CONVEXITY, DUALITY, AND OPTIMIZATION

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Abstract. These are day-by-day lecture notes for the graduate course Math 669, which I taught at the University of Michigan in the Winter term of 1998.

Contents

1. Preview: Basic Concepts

The fours basic concepts of the course are:

VECTOR SPACE, CONVEX SET, LINEAR FUNCTION, EXTREME POINT

Let us illustrate these concepts in the simple case of Euclidean space.

(1.1) Euclidean space. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. As a set, $\mathbb{R}^d$ consists of all d-tuples $x = (\xi_1, \ldots, \xi_d)$ of real numbers. An element $x \in \mathbb{R}^d$ is called a *point*. We can add points: we say that $z = x + y$ for $x = (\xi_1, \ldots, \xi_d)$, $y = (\eta_1, \ldots, \eta_d)$ and $z = (\zeta_1, \ldots, \zeta_d)$ if $\zeta_1 = \xi_1 + \eta_1$, $\zeta_2 = \xi_2 + \eta_2, \ldots, \zeta_d = \xi_d + \eta_d$. We can multiply a point by a real number: $\alpha x = (\alpha\xi_1, \ldots, \alpha\xi_d)$ for $x = (\xi_1, \ldots, \xi_d)$. We consider the *scalar product* in $\mathbb{R}^d$: $\langle x, y\rangle = \xi_1\eta_1 + \ldots + \xi_d\eta_d$, where $x = (\xi_1, \ldots, \xi_d)$ and $y = (\eta_1, \ldots, \eta_d)$, the *norm* $\|x\| = \sqrt{\xi_1^2 + \ldots + \xi_d^2}$, and the *distance*: the distance between two points x and y is $\|x - y\|$.

(1.2) Convex sets in $\mathbb{R}^d$. Let x and y be points from $\mathbb{R}^d$. The set

$$[x, y] = \Big\{\alpha x + (1-\alpha)y : \ 0 \leq \alpha \leq 1\Big\}$$

is called the *interval* between x and y. A set $A \subset \mathbb{R}^d$ is called *convex* if for any two points $x, y \in A$ we have $[x, y] \subset A$. We agree that the empty set $\emptyset$ is convex.

PROBLEMS.

1°. Intersection of convex sets is convex.

2. Let $A, B \subset \mathbb{R}^d$ be sets. Their *Minkowski sum* $A + B$ is defined as

$$A + B = \Big\{x + y : x \in A, \ y \in B\Big\}.$$

For a number $\alpha \in \mathbb{R}$ and a set $A \subset \mathbb{R}^d$ let $\alpha A = \{\alpha x : x \in A\}$. Suppose that A is a convex set and that $\alpha, \beta \geq 0$. Prove that $(\alpha + \beta)A = \alpha A + \beta A$. Show that the identity is not true for non-convex A or when α or β are allowed to be negative.

3. Let $A \subset \mathbb{R}^d$ be a convex compact set. Prove that there exists a point $x \in \mathbb{R}^d$ such that

$$-\frac{1}{d}A + x \subset A.$$

4*. For a $d \times d$ matrix $M = (m_{ij})$, let $x_M = (m_{11}, \ldots, m_{dd}) \in \mathbb{R}^d$ be the diagonal of M. Let us fix a vector $\lambda = (\lambda_1, \ldots, \lambda_d)$ of d real numbers. Consider a set $D = D_\lambda \subset \mathbb{R}^d$:

$$D_\lambda = \Big\{x_M : \quad M \text{ is a symmetric } \ d \times d \text{ matrix with the eigenvalues } \ \lambda_1, \ldots, \lambda_d\Big\}.$$

Prove that D_λ is a convex set.

(1.3) Linear functions in $\mathbb{R}^d$. A function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is called *linear*, provided $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in \mathbb{R}^d$ and any $\alpha, \beta \in \mathbb{R}$. A set $H = \{x \in \mathbb{R}^d : f(x) = \alpha\}$, where $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a linear function which is not identically zero and α is a real number, is called a *hyperplane*. Each hyperplane H dissects $\mathbb{R}^d$ into two *halfspaces*: $H^+ = \{x : f(x) \geq \alpha\}$ and $H^- = \{x : f(x) \leq \alpha\}$. We say that sets $A, B \subset \mathbb{R}^d$ are *separated* by the hyperplane H provided $A \subset H^+$ and $B \subset H^-$ (or vice versa). Similarly, a non-zero linear function f *separates* A from B if $f(a) \leq f(b)$ for all $a \in A, b \in B$.

PROBLEMS.

1°. Prove that every linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ can be written in the form $f(x) = \langle c, x \rangle$.

2. Fix sets $A, B \subset \mathbb{R}^d$. The set $F_{A,B}$ of all linear functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ that separate A from B can be considered as a subset of $\mathbb{R}^d$:

$$F_{A,B} = \Big\{ c \in \mathbb{R}^d : \langle c, a \rangle \leq \langle c, b \rangle \quad \text{for all} \quad a \in A, b \in B \Big\}.$$

Prove that $F_{A,B}$ is a convex set.

3*. Let $A, B \subset \mathbb{R}^d$ be non-empty convex sets such that $A \cap B = \emptyset$. Prove that there exists a hyperplane which separates A and B.

4°. Give an example of a non-convex connected compact set $A \subset \mathbb{R}^2$ and a point $b \notin A$ which can not be separated from A.

(1.4) Extreme points in $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$. A point $a \in A$ is called an *extreme point* of A if for any two points $b, c \in A$ such that $a = \dfrac{b+c}{2}$ one must have $a = b = c$. In other words, $a \in A$ is an extreme point of A if there is no non-trivial interval with the endpoints in A whose midpoint is a.

PROBLEMS.

1° Find the extreme points of $\mathbb{R}^n$.

2. Find the extreme points of the ball $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$.

3. Give an example of a convex compact set $A \subset \mathbb{R}^3$ whose set of extreme points is not closed.

4. Prove that any non-empty compact set $A \subset \mathbb{R}^d$ has at least one extreme point.

The three unifying ideas of the course are:

THE SET OF SOLUTIONS TO A SYSTEM OF LINEAR INEQUALITIES IS CONVEX;

GIVEN A CONVEX COMPACT SET AND A LINEAR FUNCTION, THERE IS AN EXTREME POINT OF THE SET WHERE THE FUNCTION ATTAINS ITS MAXIMUM ON THE SET;

A POINT WHICH DOES NOT BELONG TO A CONVEX CLOSED SET CAN BE SEPARATED FROM THE SET BY A HYPERPLANE.

2. Preview: Some Topics

(2.1) Classical convexity. Theorems of Radon, Carathédory and Helly.

Helly's Theorem. *Suppose that $A_i : i \in I$ is a family of convex sets in $\mathbb{R}^d$ and either I is finite or each A_i is compact (or both). If for any $d+1$ sets*

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{d+1}} \neq \emptyset$$

then

$$\bigcap_{i \in I} A_i \neq \emptyset.$$

PROBLEMS.

1. Suppose that A and B are sets in $\mathbb{R}^d$. We are looking for a hyperplane H which strictly separates A from B (that is, H separates A and B and $H \cap A = H \cap B = \emptyset$). Prove that such a hyperplane exists if and only if for every set S consisting of at most $d+2$ points there is a hyperplane H that strictly separates $S \cap A$ and $S \cap B$.
2. Let $X \subset \mathbb{R}^d$ be a set. We say that a point $a \in A$ is visible from point $b \in A$ provided $[a, b] \subset A$. Suppose that X is an infinite compact set in $\mathbb{R}^d$ such that for each $d+1$ points of X there is a point from which all $d+1$ are visible. Then there is a point from which all points of X are visible.
3. Do Problem 3, Section 1.2 by applying Helly's Theorem.

(2.2) Structural theory of convex sets. Krein-Milman Theorem, faces, polytopes and polyhedra, Weyl-Minkowski Theory.

Krein-Milman Theorem in $\mathbb{R}^d$. *Let $A \subset \mathbb{R}^d$ be a convex compact set. Then every point a of A can be represented as*

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_{d+1} e_{d+1},$$

where $e_1, \ldots, e_{d+1}$ are extreme points of A, λ_i are non-negative and $\lambda_1 + \ldots + \lambda_{d+1} = 1$.

Some interesting convex sets.

Cyclic Polytope. Let us choose n points $v_1, \ldots, v_n$ on the moment curve $\gamma(t) = (t, t^2, t^3, t^4) \in \mathbb{R}^4$ and let P be the polytope with the vertices $v_1, \ldots, v_n$. Then each pair of vertices of P is connected by an edge.

Birkhoff Polytope. Let $B_n \subset \mathbb{R}^{n^2}$ be the set of doubly stochastic matrices:

$$B_n = \left\{ x = (x_{ij}) : \ \forall j \sum_{i=1}^{n} x_{ij} = 1 \quad \forall i \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i, j \ x_{ij} \geq 0 \right\}$$

Then each vertex of B_n is a 0-1 matrix having exactly one 1 in each row and each column.

Set of positive semidefinite matrices. Let $C \subset \mathbb{R}^{n^2}$ be the set of $n \times n$ positive semidefinite matrices X such that $x_{11} + x_{22} + \ldots + x_{nn} = 1$. Then C is a convex compact set and the extreme points of C are positive semidefinite matrices X such that $x_{11} + x_{22} + \ldots + x_{nn} = 1$ and rank $X = 1$.

Set of "controls". Let us fix some functions $f_1(\tau), \ldots, f_n(\tau)$ on $[0,1]$ and some numbers $\alpha_1, \ldots, \alpha_n$. Consider the set U of functions $u(\tau)$ on $[0,1]$ such that $0 \leq u(\tau) \leq 1$ for all τ and $\int_0^1 f_i(\tau)u(\tau)\, d\tau = \alpha_i$. Then each extreme point of U is a "switching" function: for any τ either $u(\tau) = 1$ or $u(\tau) = 0$.

(2.3) Duality Theory for convex sets, linear inequalities, and linear programs. Polarity. Let $A \subset \mathbb{R}^d$ be a set. The set

$$A^\circ = \Big\{ c \in \mathbb{R}^d : \langle c, x \rangle \leq 1 \quad \text{for any} \quad x \in \mathbb{R}^d \Big\}$$

is called the *polar* of A.

"Bipolar Theorem". *If $A \subset \mathbb{R}^d$ is a closed convex set such that $0 \in A$ then $(A^\circ)^\circ = A$.*

PROBLEM.

1°. Prove that $A \subset (A^\circ)^\circ$.

