid (M, Ω) over a finite field F exist; $|M| = |P^{n-1}F| = p$ (see, for example, [7, p. 72]). A real vector $c(m)$; $m \in M$ is specified. One has to find a minimal-weight basis $\omega \in \Omega$ of the matroid M; $c(\omega) \to \min$; $\omega \in \Omega$ $c(\omega) = \Sigma c(m) : m \in \omega$. We will set $G = \mathrm{PGL}(n-1, F)$ and consider the representation π of the group G in R^p induced from the unit representation of a stabilizer of the point $P^{n-1}F$ for a natural action of G in $P^{n-1}F$: $\pi(g) e_i = e_{g(i)}$; $i \in P^{n-1}F$. Let $v \in R^p$ be the indicator of some basis $P^{n-1}F$. In this case, $\pi(g) v$ passes over the indicators of all the bases of $P^{n-1}F$ as g passes over G. Therefore, the problem of search for the minimal-weight basis of the matroid M is a π-ASSIGNMENTS problem.

Remark. Polynomial algorithms exist for problems (1), (3e), and (4). Problems (2), (3a), and (3c) are NP-complex. Problem (3d) is NP-complete.

The list of problems can be expanded. For example, many polyhedra of metroids are convex hulls of group orbits.

3. Reducibility in π-Assignments Problems

We will fix a partition of the number $k : \sum_{i=2}^{s} \lambda_i = k$, and let $\Lambda_n = (n-k, \lambda_2, \ldots, \lambda_s)$ $n > 2k$, and V_n be the sequence of spaces of the corresponding irreducible representations of the group S_n; $c_n \in (\mathrm{Hom}_Q V_n)^*$, $a_n \in Q$. We will set $\lambda_{n-k} = (n-2k+1, \lambda_2, \ldots, \lambda_s - 1)$ $(n > 3k)$.

Theorem 1. The λ_{n-k}-ASSIGNMENTS PROBLEM is polynomially in n reducible to the Λ_n-ASSIGNMENTS PROBLEM.

The proof of this theorem uses the following lemmas.

Lemma 2. Let $n > 2k$. We will fix the subgroups S_{n-k} and S_k of the group S_n that permute the first $n-k$ and the last k numbers, respectively. There exists a set $H : S_{n-k} \subset H \subset S_k \times S_{n-k}$ such that $\Lambda_n(H)$ is a face of the polyhedron P_{Λ_n}.

Proof. Let t be some table with a Young diagram Λ_n the first row of which is filled with the numbers $1, \ldots, n-k$; R_t and C_t be the row and column stabilizers, respectively; and t_{ij} be the element in the i-th row and j-th column.

We will examine the element $e_t = \Sigma (\operatorname{sgn} q) pq \in I(\Lambda_n)$, $p \in R_t$, $q \in C_t$ (see [3]) and

$$m = (e_t \Sigma g)/|S_{n-k}| = \sum_{g \in S_n} m(g) g,\ m(g) = \sum_{pqs = g} \operatorname{sgn} q : p \in R_t;$$
$$g \in C_t; \quad s \in S_{n-k}.$$

The element m determines the functional $c \in \mathrm{Hom}_Q(V_{\Lambda_n})^*$, $\langle c, \Lambda_n(g)\rangle = m(g)$. We will show that $H = \{g : g \in S_n : m(g) = 1\}$ is the required set. Since $\forall g$, $\forall s \in S_{n-k}$ there exists not more than one pair $p \in R_t$, $q \in C_t$, $pq = gs^{-1}$, then $m(g) \leq 1$. If $g \in S_{n-k}$, then, by setting $q = 1$ and $p = qs^{-1} \in R_t$, we have $m(g) = 1$. Finally, if $m(g) = 1$, then $g = p_1 q_1$, $p_1 \in R_t$, $q_1 \in C_t$. Here, if, for some element t_{1i}, the element $g(t_{1i})$ lies below the first row, then $m(g_1) < 1$ since, for $s \in S_{n-k}$ such that $s(t_{1i}) = t_{1\,n-k}$, there do not exist p, q: $pqs = g$. Therefore q_1 leaves all the elements of the first row in place and this means that $g \in S_{n-k} \times S_k$.

