CONVEX HULLS OF ORBITS OF REPRESENTATIONS OF FINITE GROUPS AND COMBINATORIAL OPTIMIZATION

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In this paper we address questions concerning the combinatorial structure of the convex hulls of orbits in representations of symmetric groups and show that with a certain exception the convex hull of the orbit of a general point is an exponentially complex polytope. The grounds for considering this question are combinatorial optimization problems, in particular, the π-assignment problem formulated below, including many combinatorial problems. It turns out that almost all of them are NP-hard. However, the questions under study are of interest for the general theory of representations and contemporary combinatorics.

1. Definitions. Let G be a finite group, and V_π the vector R-space of its rational representation π. We denote by $P_\pi = \operatorname{conv}\{\pi(g): g \in G\} \subset \operatorname{Hom} V = V \otimes V^*$ the convex, and by K_π the conical hulls of the representation operators. If π is a subrepresentation of a regular representation of G, then $\operatorname{Lin}\{\pi(g): g \in G\} = L_\pi$ is canonically isomorphic, as a bimodule, to the corresponding ideal I_π of the group algebra $\mathbf{R}[G]$, and P_π, K_π are the orthogonal projections on L_π, respectively, of the simplex $S_G = \operatorname{conv}\{g\}$ and the cone $\mathbf{R}[G]_+ = \{\Sigma\lambda(g)g;\ \lambda(g) \geqslant 0\}$. For applications it is useful to know how to prescribe P_π, K_π using linear inequalities, that is, to know how to describe the dual objects.

LEMMA 1. Let π contain the unit representation. Then the dual cone K_π^*, lying in I_π, has the form $K_\pi^* = \mathbf{R}[G]_+ \cap I_\pi$ and is the conical hull of some orbits, and its extremal rays correspond bijectively to the faces of the higher dimension of K_π and P_π.

2. The Symmetric Group $\mathfrak{S}_n$. Let λ_n be the natural representation of $\mathfrak{S}_n$ in $\mathbf{R}^n$. Then $\dim L_{\lambda_n} = (n-1)^2 + 1$, P_{λ_n} is the polytope of the histochastic matrices; $K_{\lambda_n}^* = \mathcal{K}\{\Sigma g,\ g \in h_1\mathfrak{S}_{n-1}h_2\}$. The explicit formula of the indicators of the two-sided classes is: $x_{ij} = e_{ij} + (1/(n-1))E^{ij}$; e_{ij} is the matrix unit, $(E^{ij})_{ks} = 1 - \delta_{ik}\delta_{js}$. The number of faces of P_{λ_n} and K_{λ_n} of the higher dimension is a polynomial in n.

Let us consider the representation $\pi(\Lambda_n)$ of $\mathfrak{S}_n$, induced with the unit representation of the subgroup $\mathfrak{S}_\Lambda$, corresponding to the diagram $\Lambda_n = (n-k, \lambda_2, \ldots, \lambda_s)$; $\sum_{i=2}^{s} \lambda_i = k > 1$. The absence of twofold transitivity of the action of $\mathfrak{S}_n$ on $\mathfrak{S}_n/\mathfrak{S}_\Lambda$ allows one to construct an exponential family of sets for which there exists a unique element $K_\pi^*(\Lambda)$ with given support and which belongs to the algebra of sets generated by the two-sided classes $h_1\dot{\mathfrak{S}}_\Lambda h_2$. Therefore, we have

THEOREM 1. If k > 1, then the number of faces of the higher dimension of the polytope $P_\pi(\Lambda)$ grows no slower than 2^n.

3. π-Assignment Problem. Let $c \in (\operatorname{Hom}_Q V)^*$, where V is the space of the representation π of the finite group G. Let us set up a mass π assignment problem (for short – problem 1): find $\max\{\langle c, \pi(g)\rangle;\ g \in G\} = \max\{\langle c, x\rangle;\ x \in P_\pi\}$. In the case of a natural representation – this is the assignment problem [1], and is polynomially decidable. The problem of the validity of the assignment (for short – problem 2: as regards the terminology see [2]) consists in the following: to determine whether there exists $g \in G$, for which $\langle c, \pi(g)\rangle \leqslant a$. Let us consider the sequence of diagrams $\Lambda_n = (n-k, \lambda_2, \ldots, \lambda_s)$ and the corresponding irreducible representations $\mathfrak{S}_n$ in V_n, $c_n \in (\operatorname{Hom}_Q V_n)^*$.

THEOREM 2. For k > 1 problem 1 is NP-hard, and problem 2 is NP-complete.

The proof is obtained from the following lemmas.

LEMMA 2. Let $\lambda_n = (n-2k+1, \ldots, \lambda_s - 1)$, $n > 2k$. Then the NP-completeness of problem 2 for the representations Λ_n follows from the NP-completeness of problem 2 for the representations λ_n.

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The following lemma serves as the basis of the reduction.

LEMMA 3. Problem 2 for $\Lambda_n = (n-2, 2)$, $\Lambda_n = (n-2, 1, 1)$ is NP-complete (see also Sec. 4; cf. with the problem considered in [2] concerning the faces of the convex hull of the set of admissible points of an NP-complete problem).

4. Further Examples. Problem 1 for $\pi(n-2, 1, 1)$ is the quadratic assignment problem [1]. If $\Lambda_n = (n-2, 2)$, then for a special choice of c_n we obtain the symmetric traveling salesman problem [1], and for $\Lambda_n = (n-2, 1, 1)$, the problem of searching for a Hamiltonian contour in an oriented graph. These problems are NP-hard. However, for another choice of the functional in the $(n-2, 2)$-assignment problem we obtain the polynomially decidable matching problem [1]. The problem of searching for the minimal-weight independent set of a canonical simple matroid over a finite field F is the π-assignment problem for $G = PGL(n-1, F)$, and π is the natural representation, corresponding to the action of G on $P^{n-1}F$.

The possibilities of an approximate solution of problem 1 are of interest. The following result was obtained by the first author $(G = \mathfrak{S}_n)$.

THEOREM 3. Let λ be a Young diagram with n squares. For any diagram $\mu \trianglelefteq \lambda$ with n squares there exists an algorithm polynomial in $\dim \pi(\mu)$, yielding c_μ such that

$$(\dim \lambda)/(\pi(\mu): \lambda) \geqslant c_\mu/c_0 \geqslant 1,$$

where c_0 is the true value of the objective function, and $(\pi(\mu):\lambda) \neq 0$ is the multiplicity of the irreducible representation λ in $\pi(\mu)$ (see [3]).

The proof is based on the replacement of the cone $K_{\lambda \circledast [n]}$ by a simpler cone $\tilde{K}$ dual to $K^* = \mathcal{K}\{\Sigma g;\ g \in a\mathfrak{S}_\mu b\} \cap I_{\lambda \circledast [n]}$.

5. Remarks. Let us consider the polytope $P_\pi(z) = \mathrm{conv}\{\pi(g)z\}$; $P_\pi(id) = P_\pi$. If $z = x \otimes y$, then $P_\pi(z) = \mathrm{conv}\{\pi(g)x\}$. Elucidating the dependence of the combinatorial structure and, in particular, the f-vector $P_\pi(z)$ on z is an interesting problem. There is some information [4] on the combinatorial structure of these polytopes for the natural representation $\mathfrak{S}_n$. Not eliminated is the fact that for other series of representations {π} there is a description $P_\pi(z)$ polynomial in $\dim \pi$. The approach outlined in [5] is useful for describing the structure of exponentially complex polytopes.

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