2. Let $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be the unit ball. Find B°.

3°. Suppose that $A \subset B$. Prove that $B^\circ \subset A^\circ$.

4. Suppose that $A^\circ = A$. Find A.

(2.4) Applications.

Classical linear programming results (transportation problem, diet problem, combinatorial optimization);

Quadratic convexity and semidefinite programming;

Linear programming in functional spaces and applications to Control Theory, Approximation Theory, and Statistics.

Friday, January 9

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3. Vector Spaces and Convex Sets

(3.1) Definition. We recall that a set V with the operations "+" (addition): $V \times V \longrightarrow V$ and "·" (multiplication) $\mathbb{R} \times V \longrightarrow V$ is called a *vector space* provided the following axioms are satisfied: $u + v = v + u$ for any two $u, v \in V$, $u + (v + w) = (u + v) + w$ for any three $u, v, w \in V$, there exists a zero vector $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for each $v \in V$, for each $v \in V$ there is $-v \in V$ such that $v + (-v) = \mathbf{0}$, $(\alpha\beta)v = \alpha(\beta v)$ for any $v \in V$, any $\alpha, \beta \in \mathbb{R}$, $1v = v$ for any $v \in V$, $(\alpha + \beta)v = \alpha v + \beta v$ for any $v \in V$, $\alpha, \beta \in V$, and $\alpha(v + u) = \alpha v + \alpha u$ for any $\alpha \in \mathbb{R}$ and any $u, v \in V$

We often say "points" instead of "vectors".

(3.2) Definitions. A set $A \subset V$ is called *convex*, provided for each $x, y \in A$ the interval

$$[x, y] = \big\{ \alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1 \big\}$$

is contained in A. We agree that the empty set is convex. A non-empty set $K \subset \mathbb{R}^d$ is called a *convex cone* or just *cone* provided it is convex and for each $u \in K$ and for each $\alpha \geq 0$ we have $\alpha u \in K$.

PROBLEM.

1° Prove that a non-empty set $K \subset V$ is a (convex) cone if and only if $\alpha u + \beta v \in K$ for any $u, v \in K$ and any $\alpha, \beta \geq 0$.

Let us consider some examples.

(3.3) Euclidean space $\mathbb{R}^d$. The points are all possible d-tuples $x = (\xi_1, \ldots, \xi_d)$ of real numbers with the addition $u + v = (\xi_1 + \eta_1, \ldots, \xi_d + \eta_d)$ for $x = (\xi_1, \ldots, \xi_d)$ and $y = (\eta_1, \ldots, \eta_d)$ and multiplication $\alpha x = (\alpha\xi_1, \ldots, \alpha\xi_d)$, see Section 1.1. A *polyhedron* is a set $P \subset \mathbb{R}^d$ of the type

$$P = \Big\{x \in \mathbb{R}^d : \langle c_i, x \rangle \leq \alpha_i, i = 1, \ldots, m\Big\},$$

for some $c_i \in \mathbb{R}^d$ and $\alpha_i \in \mathbb{R}$. A *polyhedral cone* is a set $K \subset \mathbb{R}^d$ of the type

$$K = \Big\{x \in \mathbb{R}^d : \langle c_i, x \rangle \leq 0, i = 1, \ldots, m\Big\}$$

for some $c_i \in \mathbb{R}^d$.

PROBLEM.

1°. Check that a polyhedron is a convex set and that a polyhedral cone is a convex cone.

(3.4) The space $\mathbb{R}^{\frac{d(d+1)}{2}}$ of $d \times d$ symmetric matrices (quadratic forms). Vectors of this space are $d \times d$ symmetric matrices $A = (a_{ij})$, $i, j = 1, \ldots, d$ (so $a_{ij} = a_{ji}$) with the usual addition: $C = A + B$ when $c_{ij} = a_{ij} + b_{ij}$ and scalar multiplication $\alpha A = (\alpha a_{ij})$, $i, j = 1, \ldots, d$. The dimension of this space is $d(d+1)/2$. With a symmetric matrix A we associate a quadratic form

$$q_A(x) = \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j,$$

in d real variables $\xi_1, \ldots, \xi_d$. We often think of points in this space as of quadratic forms in d real variables instead of symmetric $d \times d$ matrices. A form q_A is called *positive semidefinite* provided $q_A(x) \geq 0$ for each $x = (\xi_1, \ldots, \xi_d)$. Similarly, a symmetric $d \times d$ matrix is called *positive semidefinite* if q_A is a positive semidefinite quadratic form.

Let $S_+ \subset \mathbb{R}^{\frac{d(d+1)}{2}}$ be the set of all positive semidefinite quadratic forms.

PROBLEM.

1° Show that S_+ is a convex cone.

(3.5) The space $C[0,1]$ of continuous functions. Points in this space are continuous functions $f : [0,1] \longrightarrow \mathbb{R}$ on the interval $[0,1]$ with the pointwise addition: $h = f + g$ provided $h(\tau) = f(\tau) + g(\tau)$ for any $\tau \in [0,1]$ and multiplication $h = \alpha f$ provided $h(\tau) = \alpha f(\tau)$ for any $\tau \in [0,1]$.

PROBLEMS.

1°. Prove that the set $B = \Big\{f : |f(\tau)| \leq 1 \text{ for all } \tau \in [0,1]\Big\}$ is a convex set.

2°. Prove that the set $K = \Big\{f : f(\tau) \geq 0 \text{ for all } \tau \in [0,1]\Big\}$ is a convex cone.

(3.6) The space $L^1[0,1]$ of integrable functions. Points in this space are functions $f(\tau)$ on the interval $[0,1]$ such that the integral $\int_0^1 f(\tau)\,d\tau$ exists. We use the pointwise addition and multiplication (see Example 3.5).

(3.7) The space $L^\infty[0,1]$ of bounded integrable functions. Points in this space are integrable functions $f(\tau)$ on the interval $[0,1]$ such that $|f(\tau)| \leq M_f$ for some constant M_f (depending on f) and all τ.

PROBLEM.

1. Fix some functions $f_1, \ldots, f_m \in L^1[0,1]$ and some numbers $\alpha_1, \ldots, \alpha_m$. Consider the set

$$A = \Big\{ u \in L^\infty[0,1] : 0 \leq u(\tau) \leq 1 \quad \text{for all} \quad \tau$$

$$\text{and} \quad \int_0^1 f_i(\tau) u(\tau)\, d\tau = \alpha_i \quad \text{for } i = 1, \ldots, m \Big\}.$$

Show that A is a convex set.

4. Convex Hull

(4.1) Definitions. Let V be a vector space. A *convex combination* of points $v_1, \ldots, v_m \in V$ is a linear combination

$$\alpha_1 v_1 + \ldots + \alpha_m v_m, \quad \text{where} \quad \sum_{i=1}^m \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots m.$$

A *conic combination* of points $v_1, \ldots, v_m$ is a linear combination

$$\alpha_1 v_1 + \ldots + \alpha_m v_m, \quad \text{where} \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots m.$$

Let $S \subset V$ be a set. The *convex hull* of S is the set (denoted $\operatorname{conv}(S)$) of all convex combinations of points from S. In other words,

$$\operatorname{conv}(S) = \Big\{ \sum_{i=1}^m \alpha_i v_i \quad \text{where} \quad \sum_{i=1}^m \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0,\ v_i \in S : i = 1, \ldots, m \Big\}.$$

Similarly, the *conic hull* of S is the set (denoted $\operatorname{cone}(S)$) is the set of all conic combinations of points from S.

(4.2) Theorem. *The convex hull of $S \subset V$ is the intersection of all convex sets in V containing S. In other words,* $\operatorname{conv}(S)$ *is the smallest convex set containing S.*

Proof. First, we prove that $\operatorname{conv}(S)$ is a convex set itself. Indeed, let us choose two convex combinations $u = \alpha_1 u_1 + \ldots + \alpha_m u_m$ and $v = \beta_1 v_1 + \ldots + \beta_n v_n$ of points from S. The interval $[u, v]$ consists of the points $\gamma u + (1-\gamma) v$ for $0 \leq \gamma \leq 1$. Each such a point $\gamma\alpha_1 u_1 + \ldots + \gamma\alpha_m u_m + (1-\gamma)\beta_1 v_1 + \ldots + (1-\gamma)\beta_n v_n$ is a convex combination of points $u_1, \ldots, u_m, v_1, \ldots, v_n$ from S (since $\gamma\alpha_1 + \ldots + \gamma\alpha_m + (1-\gamma)\beta_1 + \ldots + (1-\gamma)\beta_n$ $= \gamma(\alpha_1 + \ldots + \alpha_m) + (1-\gamma)(\beta_1 + \ldots + \beta_n) = \gamma + (1-\gamma) = 1$). Therefore, $\operatorname{conv}(S)$ is convex.

Now we have to prove that for any convex set A such that $S \subset A$, we must have $\operatorname{conv}(S) \subset A$. Let us choose a convex combination

$$u = \alpha_1 u_1 + \ldots + \alpha_m u_m$$

of points $u_1, \ldots, u_m$ from S. We must prove that $u \in A$. Without loss of generality, we may assume that $\alpha_i > 0$ for $i = 1, \ldots, m$. We proceed by induction on m. If $m = 1$ then $u = u_1$ and $u \in A$ since $S \subset A$. Suppose that $m > 1$. Then $\alpha_m < 1$ and we may write

$$u = (1-\alpha_m)w + \alpha_m u_m, \quad \text{where} \quad w = \frac{\alpha_1}{1-\alpha_m}u_1 + \ldots + \frac{\alpha_{m-1}}{1-\alpha_m}u_{m-1}.$$

Now, w is a convex combination of $u_1, \ldots, u_{m-1}$, because

$$\sum_{i=1}^{m-1} \frac{\alpha_i}{1-\alpha_m} = \frac{1}{1-\alpha_m}\sum_{i=1}^{m-1}\alpha_i = \frac{1-\alpha_m}{1-\alpha_m} = 1.$$

Therefore, by the induction conjecture, we have $w \in A$. Since A is convex, $[w, u_m] \subset A$, so $u \in A$. □

PROBLEMS.

1. Prove that the conic hull of a set S is the intersection of all convex cones containing S.

2. Let $f(z)$ be a polynomial in one complex variable z and let $z_1, \ldots, z_m$ be the roots of f (that is, the set of all solutions to the equation $f(z) = 0$). Let us interpret a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Prove that each root of the derivative $f'(z)$ lies in the convex hull $\operatorname{conv}(z_1, \ldots, z_m)$.