Lemma 3. Let $\Lambda_n = (n-k, \lambda_2, \ldots, \lambda_s)$; $n > 3k$. $\lambda_{n-k} = (n-2k+1, \lambda_2, \ldots, \lambda_s - 1)$. Then, $\forall c \in (\mathrm{Hom}_Q V_\lambda)^*$; $\forall z \in Q$. $\exists C \in (\mathrm{Hom}_Q V_\Lambda)^*$ such that $\langle C, \Lambda(g)\rangle \geq z$, if $g \notin H$; $\langle C, \Lambda(g)\rangle = \langle c, \lambda(p)\rangle$ if $g = pq \in H$; $p \in S_{n-k}$, $q \in S_k$.

Proof. We will examine the contraction of the representation Λ_n to the subgroup $S_{n-k} \subset S_n$ and the isotypical component X_λ of the representation λ_{n-k} in it. We have $X_\lambda = V_\lambda \otimes W$; dim $W = k$. The group S_{n-k} acts in the first factor (representation λ_{n-k}), and the group S_k in the second (natural representation). We will set $C = c \otimes W^* + h$, where W^* is the projector on the stationary straight line of the representation of S_k in W, and h is a functional which singles out the face $\Lambda_n(H)$, $h(g) = 0$: $g \in H$; $h(g) > z_* - \min_{g \in H} \langle c \otimes W^*, \Lambda_n(g) \rangle$ when $g \notin H$. This proves the lemma.

Theorem 1 follows from Lemmas 2 and 3 and the polynomiality of the operations of constructing a functional that separates a face and of singling out the isotypical component.

Theorem 2. For $k > 1$, Problem 1 is NP-complex and Problem 2 is NP-complete.

The NP-completeness of Problem 2 follows from Theorem 1 and from the fact that Problem 2 is NP-complete for $\Lambda_n = (n-2, 2)$ and $\Lambda_n = (n-2, 1, 1)$ since the symmetric travelling-salesman problem and the problem of the search for a Hamiltonian loop in an oriented graph (see section 2, Example 3, part d) are reduced in it. The NP-complexity of Problem 1 is derived, as usual (compare with [5]), from the NP-completeness of Problem 2.

4. Approximations

Approximate solution methods for Problem 1 are of interest in connection with Theorem 2. We will fix $\varkappa = (\varkappa_1, \dots \varkappa_l)$, the irreducible representation of the group S_n in R^m. In this section, we shall denote, for convenience, $P = P_\varkappa = \text{conv}\{\varkappa(g) : g \in G\} \subset \text{Hom}(V_\varkappa)$. The basic idea of the construction of approximate algorithms in Problem 1 consists of replacing the polyhedron P by a simpler polyhedron $P(\lambda)$ with the aid of a suitable system S_n of invariant cones in $\mathbf{R}[S_n]$.

Definition. A cone $K \subset \mathbf{R}[G]_*$ is called G-invariant if, $\forall g \in G$, $gK = Kg = K$. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be some partition of the number n. We will set $K(\lambda) = \text{co}\{\sum g : g \in a S_\lambda b;\ a, b \in S_n\}$. Obviously, the cone $K(\lambda)$ is S_n-invariant

$$|\text{ext}\, K(\lambda)|,\ \dim K(\lambda) \leqslant (n!/|S_\lambda|)^2.$$

Consider the polyhedra $\bar{P}(\lambda)^* = K(\lambda) \cap T_{S_n}$; $P^*(\lambda) = K(\lambda) \cap T_{S_n} \cap I(\varkappa \oplus \varepsilon)$ (ε is the unit representation). Obviously, $P^*(\lambda) \subset P^*$. The polyhedron $P(\lambda) \subset \text{Hom}(V_\varkappa)$ which is dual to $P^*(\lambda)$ is the projection $\bar{P}(\lambda) \subset \text{Hom}(V\pi(\lambda))$ of the polyhedron dual to $\overline{P}^*(\lambda)$ on $\text{Hom}(V_\varkappa)$. Thus, we have $pr: \bar{P}(\lambda) \to P(\lambda) \supset P$. Let $c \in (\text{Hom}\, V_\varkappa)^*$.

We replace Problem 1 $\min \langle c, x \rangle : x \in P$ by Problem 1 (λ):

$$\min_{x \in \bar{P}(\lambda)} \langle \text{pr}^* c, x \rangle. \tag{1(λ)}$$

Since $\dim \bar{P}(\lambda) \le \dim K(\lambda) \le (n!/|S_\lambda|)^2$ and the number of senior-dimension faces is $f(\bar{P}_\lambda) \le |\text{ext}(K(\lambda))| \le (n!/|S_\lambda|)^2$. Problem 1 is replaced by the linear programming problem of dimension not exceeding $(n!/|S_\lambda|)^2 \times (n!/|S_\lambda|)^2$.