Hint: Without loss of generality we may suppose that $f(z) = (z-z_1)\cdots(z-z_m)$. If w is a root of $f'(z)$ then $\sum_{i=1}^m \prod_{j\neq i}(w - z_j) = 0$, and, therefore, $\sum_{i=1}^m \prod_{j\neq i}(\overline{w - z_j}) = 0$, where $\overline{z}$ is the complex conjugate of z. Multiply both sides of the last identity by $(w-z_1)\cdots(w-z_n)$ and express w as a convex combination of $z_1, \ldots, z_m$.

(4.3) Definition. Convex hull of a finite set of points in a vector space is called a *polytope*.

PROBLEMS.

1. Prove that the set

$$\Delta = \Big\{x \in \mathbb{R}^d : \xi_1 + \ldots + \xi_d = 1 \quad \text{and} \quad \xi_i \geq 0 : \ i = 1, \ldots, d\Big\}$$

is a polytope in $\mathbb{R}^d$. This polytope is called a *simplex*.

2. Prove that the set

$$I = \Big\{x \in \mathbb{R}^d : 0 \leq \xi_i \leq 1 : \ i = 1, \ldots, d\Big\}$$

is a polytope. This polytope is called a *cube*.

3. Prove that the set $A \subset C[0,1]$:

$$A = \Big\{f \in C[0,1] : 0 \leq f(\tau) \leq 1 \quad \text{for each } \tau \in [0,1]\Big\}$$

is not a polytope.

4. Let $P \subset \mathbb{R}^d$ be a polytope. Prove that P is a bounded closed set.

5. Theorems of Carathéodory and Radon

(5.1) Carathéodory's Theorem. *Let $S \subset \mathbb{R}^d$ be a set. Then every point $\mathbf{u} \in \operatorname{conv}(S)$ can be represented as a convex combination of $d+1$ points from S:*

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_{d+1} \mathbf{v}_{d+1}, \quad \text{where} \quad \sum_{i=1}^{d+1} \alpha_i = 1, \quad \alpha_i \geq 0$$

and $\mathbf{v}_i \in S$ for $i = 1, \ldots, d+1$.

Proof. Every point $\mathbf{u} \in \operatorname{conv}(S)$ can be written as a convex combination

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m$$

of some points $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S$. We can assume that $\alpha_i > 0$ for all $i = 1, \ldots, m$. If $m < d+1$, we can add terms $0\mathbf{v}_1$, say, to get a convex combination with $d+1$ terms. Suppose that $m > d+1$. Let us show that we can construct a convex combination with fewer terms. Let us consider a system of linear homogeneous equations in m real variables $\gamma_1, \ldots, \gamma_m$:

$$\gamma_1 \mathbf{v}_1 + \ldots + \gamma_m \mathbf{v}_m = \mathbf{0} \quad \text{and} \quad \gamma_1 + \ldots + \gamma_m = 0.$$

The first vector equation reads as d real linear equations

$$\gamma_1 v_{1j} + \ldots + \gamma_m v_{mj} = 0 : \quad j = 1, \ldots, d,$$

where v_{ij} are the coordinates of $\mathbf{v}_i$: $\mathbf{v}_i = (v_{i1}, \ldots, v_{id})$. Altogether, we have $d+1$ linear homogeneous equations in m variables $\gamma_1, \ldots, \gamma_m$. Since $m > d+1$, there must be a non-trivial solution $\gamma_1, \ldots, \gamma_m$. Since $\gamma_1 + \ldots + \gamma_m = 0$, some γ_i are strictly positive and some are strictly negative. Let

$$\tau = \min\{\alpha_i / \gamma_i : \gamma_i > 0\} = \alpha_{i_0} / \gamma_{i_0}.$$

Let $\widetilde{\alpha}_i = \alpha_i - \tau\gamma_i$ for $i = 1, \ldots, m$. Then $\widetilde{\alpha}_i \geq 0$ for all $i = 1, \ldots, m$ and $\alpha_{i_0} = 0$. Furthermore,

$$\widetilde{\alpha}_1 + \ldots + \widetilde{\alpha}_m = (\alpha_1 + \ldots + \alpha_m) - \tau(\gamma_1 + \ldots + \gamma_m) = 1$$

and

$$\widetilde{\alpha}_1 \mathbf{v}_1 + \ldots + \widetilde{\alpha}_m \mathbf{v}_m = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m - \tau(\gamma_1 \mathbf{v}_1 + \ldots + \gamma_m \mathbf{v}_m) = \mathbf{u}$$

Therefore, we represented u as a convex combination

$$\mathbf{u} = \sum_{i \neq i_0} \widetilde{\alpha}_i \mathbf{v}_i$$

of $m-1$ points $\mathbf{v}_1, \dots, \widehat{\mathbf{v}_{i_0}}, \dots, \mathbf{v}_m$ ($\mathbf{v}_{i_0}$ omitted).

So, if $\mathbf{u}$ is a convex combination of $m > d+1$ points, it can be written as a convex combination of fewer points. Iterating this procedure, we get $\mathbf{u}$ as a convex combination of $d+1$ (or fewer) points from S. $\square$

PROBLEMS.

1. Let $S \subset \mathbb{R}^d$ be a set. Prove that every point $u \in \operatorname{cone}(S)$ is a conic combination of d points from S.

Some general problems about convex hulls.

2°. Prove that if $A \subset B$ then $\operatorname{conv}(A) \subset \operatorname{conv}(B)$.

3°. Prove that $\Big(\operatorname{conv}(A) \cup \operatorname{conv}(B)\Big) \subset \operatorname{conv}(A \cup B)$.

4. Let V be a vector space, let $S \subset V$ be a set and let $u, v \in V$ be points such that $u \notin \operatorname{conv}(S)$ and $v \notin \operatorname{conv}(S)$. Prove that if $u \in \operatorname{conv}\big(S \cup \{v\}\big)$ and $v \in \operatorname{conv}\big(S \cup \{u\}\big)$ then $u = v$.

A generalization of Carathéodory's Theorem:

5*. Let $S_1, \dots, S_{d+1}$ be subsets of $\mathbb{R}^d$. If $u \in \operatorname{conv}(S_i)$ for each S_i, then there exist points $v_i \in S_i$ such that $u \in \operatorname{conv}\big(v_1, \dots, v_{d+1}\big)$. See also I. Bárány and S. Onn, Colourful linear programming and its relatives, *Math. Oper. Res.* **22**(1997), no. 3, 550–567.

6. Let $S \subset \mathbb{R}^d$ be a set and let u be a point *in the interior* of $\operatorname{conv}(S)$. Then one can choose $2d$ points $v_1, \dots, v_{2n} \in S$ such that u lies *in the interior* of $\operatorname{conv}\big(v_1, \dots, v_{2n}\big)$.

7°. Show by an example that the constant $d+1$ in Carathéodory's theorem can not be improved to d.

8*. Suppose that $S \subset \mathbb{R}^d$ is a connected set (or even a union of d connected sets). Prove that every point $u \in \operatorname{conv}(S)$ is a convex combination of some d points of S.

(5.2) Corollary. *Suppose that $S \subset \mathbb{R}^d$ is a compact set. Then $\operatorname{conv}(S)$ is a compact set.*

Proof. Let $\Delta \subset \mathbb{R}^{d+1}$ be the set

$$\Delta = \Big\{(\alpha_1, \dots, \alpha_{d+1}) : \sum_{i=1}^{d+1} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \text{ for } \ i = 1, \dots, d+1\Big\}.$$

Then Δ is compact and so is

$$S^{d+1} \times \Delta = \Big\{\big(u_1, \dots, u_{d+1}; \alpha_1, \dots, \alpha_{d+1}\big) : u_i \in S; (\alpha_1, \dots, \alpha_{d+1}) \in \Delta\Big\}.$$

Consider the map $\Phi : S^{d+1} \times \Delta \longrightarrow \mathbb{R}^d$, $\Phi(u_1, \dots, u_{d+1}; \alpha_1, \dots, \alpha_{d+1}) = \alpha_1 u_1 + \ldots + \alpha_{d+1} u_{d+1}$. Theorem 5.1 implies that the image of Φ is $\operatorname{conv}(S)$. Since Φ is continuous, the proof follows. $\square$

PROBLEMS.

1. Give an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.
2. Prove that the convex hull of an open set in $\mathbb{R}^d$ is open.

(5.3) Radon's Theorem. *Let $S \subset \mathbb{R}^d$ be a set containing at least $d+2$ points. Then there are two non-intersecting subsets $A, B \subset S$: $A \cap B = \emptyset$ such that*

$$\operatorname{conv}(A) \cap \operatorname{conv}(B) \neq \emptyset.$$

Proof. Let $v_1, \ldots, v_m$, $m \geq d+2$ be distinct points from S. Consider the system of homogeneous linear equations in variables $\gamma_1, \ldots, \gamma_m$:

$$\gamma_1 v_1 + \ldots + \gamma_m v_m = 0, \quad \gamma_1 + \ldots + \gamma_m = 0.$$

Since $m \geq d+2$ there is a non-trivial solution to this system. Let

$$A = \big\{ v_i : \ \gamma_i > 0 \big\} \quad \text{and} \quad B = \big\{ v_i : \ \gamma_i < 0 \big\}.$$

Then $A \cap B = \emptyset$.

Let $\alpha = \sum_{i:\gamma_i > 0} \gamma_i$. Then $\beta > 0$ and $\sum_{i:\gamma_i<0} \gamma_i = -\beta$, since γ's sum up to zero. Since $\gamma_1 v_1 + \ldots + \gamma_m v_m = 0$, we have

$$\sum_{i:\gamma_i>0} \gamma_i v_i = \sum_{i:\gamma_i<0} (-\gamma_i) v_i.$$

Let

$$v = \sum_{i:\gamma_i>0} \frac{\gamma_i}{\beta} v_i = \sum_{i:\gamma_i<0} \frac{-\gamma_i}{\beta} v_i.$$

It is seen that v is a convex combination of the points from A and a convex combination of the points from B. In other words, $v \in \operatorname{conv}(A)$ and $v \in \operatorname{conv}(B)$. $\square$

PROBLEMS.

1*. Prove the following Tverberg's Theorem: Given a set S of $(k-1)(d+1)+1$ or more points in $\mathbb{R}^d$, one can find k pairwise non-intersecting subsets $A_1, \ldots, A_k \subset S$ such that the intersection

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \cap \ldots \cap \operatorname{conv}(A_k)$$

is not empty.

For further generalizations, see I. Bárány, Geometric and combinatorial applications of Borsuk's theorem, in J. Pach, editor, *New Trends in Discrete and Computational Geometry*, Algorithms in Combinatorics **10**, 235–249, Springer-Verlag, 1993.

2°. Show by an example that the constant $d+2$ in Radon's Theorem can not be improved to $d+1$.

6. Helly's Theorem

(6.1) Helly's Theorem. *Let $A_1, \ldots, A_m$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that every $d+1$ of the sets have a common point:*

$$A_{i_1} \cap \ldots \cap A_{i_{d+1}} \neq \emptyset.$$

Then all the sets have a common point:

$$A_1 \cap \ldots \cap A_m \neq \emptyset.$$

Proof. The proof is by induction on m (starting with $m = d+1$). Suppose that $m > d+1$. Then, by the induction conjecture, for every $i = 1, \ldots, m$ there is a point p_i in the intersection $A_1 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_m$ (A_i is missing). Altogether, we have $m > d+1$ points p_i, each of which belongs to all the sets, except, perhaps, A_i. By Radon's Theorem (Theorem 5.3) there are non-intersecting subsets $R = \{p_i : i \in I\}$ and $B = \{p_j : j \in J\}$ such that there is a point

$$p \in \operatorname{conv}(R) \cap \operatorname{conv}(B).$$

We claim that p is the common point of $A_1, \ldots, A_m$. Indeed, all the points $p_i : i \in I$ of R belong to the sets $A_i : i \notin I$. All the points $p_j : j \in J$ of B belong to the sets $A_j : j \notin J$. Since the sets A_i are convex, every point from $\operatorname{conv}(R)$ belongs to the sets $A_i : i \notin I$. Similarly, every point from $\operatorname{conv}(B)$ belongs to the sets $A_j : j \notin J$. Therefore,

$$p \in \bigcap_{i \notin I} A_i \quad \text{and} \quad p \in \bigcap_{j \notin J} A_j.$$

Since $I \cap J = \emptyset$ we have

$$p \in \bigcap_{i=1}^{m} A_i$$

and the proof follows. $\square$

PROBLEMS.

1°. Show that the theorem does not hold for non-convex sets A_i.

2°. Construct an example of convex sets A_i in $\mathbb{R}^2$, such that every two sets have a common point, but there is no point which would belong to all sets A_i.

3°. Give an example of an infinite family $\{A_i : i = 1, 2, \ldots\}$ of convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have a common point but there isn't a point common to all sets A_i.

(6.2) Corollary. *Let $\{A_i : i \in I\}$ be a (possibly infinite) family of convex compact sets in $\mathbb{R}^d$ such that the intersection of any $d+1$ sets is not empty:*

$$A_{i_1} \cap \ldots \cap A_{i_{d+1}} \neq \emptyset.$$

Then the intersection of all the sets A_i is not empty:

$$\bigcap_{i \in I} A_i \neq \emptyset.$$

Proof. By Theorem 6.1, for any finite subfamily $J \subset I$ the intersection $\bigcap_{i \in J} A_i$ is not empty. We use the fact that if the intersection of a family of compact sets is empty, then the intersection of the sets from some finite subfamily is empty. $\square$

Here is an example demonstrating one of the numerous applications of Helly's Theorem.

(6.3) Example [Kirchberger]. Suppose that there is a finite set R of red points in $\mathbb{R}^d$ and a finite set B of blue points in $\mathbb{R}^d$. We say that a hyperplane $H \subset \mathbb{R}^d$ strictly separates red and blue points if the sets R and B belong to different open halfspaces H^+ and H^-.

Claim. *Suppose that for any set $S \subset \mathbb{R}^d$ of $d+2$ points (or fewer), there exists a hyperplane which strictly separates the sets $S \cap R$ and $S \cap B$ of red, resp. blue points in S. Then there exists a hyperplane which strictly separates the sets R and B.*

Proof. A hyperplane $H \subset \mathbb{R}^d$ can be written in the form $H = \big\{x \in \mathbb{R}^d : f(x) = \alpha\big\}$, where $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a non-zero linear function, $f(x) = \langle c, x \rangle$ for $c = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$. Such a hyperplane H can be encoded by a point $(c, \alpha) = (\gamma_1, \dots, \gamma_d, \alpha) \in \mathbb{R}^{d+1}$. For every point $r \in R$ we define a set $A_r \subset \mathbb{R}^{d+1}$:

$$A_r = \Big\{(c, \alpha) \in \mathbb{R}^{d+1} : \langle c, r \rangle < \alpha\Big\}$$

and for every point $b \in B$ we define a set $A_b \subset \mathbb{R}^{d+1}$:

$$A_b = \Big\{(c, \alpha) \in \mathbb{R}^{d+1} : \langle c, r \rangle > \alpha\Big\}.$$

It is clear that A_b and A_r are convex sets in $\mathbb{R}^{d+1}$. Therefore, by Helly's Theorem the intersection

$$\Big(\bigcap_{r \in R} A_r\Big) \cap \Big(\bigcap_{b \in B} A_b\Big)$$

is non-empty, provided for any subset $S \subset R \cup B$ of $d+2$ points the intersection

$$\Big(\bigcap_{r \in S \cap R} A_r\Big) \cap \Big(\bigcap_{b \in S \cap B} A_b\Big)$$

is non-empty. Since the sets A_r and A_b are open, their intersection is an open set; therefore an intersection of sets A_r and A_b is non-empty if and only if it contains a point (c, α) with $c \neq 0$. Hence for any subset $S \subset R \cup B$, the intersection

$$\Big(\bigcap_{r \in S \cap R} A_r\Big) \cap \Big(\bigcap_{b \in S \cap B} A_b\Big)$$

is not empty if and only if there is a point $(c, \alpha) \in \mathbb{R}^{d+1}$ such that the hyperplane

$$H = \{x \in \mathbb{R}^d : \langle c, x \rangle = \alpha\}$$

strictly separates the sets $B \cap S$ and $R \cap S$ The proof now follows. $\square$

PROBLEMS.

1. Prove that, if a convex set is contained in the union of a finite family of halfspaces in $\mathbb{R}^d$ (sometimes we say *covered* by a finite family of halfspaces), then it is contained in the union of some $d+1$ (or fewer) halfspaces from the family (covered by some $d+1$ subspaces).
2. Let $A_1, \ldots, A_m$ and C be convex sets in $\mathbb{R}^d$. Suppose that, for any $d+1$ sets $A_{i_1}, \ldots, A_{i_{d+1}}$ there is a translate $C + u : u \in \mathbb{R}^d$ of C which *intersects* all $A_{i_1}, \ldots, A_{i_{d+1}}$. Prove that there is a translate $C + u$ of C which *intersects* all sets $A_1, \ldots, A_m$.
3. In Problem 2, replace *intersects* by *contains.*
4. In Problem 2, replace *intersects* by *is contained in.*
5. Let $I_1, \ldots, I_m$ be parallel line segments in $\mathbb{R}^2$, such that for every three $I_{i_1}, I_{i_2}, I_{i_3}$ there is a straight line that intersects all the three. Then there is a straight line that intersects all the segments $I_1, \ldots, I_m$.
6. Let $A_i : i = 1, \ldots, m$ be convex sets in $\mathbb{R}^2$ such that for every two sets A_i and A_j there is a line parallel to x-axis which intersects them both. Prove that there is a line parallel to x-axis which intersects all sets A_i.
7. Let $A_1, \ldots, A_m$ be convex sets in $\mathbb{R}^d$ and let $k \leq d+1$. Prove that, if every k of the sets have a point in common, then for every $(d-k+1)$-dimensional subspace L in $\mathbb{R}^d$ there exists a translate $L + u : u \in \mathbb{R}^d$ which intersects every set $A_i : i = 1, \ldots, m$.

Helly's Theorem has numerous generalizations, extensions, and ramifications. Here are some of them.

(6.4) Generalizations, extensions, and ramifications

Generalizations to non-convex sets. Let $\mathcal{C}_j^d$ be the family of all sets of $\mathbb{R}^d$ that are the unions of j or fewer convex sets. Then for every $j, d \geq 1$ there exists an integer $c(j, d)$ with the following property:

Let $A_i \in \mathcal{C}_j^d : i = 1, \ldots, m$ be a finite family of subsets such that the intersection of any subfamily is also in $\mathcal{C}_j^d$ and such that every $c(j, d)$ members of the family have a point in common. Then there is a point common to all A_i.

The following generalization is for topologists:

Let $A_i : i = 1, \ldots, m$ be a finite family of closed homology cells in $\mathbb{R}^d$. If the intersection of every $d+1$ or fewer sets A_i is a homology cell, then the intersection of all the sets A_i is a homology cell.

Let $A_i : i = 1, \ldots, m$ be a finite family of algebraic varieties in $\mathbb{R}^d$, each of which is given as a set of solutions to a system of polynomial equations of degree at most k. Suppose that every $\binom{d+k}{k}$ sets A_i have a point in common. Then all the sets A_i have a point in common.

Reducing $d+1$. Let $A_1, \ldots, A_m$ be convex sets in $\mathbb{R}^d$. Let f be the number of $(d+1)$-subfamilies $A_{i_1}, \ldots, A_{i_{d+1}}$ that have a point in common. For any $0 \leq \rho \leq 1$

if $f > \left(1-(1-\rho)^{d+1}\right)\binom{m}{d+1}$, then some $[\rho m]+1$ sets A_i have a point in common (*The fractional Helly Theorem*).

For any $p \geq q \geq d+1$, there exists a positive integer $c(p,q,d)$ with the following property.

Let $A_1, \ldots, A_m$ be convex sets in $\mathbb{R}^d$ and suppose that out of every p sets A_i some q have a point in common. Then some set of $c(p,q,d)$ points intersects every $A_i : i = 1, \ldots, m$.

Measuring the intersection. Let $A_1, \ldots, A_m$ be convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ sets A_i has volume at least 1, then the intersection of all A_i has volume at least d^{-2d^2}.

See J. Eckhoff, Helly, Radon, and Carathédory type theorems, in P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, 389–448, North-Holland, Amsterdam, 1993.

PROBLEM.

1**. Let $A \subset \mathbb{R}^2$ be a convex set and let $A+u_1, A+u_2, \ldots, A+u_m$ for $u_1, \ldots, u_m \in \mathbb{R}^2$ be a family of its translates. Is it true that if every two translates $A+u_i$ and $A+u_j$ intersect, then there is a set of three points $\{p_1, p_2, p_3\}$ such that every $A+u_i$ contains at least one point p_1, p_2 or p_3?

Friday, January 16

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7. An Application to Approximation

(7.1) Chebyshev approximations.