We will describe the polyhedron $\bar{P}(\lambda)$ explicitly. One can easily see that $\bar{P}(\lambda) = \text{aff}\{\pi(\lambda)(g) : g \in S_n\} \cap \text{Hom} \times (V_{\pi(\lambda)})_+$, where aff denotes the affine hull and $\text{Hom}(V_{\pi(\lambda)})$ is the nonnegative orthant in the space $\text{Hom}(V_{\pi(\lambda)}) = V^*_{\pi(\lambda)} \otimes V^*_{\pi(\lambda)}$ with the standard basis $a_i^* \otimes a_j^*$, where a_i and a_j are (left) adjacent classes S_n / S_λ. The presence of the scalar product enables one to identify $V^*_{\pi(\lambda)}$ with $(V_{\pi(\lambda)})$. Let $\theta_T : V_\mu \to V_{\pi(\lambda)}$ be different semistandard homomorphisms (see [4]). Then, polyhedron $\bar{P}(\lambda)$ is described by the following equations and inequalities $(\theta_T^* \otimes \theta_S^*)\, x = 0$ if θ_T and θ_S are different semistandard homomorphisms; $(\theta_T^* \otimes \theta_T^* - \theta_S^* \otimes \theta_S^*)\, x = 0$ if $\theta_T, \theta_S : V_\mu \to V_{\pi(\lambda)}$; $\sum_{i,j} \langle x, a_i^* \otimes a_j^* \rangle = \dim V_{\pi(\lambda)}$;

$\langle x, a_i^* \otimes a_j^* \rangle \ge 0$. Semistandard homomorphism formulas imply that the coefficients in the equations presented in the preceding are 0 and ± 1. Hence, Problem 1 (λ) is solvable in time that is polynomial in $n!/|S_\lambda|$ (see [2]). We will calculate an estimate of the error that arises when Problem 1 is replaced by Problem 1 (λ). Let c_0 be a solution of Problem 1 and $c(\lambda)$ a solution of Problem 1 (λ). Since $\sum \langle c, \varkappa(g) \rangle = \langle c, \sum \varkappa(g) \rangle = 0$, then $c_0 \le 0$. Since $P(\lambda) \supset P$, then $c(\lambda) \le c_0 \le 0$. If $c_0 = 0$, then the functional c is constant on $\text{Hom}(V_\varkappa)$ and therefore $c(\lambda) = 0$. Consequently, we shall assume that $c_0 < 0$ and $c(\lambda) < 0$. We set $\beta = c(\lambda)/c_0 \ge 1$; $e = \dfrac{1}{n!} \sum g : g \in S_n$.

Lemma 4. Let $k \ge 0$ be a number such that $P^* + ke \subset K(\lambda)$. Then, $(k+1) \ge \beta \ge 1$.

Proof. Since $P^* + ke \subset K(\lambda)$, then $(P^* - e)/(k+1) + e \subset P^*(\lambda)$. For dual polyhedra we have $P(\lambda)/(k+1) \subset P \subset$

$P(\lambda)$. The required inequality now follows.

Lemma 5. Let u be an element of the group algebra R $[S_n]$ such that

1) $\forall r \in I(\varkappa \oplus \varepsilon)\ ru = r$,

2) $u + ke \in K(\lambda)$.

Then $P^* + ke \subset K(\lambda)$.

Proof. We have $r + ke = ru + kre = \sum r(g)\, g\,(u + ke) \in K(\lambda)$. The following lemma uses some structural theorems on the group algebra R $[S_n]$ as well as the partial order $\leq$ on partitions of the given number n (see [3, 4]). If some representation of λ is decomposed (uniquely) into a sum of irreducible representations, then the number of times the irreducible representation μ appears as a direct addend in the representation of λ is called its multiplicity $(\lambda : \mu)$.

Lemma 6. Let λ be a partition such that $\lambda \leq \varkappa$ (and consequently $(\pi(\lambda) : \varkappa) \neq 0$). We set

$$u = \frac{\dim \varkappa}{(\pi(\lambda) : \varkappa)\, n!} \sum_{t \in S_n} \sum_{g \in t^{-1} S_\lambda t} g - \left(\frac{\dim \varkappa}{(\pi(\lambda) : \varkappa)} - 1 \right) e, \quad k = \dim \varkappa / (\pi(\lambda) : \varkappa) - 1.$$

Then, u and k satisfy the conditions of Lemma 5.