Let us fix some real-valued functions $\phi_0(x), \phi_1(x), \ldots, \phi_d(x)$ on the interval $[0,1]$. Given a function $f : [0,1] \longrightarrow \mathbb{R}$ and a number $\epsilon \geq 0$, we want to construct a linear combination $\Phi_a(x) = \alpha_0\phi_0(x) + \alpha_1\phi_1(x) + \ldots + \alpha_d\phi_d(x)$: $a = (\alpha_0, \ldots, \alpha_d)$ such that

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \text{for each } 0 \leq x \leq 1.$$

This is is the problem of the *uniform* or *Chebyshev* approximation. Sometimes we want to approximate f on a grid: we fix N points $x_1, \ldots, x_N$ on the interval $[0,1]$ and we want to find $\alpha_0, \ldots, \alpha_d$ such that

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \text{for } x = x_1, \ldots, x_N.$$

(7.2) Definition. For an $x \in [0,1]$ let us define a set $A(x) \subset \mathbb{R}^{d+1}$:

$$A(x) = \Big\{(\alpha_0, \ldots, \alpha_d) : \big|f(x) - \alpha_0\phi_0(x) - \ldots - \alpha_d\phi_d(x)\big| \leq \epsilon\Big\}.$$

In other words, $A(x)$ is the set of functions Φ_a that approximate f within ϵ at the point x.

PROBLEMS.

1°. Prove that $A(x) \subset \mathbb{R}^{d+1}$ is a closed convex set.

2°. Show that for $d \geq 1$ the set $A(x)$ is *not* compact.

First, we discuss approximations on the grid.

(7.3) Proposition. *Let us fix an $\epsilon \geq 0$. Suppose that for any $d+2$ points $x_{i_1}, \ldots, x_{i_{d+2}}$ of the grid one can construct a function $\Phi_a(x)$ (with $a = (\alpha_0, \ldots, \alpha_d)$ depending on $x_{i_1}, \ldots, x_{i_{d+2}}$) such that*

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \textit{for } x = x_{i_1}, \ldots, x_{i_{d+2}}.$$

Then there exists a function $\Phi_a(x)$ such that

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \textit{for all} \quad x = x_1, \ldots, x_N.$$

Proof. We know that

$$A(x_{i_1}) \cap \ldots \cap A(x_{i_{d+2}}) \neq \emptyset$$

for all possible choices of $d+2$ points in the grid. By Problem 1, Section 7.2 all the sets $A(x)$ are convex. Therefore, by Helly's Theorem (Theorem 6.1), the intersection of all sets $A(x_i)$ is non-empty:

$$A(x_1) \cap \ldots \cap A(x_N) \neq \emptyset.$$

It follows then that for a point $a \in A(x_1) \cap \ldots \cap A(x_N)$ we have $|f(x) - \Phi_a(x)| \leq \epsilon$ for all $x = x_1, \ldots, x_N$. □

Now we turn to uniform approximations.

We suppose that the functions $\phi_0, \ldots, \phi_d$ are not too "weird".

(7.4) Definition. We say that the system of functions $\phi_0, \ldots, \phi_d$ is *somewhat regular* provided there is a finite set $y_1, \ldots, y_M$ of points of the interval $[0,1]$ such that the following condition is satisfied:

if for some $\Phi_a = \alpha_0\phi_0 + \ldots + \alpha_d\phi_d$ we have $\Phi_a(y_1) = \ldots = \Phi_a(y_M) = 0$ then we must have $\alpha_0 = \ldots = \alpha_d = 0$.

PROBLEM.

1°. Prove that the system $\phi_0(x) = 1$, $\phi_1(x) = x, \ldots, \phi_d(x) = x^d$ of functions is somewhat regular with $M = d+1$ for any choice of $y_1, \ldots, y_{d+1}$.

2. Prove that the intersection $A(y_1) \cap \ldots \cap A(y_M)$ is compact.

Hint: The intersection $A(y_1) \cap \ldots \cap A(y_M)$ is closed (see Problem 1, Section 7.2). It remains to show that it is bounded. Let us define a function

$$G : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}, \quad G(a) = \max\big\{|\Phi_a(x)| : x = y_1, \ldots, y_M\big\}.$$

Then $G(\lambda a) = |\lambda| G(a)$ for $\lambda \in \mathbb{R}$, $G(a) > 0$ for $a \neq 0$ and G is continuous. Therefore,

$$\min\big\{G(a) : \|a\| = 1\big\} = \delta > 0$$

and $G(a) > \delta R$ provided $\|a\| > R$. Now, if $|f(x) - \Phi_a(x)| \leq \epsilon$ for $x = y_1, \ldots, y_M$, we have $|\Phi_a(x)| \leq |f(x)| + \epsilon$ for $x = y_1, \ldots, y_M$ and $\|a\| \leq R/\delta$, where $R = \epsilon + \max\{|f(y_i)| : i = 1, \ldots, M\}$.

(7.5) Proposition. *Let us fix an $\epsilon \geq 0$. Suppose that the system of functions $\phi_0, \ldots, \phi_d$ is somewhat regular. Suppose further, that for any set of $d+2$ points $x_1, \ldots, x_{d+2}$ of $[0,1]$, one can construct a function $\Phi_a(x)$ (with $a = (\alpha_0, \ldots, \alpha_d)$ depending on $x_1, \ldots, x_{d+2}$) such that*

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \textit{for} \;\; x = x_1, \ldots, x_{d+2}.$$

Then there exists a function $\Phi_a(x)$ such that

$$|f(x) - \Phi_a(x)| \leq \epsilon \quad \textit{for all} \;\; x \in [0,1].$$

Proof. For each $x \in [0,1]$ let $A(x)$ be the set of Definition 7.2. Let

$$A = A(y_1) \cap \ldots \cap A(y_m),$$

where $y_1, \ldots, y_m$ are the points from Definition 7.4. For $x \in [0,1]$ let $\overline{A(x)} = A(x) \cap A$. Then, by Problem 2, Section 7.4, the set $\overline{A(x)}$ is compact. Applying Helly's Theorem as in the proof of Proposition 7.3, we conclude that every intersection of *finite* family of sets $A(x)$ is non-empty. Therefore, every intersection $\overline{A(x_1)} \cap \ldots \overline{A(x_{d+2})}$ is a non-empty compact convex set. Therefore, By Corollary 6.2, the intersection of all the sets $\overline{A(x)}$ is non-empty and so is the intersection of all the sets $A(x)$. A point

$$a = (\alpha_0, \ldots, \alpha_d) \in \bigcap_{x \in [0,1]} A(x)$$

gives rise to a function

$$\Phi_a = \alpha_0 \phi_0 + \ldots + \alpha_d \phi_d$$

which approximates $f(x)$ uniformly within the error ϵ. □

PROBLEM.

1°. Let $f(x) = e^x$ and let $\phi_i(x) = x^i$ for $i = 0, \ldots, d$. Let us choose $\epsilon = 0$. Check that each intersection $A(x_1) \cap \ldots A(x_{d+1})$ is not empty for any choice of $x_1, \ldots, x_{d+1} \in [0,1]$, but $\bigcap_{x \in [0,1]} A_x = \emptyset$. In other words, for every $d+1$ points $x_1, \ldots, x_{d+1}$ there is a polynomial $p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_d x^d$ such that $p(x) = e^x$ for $x = x_1, \ldots, x_{d+1}$ but there isn't a polynomial $p(x)$ such that $e^x = p(x)$ for all $x \in [0,1]$.

(7.6) Constructing the best Chebyshev approximation. Suppose that the functions $f(x), \phi_0(x), \ldots, \phi_d(x)$ are continuous and somewhat regular. We want to find *the smallest possible* $\epsilon \geq 0$ and a linear combination $\Phi_a = \alpha_0 \phi_0 + \ldots + \alpha_d \phi_d$ such that

$$\max\Big\{|f(x) - \Phi_a(x)| : 0 \leq x \leq 1\Big\} \leq \epsilon.$$

In other words, we want to construct a function Φ_a that approximates $f(x)$ best uniformly on $[0,1]$. Proposition 7.5 gives us a clue how to do that. For every $d+2$ points $x_1, \ldots, x_{d+2}$ of $[0,1]$ we solve a *finite dimensional* optimization problem (in fact, *a linear programming problem*):

Find $\min \epsilon$

Subject to:

$$-\epsilon \leq f(x) - \alpha_0\phi_0(x) - \ldots - \alpha_d\phi_d(x) \leq \epsilon \qquad \text{for } \; x = x_1, x_2, \ldots, x_{d+2}$$

Let $\epsilon(x_1, \ldots, x_{d+2})$ be the solution of the problem. Then

$$\epsilon_0 = \max\Big\{\epsilon(x_1, \ldots, x_{d+2}) : 0 \leq x_1 < x_2 < \ldots < x_{d+2} \leq 1\Big\}$$

is the best possible error of the uniform approximation.

If the function $\epsilon(x_1, \ldots, x_{d+2})$ is known, finding ϵ_0 is a *finite-dimensional optimization problem* again.

In many important cases the function $\epsilon(x_1, \ldots, x_{d+2})$ can be found.

PROBLEM.

1*. Suppose that $\phi_i(x) = x^i$, $i = 0, \ldots, d$ and that $f(x)$ is any function. Prove that for any $d+2$ points $0 \leq x_1 < x_2 < \ldots < x_{d+2} \leq 1$ there is a unique polynomial $p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_d x^d$, such that

$$|f(x_1) - p(x_1)| = |f(x_2) - p(x_2)| = \ldots = |f(x_{d+2}) - p(x_{d+2})|$$

and the signs of the differences

$$f(x_1) - p(x_1), f(x_2) - p(x_2), \ldots, f(x_{d+2}) - p(x_{d+2})$$

alternate. The polynomial p gives the unique best uniform approximation to f on the set of $d+2$ points $x_1, \ldots, x_{d+2}$. The error ϵ of this approximation can be therefore found as $\epsilon = |\beta|$, where $\alpha_0, \ldots, \alpha_d, \beta$ is the (necessarily unique) solution to the system of $d+2$ linear equations

$$f(x_1) - p(x_1) = \beta, \;\; f(x_2) - p(x_2) = -\beta, \ldots, f(x_{d+2}) - p(x_{d+2}) = (-1)^{d+1}\beta$$

in $d+2$ variables $(\alpha_0, \ldots, \alpha_d, \beta)$.

Wednesday, January 21

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8. Some Linear Algebra

(8.1) Affine subspaces

Let V be a vector space and let $L \subset V$ be a subspace of V. For a vector $u \in V$ let

$$L + u = \big\{x + u : x \in L\big\}$$

be a translation of L. The set $A = L + u$ is called an *affine subspace* of V. The dimension of A is the dimension of L. In particular, if $\dim \; A = 1$, the set A is called a *straight line*. A straight line can be written in the parametric form $A = \big\{u + \alpha v : \alpha \in \mathbb{R}\big\}$, where $u, v \in V$ are vectors and $v \neq 0$. We say that $A = L + v$ is *parallel* to L. Another important example of an affine subspace is an affine hyperplane: Let

$f : V \longrightarrow \mathbb{R}$ be a linear function (that is, $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for each $u, v \in V$ and each $\alpha, \beta \in \mathbb{R}$), which is not identically zero. The set

$$H = \{v \in V : f(v) = \alpha\}, \quad \text{where } \alpha \in \mathbb{R}$$

is called an *affine hyperplane*.

A linear combination

$$v = \alpha_1 v_1 + \ldots + \alpha_m v_m \quad \text{where} \quad \alpha_1 + \ldots + \alpha_m = 1$$

is called an *affine combination*. Similarly, points $v_1, \ldots, v_m \in V$ are said to be *affinely independent* if, whenever

$$\alpha_1 v_1 + \ldots + \alpha_m v_m = 0 \quad \text{and} \quad \alpha_1 + \ldots + \alpha_m = 0$$

we must have $\alpha_1 = \ldots = \alpha_m = 0$.

Some easy problems.

PROBLEMS.

1°. Prove that an affine combination of vectors from an affine subspace is a vector from the subspace.

2°. Prove that the intersection of affine subspaces is an affine subspace.

3°. Prove that an affine hyperplane is an affine subspace.

4°. Prove that an affine subspace is a subspace if and only if it contains 0.

5°. Let $A \subset V$ be an affine subspace of dimension n. Prove that the maximum number of affinely independent points in A is $n + 1$.

6°. Let $L \subset V$ be a subspace and let $v, u \in V$ be vectors. Then $L + u = L + v$ if and only if $u - v \in L$.

(8.2) Quotients, projections, and codimension

Let V be a vector space and let $L \subset V$ be a subspace. We can form the *quotient space* V/L as follows: the points of V/L are affine subspaces of the type $L + v$, where $v \in V$. The addition in V/L is defined as follows: $A_1 + A_2 = A_3$ provided $A_1 = L + v_1$, $A_2 = L + v_2$, $A_3 = L + v_3$ and $v_1 + v_2 = v_3$ for some v_1, v_2 and v_3. The scalar multiplication in V/L is defined as follows: $\alpha A_1 = A_2$ provided $A_1 = L + v_1$, $A_2 = L + v_2$ and $v_2 = \alpha v_1$ for some $v_1, v_2 \in V$. Thus L itself is the 0 of the quotient V/L. The dimension of V/L is called the *codimension* of L (denoted codim L). If $A = L + u$ is an affine subspace, the codimension of A is defined to be the codimension of L. There is a linear transformation, called *projection* $pr : V \longrightarrow V/L$, where $pr(v) = L + v$.

Some easy problems.

PROBLEMS.

1°. Prove that the addition and scalar multiplication in V/L are well-defined (do not depend on the particular choice of v_1 and v_2).

Addition: let $L \subset V$ be a subspace. Suppose there are two vectors u_1, v_1 such that $L + u_1 = L + v_1 = A_1$ and two vectors u_2, v_2 such that $L + u_2 = L + v_2 = A_2$. Let $u_3 = u_1 + u_2$ and $v_3 = v_1 + v_2$. Then $L + v_3 = L + u_3 = A_3$.

Multiplication: let $L \subset V$ be a subspace. Suppose there are two vectors u_1, v_1 such that $L + u_1 = L + v_1 = A_1$. For $\alpha \in \mathbb{R}$ let $v_2 = \alpha v_1$ and $u_2 = \alpha u_1$. Then $L + u_2 = L + v_2 = A_2$.

2°. Prove that the affine hyperplanes are exactly the affine subspaces of codimension 1.

3°. Prove that the projection $V \longrightarrow V/L$ is indeed a linear transformation, that its image is the whole space V/L, and that its kernel is L.

4°. Let $L \subset \mathbb{R}^d$ be a subspace. Prove that $\dim\ L + \operatorname{codim}\ L = d$.

Friday, January 23

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9. The Isolation Theorem

(9.1) Halfspaces

Let V be a vector space and let $H \subset V$ be an affine hyperplane. Then the complement of H in V is the union of two convex sets, called *open halfspaces*: $V \setminus H = H_+ \cup H_-$. Indeed, suppose that $H = \{x \in \mathbb{R}^d : f(x) = \alpha\}$, where $f : V \longrightarrow \mathbb{R}$ is a linear function and $\alpha \in \mathbb{R}$ is a number. We let $H_+ = \{x \in \mathbb{R}^d : f(x) > \alpha\}$ and $H_- = \{x \in \mathbb{R}^d : f(x) < \alpha\}$. Of course, if we choose a different equation for H (say, $-f(x) = -\alpha$) then H_+ and H_- interchange. The sets $\overline{H_+} = H \cup H_+$ and $\overline{H_-} = H \cup H_-$ are called *closed subspaces.* We can write $\overline{H_+} = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}$ and $\overline{H_-} = \{x \in \mathbb{R}^d : f(x) \leq \alpha\}$.

PROBLEMS.

1°. Check that open halfspaces and closed halfspaces are convex.

2°. Check that an open halfspace of $\mathbb{R}^d$ is an open subset of $\mathbb{R}^d$ and that a closed halfspace of $\mathbb{R}^d$ is a closed subset of $\mathbb{R}^d$.

(9.2) Definitions. Let V be a vector space, let $A \subset V$ be a set and let $H \subset V$ be a hyperplane. We say that H *isolates* A if A is contained in one of the closed subspaces $\overline{H_-}$ or $\overline{H_+}$. We say that H *strictly isolates* A if A is contained in one of the open halfspaces H_- or H_+.

Let V be a vector space, let $A, B \subset V$ be sets and let $H \subset V$ be a hyperplane. We say that H *separates* A and B if A is contained in one closed halfspace and B is contained in the other. We say that H *strictly separates* A and B if A is contained in one open halfspace and B is contained in the other open halfspace.

PROBLEM.

1°. Prove that sets $A, B \subset V$ can be separated, resp. strictly separated, by a hyperplane if and only if there is a linear function $f : V \longrightarrow \mathbb{R}^d$ and a number $\alpha \in \mathbb{R}$ such that $f(x) \leq \alpha \leq f(y)$, resp. $f(x) < \alpha < f(y)$ for any $x \in A$ and any $y \in B$.

(9.3) Definition. Let V be a vector space and let $A \subset V$ be a set. The set A is called *algebraically open* if the intersection of A with every straight line in V is a union of open intervals (possibly empty). In particular, if A is convex and $L = \{v + \tau u : \tau \in \mathbb{R}\}$ is a straight line in V, where $v, u \in V$, then

$$A \cap L = \{v + \tau u : \alpha < \tau < \beta\}, \quad \text{where}$$

either $-\infty < \alpha < \beta < +\infty$ (the intersection is a non-empty open interval);
or $\alpha = -\infty < \beta < +\infty$ (the intersection is an open ray);

or $-\infty < \alpha < \beta = +\infty$ (the intersection is an open ray);
or $-\infty = \alpha < \beta = +\infty$ (the intersection is the whole straight line);
or $\alpha \geq \beta$ (the intersection is empty).

PROBLEMS.

1°. Prove that open sets in $\mathbb{R}^d$ are algebraically open.

2. Construct an example of a set $A \subset \mathbb{R}^2$ such that the intersection of A with every line through the origin is an open interval, but A is not open.

3. Let $A \subset \mathbb{R}^d$ be a convex set. Prove that it is open if and only if it is algebraically open.

4. Prove, that if an algebraically open set is isolated by a hyperplane, it is strictly isolated by the hyperplane.

5. Let V be a vector space, let $A, B \subset V$ be algebraically open sets. Prove that if A and B are separated by a hyperplane H then A and B are strictly separated by H.

6°. Prove that the union of algebraically open sets is algebraically open and that the intersection of a finite family of algebraically open sets is algebraically open.

7. Let V and W be vector spaces, let $T : V \longrightarrow W$ be a linear transformation whose image is W. Let $A \subset V$ be an algebraically open set in V. Then the image $T(A)$ is algebraically open in W.

8°. Let $A \subset V$ be an algebraically open set and let $L \subset V$ be a subspace. Prove that $A \cap L$ is algebraically open as a set in L.

9. Let V be a vector space. Problem 7 implies that all algebraically open sets constitute a *topology* in V. Prove that every linear function $f : V \longrightarrow \mathbb{R}$ is continuous in this topology. Furthermore, prove that any linear transformation $T : V \longrightarrow W$, where V and W are vector spaces, is continuous in the topology of algebraically open sets.

(9.4) The Isolation Theorem. *Let V be a vector space, let $A \subset V$ be an algebraically open convex set and let $u \notin A$ be a point. Then there exists a hyperplane H that contains u and strictly isolates A.*

Proof. Without loss of generality we may assume that $u = 0$ is the origin.

First, we prove the result in the case of $V = \mathbb{R}^2$. Let $S = \big\{x \in \mathbb{R}^2 : \|x\| = 1\big\}$ be the circle centered at the origin of radius 1. Let us project radially A into S: $v \longrightarrow v/\|v\|$. Since A is convex, it is connected, and, therefore, the image of this projection is a connected arc Γ of S. Furthermore, since A is algebraically open, Γ must be an open arc

$$\Gamma = \Big\{(\cos\phi,\ \sin\phi) : \alpha < \phi < \beta\Big\}$$

of the circle S. Indeed, let $x \in \Gamma$ be a point. Then $x = v/\|v\|$ for some $v \in A$, so we can choose a straight line L through v parallel to the tangent line to S at x. Then the intersection $L \cap A$ will be an open interval containing v, so the radial projection of A will contain an open arc containing x. Next, we observe that the length of Γ can not be greater then π, because otherwise Γ would have contained two antipodal points x and $-x$ and 0 would have been in A, since A is convex. Now, let v be an endpoint of Γ (which is not in Γ, since Γ is open). The straight line through 0 and v is the desired hyperplane, containing 0 and strictly isolating A.

Next, suppose that dim $V \geq 2$. We prove, that there is a straight line L, such that $0 \in L$ and $L \cap A = \emptyset$. To prove this, let us consider any 2-dimensional plane

P containing 0. The intersection $B = P \cap A$ is a convex algebraically open subset of P (possibly empty, see Problem 8 of Section 9.3) and as we proved, there is a line $L \subset P$ such that $0 \in L$ and $L \cap B = \emptyset$. Then L is the desired straight line.