Proof. The central idempotent of the ring R $[S_n]$ which corresponds to an irreducible representation π is calculated from the formula

$$e_\pi = \frac{\dim \pi}{n!} \sum_i \chi_\pi(i)\, x_i,$$

where i scans all the distinct conjugacy classes C_i in S_n, $x_i = \sum g : g \in C_i$, and $\chi_\pi(i)$ is the value of the character of the representation π on the i-th conjugacy class. We rewrite the element u in the form

$$u = \frac{\dim \varkappa}{(\pi(\lambda) : \varkappa)\, n!} \sum_i \chi_{\pi(\lambda)}(i)\, x_i - \left(\frac{\dim \varkappa}{(\pi(\lambda) : \varkappa)} - 1 \right) e.$$

Hence it follows that u lies in the center of the group algebra R $[S_n]$. Expanding u into a linear combination of central idempotents and noting that

$$\chi_{\pi(\lambda)} = (\pi(\lambda) : \varkappa)\chi_\varkappa + \chi_\varepsilon + \sum \alpha_\pi \chi_\pi : \pi \neq \varkappa,\ \varepsilon,$$

we have $u = u_\varkappa + u_\varepsilon + \sum \alpha_\pi e_\pi;\ \pi \neq \varkappa, \varepsilon$. Condition (1) is satisfied. The satisfaction of condition (2) is obvious. Thus, we have proven

Theorem 3. Let $\varkappa$ be a Young diagram with n cells. For any diagram λ with n cells there exists an approximate algorithm, polynomial in $\dim \pi(\lambda)$ in the $\varkappa$-ASSIGNMENTS problem such that the value of $c(\lambda)$ yielded by it is connected with the optimal value c_0 of the objective function by the inequality $\gamma^1 \dim \varkappa \geq c(\lambda)/c_0 \geq 1$, where γ is the multiplicity of the representation $\varkappa$ in $\pi(\lambda)$. If $c_0 = 0$, then $c(\lambda) = 0$ and vice versa.

Remark. The multiplicity of $(\pi(\lambda) : \varkappa)$ has the following combinatorial meaning. The number of ways in which λ_1 units, λ_2 pairs, λ_3 triplets, etc. can be arranged in the Young diagram $\varkappa$ so that the numbers will not decrease (from left to right) in the rows and will increase in the columns (from top to bottom) is equal to $(\pi(\lambda) : \varkappa)$. We will apply the obtained result to the first of the nonpolynomial representations $\varkappa_n = (n-2, 2)$. We will use $(\lambda_n)_j = (n-j, 1^j)$ as an approximating sequence. Varying j, we obtain

$$\dim \varkappa_n = (n^2 - 3n)/2,\ (\pi(\lambda_n) : \varkappa) = (j^2 - j)/2,\ n - 1 \geqslant j \geqslant 1.$$

Corollary. In Problem 1 with $\varkappa_n = (n-2, 2)$:

(a) For any second-degree polynomial $f(n) = \alpha n^2$, $\alpha > 0$, there exists a polynomial algorithm which yields a relative error which does not exceed $f(n)$.

(b) For any power function $f(n) = n^\alpha$, $\alpha > 0$, there exists a subexponential algorithm (of complexity $O(n^{n^\beta})$ $0 < \beta < 1$), which yields a relative error that does not exceed $f(n)$.

5. Concluding Remarks and Posings of Questions

1. The theorem of section 3 does not exclude the existence of a description, polynomial in dim π_n, of the polyhedron P_{π_n} for other series of symmetric group representations.

2. Exponentiality of the number of faces of the polyhedron $P_\pi(v) = \text{conv}\{\pi(g)\, v\}$ does not in itself mean the absence of a polynomial algorithm in the corresponding assignment problem (see section 2, example (3e) and [5]).

3. The question how a combinatorial type of polyhedron $P_\pi(v)$, in particular, its f-vector, depends on v is of interest. For a natural representation of a symmetric group, some information is contained in [8]. For other polyhedra, the approach described in [9] may turn out to be useful.

4. The problem of calculating a dual polyhedron is contained in the more general problem of calculating the polar to a semialgebraic set.

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