Now, we prove the theorem. Let $H \subset V$ be a largest subspace such that $0 \in H$ and $H \cap A = \emptyset$. By "largest", we mean a subspace which has these properties and is not contained in a larger subspace with the same properties. If V is finite-dimensional, we could chose H to be a subspace of the largest possible dimension such that $0 \in H$ and $H \cap A = \emptyset$. If V is arbitrary, the existence of such an H is ensured by Zorn's Lemma. We claim that H is a hyperplane. To prove this, consider the quotient V/H and let $pr : V \longrightarrow V/H$ be the projection. If H is not a hyperplane, then $\dim\ V/H \geq 2$ and $pr(A)$ is an algebraically open subset in V/H (see Problem 7, Section 9.3). Then, as we proved, there is a straight line $L \subset V/H$ such that $0 \in L$ and $L \cap pr(A) = \emptyset$. Then the preimage $G = pr^{-1}(L) = \big\{x : pr(x) \in L\big\}$ is a subspace in V, such that $0 \in G$, $G \cap A = \emptyset$, $H \subset G$ and G is strictly larger than H. This contradiction shows that H is a hyperplane. □

PROBLEMS.

1°. Construct an example of a non-convex algebraically open set $A \subset \mathbb{R}^2$ such that $0 \notin A$ and there isn't a hyperplane H such that $0 \in H$ and H strictly isolates A.

2. Let V be a vector space of all polynomials in one variable. Let $A \subset V$ be the set of all polynomials whose highest coefficient is positive. Prove that $0 \notin A$, that A is convex, that A is not algebraically open and that there isn't a hyperplane H such that $0 \in H$ and H isolates A.

3. Prove the following generalization of Theorem 9.4: Let V be a vector space, let $A \subset V$ be an algebraically open convex set, and let $L \subset V$ be an affine subset such that $L \cap A = \emptyset$. Then there exists a hyperplane H containing L that strictly isolates A.

Monday, January 26

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10. Convex Sets in Euclidean Space

(10.1) Definition. Let $A \subset \mathbb{R}^d$ be a set. A point $u \in A$ is called an *interior* point of A if there exists an $\epsilon > 0$ such that the (open) ball $B(u, \epsilon) = \big\{x : \|x - u\| < \epsilon\big\}$ centered at u and of the radius ϵ is contained in A: $B(u, \epsilon) \subset A$. The set of all interior points of A is called the *interior* of A and denoted $\operatorname{int}(A)$. The set of all non-interior points of A is called the *boundary* of A and denoted ∂A.

(10.2) Lemma. *Let $A \subset \mathbb{R}^d$ be a convex set and let $u_0 \in \operatorname{int}(A)$ be an interior point of A. Then, for any point $u_1 \in A$ and any $0 \leq \alpha < 1$, the point $u_\alpha = (1-\alpha)u_0 + \alpha u_1$ is an interior point of A.*

Proof. Let $B(u_0, \epsilon) \subset A$ be a ball centered at u_0 and contained in A. Then elementary geometry shows that $B\big(u_\alpha, (1-\alpha)\epsilon\big) \subset A$. □

(10.3) Corollary. *Let $A \subset \mathbb{R}^d$ be a convex set. Then $\operatorname{int}(A)$ is a convex set.*

Proof. Let $u, v \in \operatorname{int}(A)$ be points and let $w = \alpha u + (1-\alpha)v$ for $0 \leq \alpha \leq 1$. If $\alpha < 1$ we apply Lemma 10.2 with $u_0 = v$, $u_1 = u$ and $w = u_\alpha$ to show that $w \in \operatorname{int}(A)$. If $\alpha = 1$ then $w = u \in \operatorname{int}(A)$. □

PROBLEMS.

1. Let $v_1, \ldots, v_{d+1} \in \mathbb{R}^d$ be affinely independent points in $\mathbb{R}^d$. The polytope $\Delta = \operatorname{conv}\big(v_1, \ldots, v_{d+1}\big)$ is called a d-dimensional simplex. Prove that Δ has non-empty interior.

Hint: Let $u = (v_1 + \ldots + v_{d+1})/(d+1) \in \Delta$. We claim that for a sufficiently small $\epsilon > 0$ we have $B(u, \epsilon) \subset \Delta$. Indeed, the matrix of the system of $(d+1)$ linear equations in $(d+1)$ variables

$$\gamma_1 v_1 + \ldots + \gamma_{d+1} v_{d+1} = w, \quad \gamma_1 + \ldots + \gamma_{d+1} = 1$$

is non-degenerate. Therefore, for each $w \in \mathbb{R}^d$, there is a unique solution $\gamma_1, \ldots, \gamma_{d+1}$ and the solution depends on w continuously. If $w = u$ then $\gamma_1 = \ldots = \gamma_{d+1} = 1/(d+1) > 0$. Therefore, if w is sufficiently close to u, all the γ's are non-negative and $w \in \Delta$.

2. Let $A \subset \mathbb{R}^d$ be a convex set, such that $\operatorname{int}(A) \neq \emptyset$ and let $H \subset \mathbb{R}^d$ be a hyperplane isolating $\operatorname{int}(A)$. Prove that H isolates A.

(10.4) Theorem. *Let $A \subset \mathbb{R}^d$ be a convex set. If* $\operatorname{int}\ A = \emptyset$*, there exists an affine subspace $L \subset \mathbb{R}^d$ such that $A \subset L$ and* $\dim\ L < d$.

Proof. First, we claim there are no $d+1$ affinely independent points $v_1, \ldots, v_{d+1} \in A$. For if there were such points, then $\Delta = \operatorname{conv}\big(v_1, \ldots, v_{d+1}\big) \subset A$ and Problem 1, Section 10.3 would have implied that Δ contains an interior point. Let $k < d+1$ be the maximum number of affinely independent points in A and let $v_1, \ldots, v_k$ be such points. Then, for each point $v \in A$ there is a non-trivial solution to the system $\gamma_1 v_1 + \ldots + \gamma_k v_k + \gamma v = 0$, $\gamma_1 + \ldots + \gamma_k + \gamma = 0$ with $\gamma \neq 0$, so $v \in A$ can be expressed as an affine combination of $v_1, \ldots, v_k$. Therefore, A is contained in the affine subspace L that is the affine hull (the set of all affine combinations) of $v_1, \ldots, v_k$. So, $\dim\ L = k - 1$. □

The *dimension* of a convex set $A \subset \mathbb{R}^d$ is the dimension of the smallest affine subspace that contains A.

(10.5) Definitions. Let $K \subset \mathbb{R}^d$ be a closed convex set. A set $F \subset K$ is called a *face* of K if there exists an affine hyperplane H which isolates K and such that $F = K \cap H$. If F is a point, then F is called an *exposed point* of K. The dimension of the empty face is $-\infty$.

PROBLEMS.

1°. A face is a closed convex set. A face of a compact convex set is a compact convex set.

2. Find the faces of the unit ball $B = \big\{x \in \mathbb{R}^d : \|x\| \leq 1\big\}$.

3. Describe the faces of the unit d-dimensional cube $I = \Big\{x = (\xi_1, \ldots, \xi_d) : 0 \leq \xi_k \leq 1 \text{ for } k = 1, \ldots, d\Big\}$.

4. Describe the faces of a d-dimensional simplex $\Delta = \operatorname{conv}\big(v_1, \ldots, v_{d+1}\big)$, where $v_1, \ldots, v_{d+1}$ are affinely independent points in $\mathbb{R}^d$.

5. Let $K \subset \mathbb{R}^d$ be a convex compact set. Prove that the intersection of any two faces of K is a face of K.

6. Construct an example of a convex compact set $K \subset \mathbb{R}^2$, a face F of K and a face G of F such that G is not a face of K.

7. Prove that every non-empty convex compact set in $\mathbb{R}^d$ has an exposed point.

8. Construct a convex compact set $A \subset \mathbb{R}^2$ whose set of exposed points is not compact.

9*. Prove Straszewicz' theorem: any convex compact set $A \subset \mathbb{R}^d$ is the closure of the convex hull of the set of its exposed points.

(10.6) Theorem. *Let $K \subset \mathbb{R}^d$ be a closed convex set with non-empty interior and let $u \in \partial K$ be a point. Then there exists a hyperplane H, called a support hyperplane at u, such that $u \in H$ and H isolates K.*

Proof. By Corollary 10.3, $\operatorname{int}(K)$ is a non-empty convex open set. Therefore, $\operatorname{int}(K)$ is a convex, algebraically open set, such that $u \notin \operatorname{int}(K)$. Therefore, by Theorem 9.4, there is an affine hyperplane H containing u and isolating $\operatorname{int}(K)$. Then by Problem 2, Section 10.3, H isolates K, so H is a support hyperplane at u. □

PROBLEM.

1°. Construct an example of a convex closed set $K \subset \mathbb{R}^2$ with non-empty interior and a point $u \in \partial K$ such that a support hyperplane of K at u is not unique.

2°. Let $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be the unit ball and let $u \in \partial B$ be a point. Find the support hyperplane to B at u.

(10.7) Corollary. *Let $K \subset \mathbb{R}^d$ be a closed convex set with non-empty interior and let $u \in \partial K$ be a point. Then there is a face F of K such that $u \in F$.*

Proof. Let H be a support hyperplane of K at u. Let $F = H \cap K$. □

Finally, we prove a version of the isolation theorem for convex sets in $\mathbb{R}^d$.

(10.8) Theorem. *Let $A \subset \mathbb{R}^d$ be a non-empty convex set and let $u \notin A$ be a point. Then there is an affine hyperplane $H \subset \mathbb{R}^d$ such that $u \in H$ and H isolates A.*

Proof. Let us choose the minimal affine subspace $L \subset \mathbb{R}^d$ such that $A \subset L$. Theorem 10.4 implies that A has a non-empty interior as a subset of L. If $u \notin L$, we can choose H parallel to L. Thus, restricting ourselves to L, we see that $\operatorname{int}(A) \neq \emptyset$ (in L) and that $u \in L$. Then, by Theorem 9.4, there is an affine hyperplane $\hat{H}$ in L, such that $u \in \hat{H}$ and $\hat{H}$ isolates $\operatorname{int}(A)$. Then, by Problem 2, Section 10.3, $\hat{H}$ isolates A. Then we choose any hyperplane H such that $H \cap L = \hat{H}$. □

PROBLEM.

1. Let $A \subset \mathbb{R}^d$ be a convex set and let $L \subset \mathbb{R}^d$ be an affine subspace such that $L \cap A = \emptyset$. Then there exists an affine hyperplane H such that $L \subset H$ and H isolates A.

11. The Krein-Milman Theorem in Euclidean Space

(11.1) Extreme points

Let V be a vector space and let $A \subset V$ be a set. We recall that a point $a \in A$ is called an *extreme* point of A provided for any $b, c \in A$ such that $(b + c)/2 = a$, one must have $b = c = a$. The set of all extreme points of A is denoted $\operatorname{ex}(A)$.

(11.2) Theorem. *Let V be a vector space, let $A \subset V$ be a non-empty set, and let $f : V \longrightarrow \mathbb{R}$ be a linear function.*

(1) *Suppose that f attains its unique maximum on A at a point $u \in A$, that is $f(u) > f(v)$ for any $v \neq u, v \in A$. Then u is an extreme point of A;*

(2) *Suppose that f attains its maximum α on A and suppose that $B = \{x \in A : f(x) = \alpha\}$ is the set where the maximum is attained. Let u be an extreme point of B. Then u is an extreme point of A.*

Proof. Let us prove the first part. If $u = (a+b)/2$ then $f(u) = (f(a)+f(b))/2$, where $f(a) \leq f(u)$ and $f(b) \leq f(u)$. Therefore, $f(a) = f(b) = f(u)$ and we must have $a = b = u$, because the maximum is unique. Let us prove the second part. Suppose that $u = (a+b)/2$ for $a, b \in A$. Then $\alpha = f(u) = (f(a)+f(b))/2$ and $f(a), f(b) \leq \alpha$. Thus we must have $f(a) = f(b) = \alpha$, so $a, b \in B$. Then $a = b = u$ since u is an extreme point of B. □

PROBLEMS.

1°. Let $K \subset \mathbb{R}^d$ be a convex compact set and let $F \subset K$ be a face. If $u \in F$ is an extreme point of F then u is an extreme point of K.

2. Let $K \subset \mathbb{R}^d$ be a compact convex set and let $u \in K$ be a point such that $\|u\| \geq \|v\|$ for each $v \in K$. Prove that u is an extreme point of K.

(11.3) Krein-Milman Theorem. *Let $K \subset \mathbb{R}^d$ be a convex compact set. Then K is the convex hull of the set of its extreme points: $K = \operatorname{conv}(\operatorname{ex}(K))$.*

Proof. We proceed by induction on the dimension d. If $d = 0$ then K is a point and the result follows. Suppose that $d > 0$. Without loss of generality we may suppose that $\operatorname{int}(K) \neq \emptyset$ (otherwise, K lies in an affine subspace of a smaller dimension and the result follows by the induction conjecture). It suffices to prove that every point $u \in K$ can be represented as a convex combination of extreme points of K. If $u \in \partial K$, then, by Corollary 10.7, there exists a face F of K such that $u \in F$. Then F lies in an affine subspace of a smaller dimension, and by the induction conjecture $u \in \operatorname{conv}(\operatorname{ex}(F))$, so the result follows since $\operatorname{ex}(F) \subset \operatorname{ex}(K)$ (see Problem 1, Section 11.2). Suppose that $u \in \operatorname{int}(K)$. Let us draw a straight line L trough u. The intersection $L \cap K$ is an interval $[a, b]$, where $a, b \in \partial K$ and u is an interior point of $[a, b]$. As we already proved, $a, b \in \operatorname{conv}(\operatorname{ex}(K))$. Since u is a convex combination of a and b, the proof follows. □

PROBLEMS.

1. Prove that the set of extreme points of a closed convex set in $\mathbb{R}^2$ is closed.

2. Construct an example of a compact convex set $K \subset \mathbb{R}^3$ such that $\operatorname{ex}(K)$ is not closed.

3. Let $A \subset \mathbb{R}^d$ be a set. Prove that u is an extreme point of $\operatorname{conv}(A)$ if and only if $u \in A$ and $u \notin \operatorname{conv}(A \setminus \{u\})$.

4. Construct an example of a convex compact set $K \subset \mathbb{R}^2$ and a point $u \in K$ such that u is an extreme point of K, but not an exposed point of K.

5°. Prove that exposed point is an extreme point.

6*. Let $A \subset \mathbb{R}^d$ be a closed convex set. Prove that each extreme point of A is a limit of exposed points of A.

(11.4) Corollary. *Let $K \subset \mathbb{R}^d$ be a convex compact set and let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. Then there exists an extreme point u of K such that $f(u) \geq f(x)$ for any $x \in K$.*

Proof. Clearly, f attains its maximum value, say, α on K. Let $F = \{x \in K : f(x) = \alpha\}$ be the corresponding face of K. Then $\operatorname{ex}(F) \neq \emptyset$ and any $u \in \operatorname{ex}(F)$ is an extreme point of K. □

12. Extreme Points of Polyhedra

(12.1) Definition. A *polyhedron* in $\mathbb{R}^d$ is the set of solutions to a system of finitely many linear inequalities: $P = \Big\{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \beta_i : \ i = 1, \ldots, m\Big\}$, where $c_i \in \mathbb{R}^d$ and $\beta_i \in \mathbb{R}$.

(12.2) Theorem. *Let $P \subset \mathbb{R}^d$ be a polyhedron*

$$P = \Big\{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \beta_i : \ i = 1, \ldots, m\Big\}.$$

For $u \in P$ let

$$I(u) = \big\{i : \langle c_i, u\rangle = \beta_i\big\}$$

be the set of the inequalities that are "active" on u. Then u is an extreme point of P if and only if the set of vectors $\{c_i : i \in I(u)\}$ spans the vector space $\mathbb{R}^d$. In particular, if u is an extreme point, the set $I(u)$ contains at least d indices: $|I(u)| \geq d$.

Proof. Suppose that the vectors $c_i : i \in I(u)$ do not span $\mathbb{R}^d$. Then there is a non-zero $y \in \mathbb{R}^d$ such that $\langle y, c_i\rangle = 0$ for all $i \in I(u)$. We note that $\langle c_i, u\rangle < \beta_i$ for $i \notin I(u)$. For $\epsilon > 0$ let $u_+(\epsilon) = u + \epsilon y$ and let $u_-(\epsilon) = u - \epsilon y$. Then $u = (u_+(\epsilon) + u_-(\epsilon))/2$, $u_+(\epsilon) \neq u_-(\epsilon)$ and for sufficiently small $\epsilon > 0$ the points $u_-(\epsilon)$ and $u_+(\epsilon)$ belong to the polyhedron P. Hence u is not an extreme point of P.

Suppose now that $u \in P$ and the vectors $c_i : i \in I(u)$ span $\mathbb{R}^d$. Suppose that $u = (v+w)/2$ for $v, w \in P$. Then $\langle c_i, v\rangle \leq \beta_i$ and $\langle c_i, w\rangle \leq \beta_i$. Since $\langle c_i, u\rangle = \beta_i$ for $i \in I(u)$, we must have $\langle c_i, v\rangle = \langle c_i, w\rangle = \beta_i$ for $i \in I(u)$. Since vectors $c_i : i \in I(u)$ span $\mathbb{R}^d$, the system of linear equations must have a unique solution. Therefore, $v = w = u$ and u is an extreme point. $\square$

(12.3) Corollary. *Bounded polyhedron is a polytope, that is the convex hull of finitely many points.*

Proof. Every set of linear equations $\langle c_i, x\rangle = \beta_i : i \in I$, where $\{c_i : i \in I\}$ span $\mathbb{R}^d$, has at most one solution. Therefore, by Theorem 12.2, the number of extreme points of a polyhedron in $\mathbb{R}^d$, defined by a set of m inequalities, does not exceed $\binom{m}{d}$, and hence is finite. By the Krein-Milman Theorem (Theorem 11.3), P is the convex hull of the set of its extreme points and the proof follows. $\square$

An extreme point of a polyhedron is called a *vertex*.

PROBLEMS.

1. Prove that a polyhedron has finitely many faces.

2°. Prove that a face of a polyhedron is a polyhedron.

3. Prove that polytopes have finitely many faces.

Effect of "unrealistic solutions" in linear programming problems

Let $P \subset \mathbb{R}^d$ be a polyhedron defined by a system of m linear inequalities. Suppose we want to solve a *linear programming problem*

Find $\max\langle c_0, x\rangle$
Subject to: $x \in P$,

where $c_0 \in \mathbb{R}^d$ is the vector of the objective function. If the point $u \in P$, where the maximum is attained, is unique, then by Part 1 of Theorem 11.2, u must be an extreme point of P. Theorem 12.2 then implies that at least d of the m inequality constraints turn equalities on u. This sometimes is not at all desired.

(12.4) Example. The Diet Problem. Suppose we have m different types of food, the unit price of the i-th product is c_i, $i = 1, \ldots, m$. We want the diet to be balanced with respect to k vitamins (or otherwise useful substances). Suppose that a_{ij} is the content of the j-th vitamin in the i-th product. Let $x_i : i = 1, \ldots, m$ be the quantity of the i-th product in the diet and let $p_j : j = 1, \ldots, k$ be the target quantity of the j-th vitamin.

Trying to find the cheapest balanced diet, we come to a linear programming problem:

$$\text{Find } \min \sum_{i=1}^{m} c_i x_i$$

$$\text{Subject to: } \sum_{i=1}^{m} a_{ij} x_i = p_j \text{ for } j = 1, \ldots k;$$

$$x_i \geq 0 \text{ for } i = 1, \ldots, m.$$

Let $P \subset \mathbb{R}^d$ be the polyhedron of all feasible diets $\mathbf{x} = (x_1, \ldots, x_d)$. Clearly, P lies in the affine subspace defined by the k balance constraints $\sum_{i=1}^{m} a_{ij} x_i = p_j$. We may expect that these constraints are sufficiently generic, so $\dim\ P = d - k$. If the optimal diet $\mathbf{x} = (x_1, \ldots, x_d)$ is unique, it has to be an extreme point of P, so at least $d - k$ of the coordinates $x_1, \ldots, x_d$ are zero. This, in turn, means that the optimal diet would consist of at most k products. For example, if we balancing the diet by the content of 5 substances, we should expect to get the menu consisting of 5 products only.

13. The Birkhoff Polytope and the Assignment Problem

(13.1) Definitions. Let σ be a permutation of the set $\{1, \ldots, n\}$. The *permutation matrix* X^σ is an $n \times n$ matrix $X^\sigma = (x^\sigma_{ij}) : i, j = 1, \ldots, n$, defined as follows:

$$x^\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

An $n \times n$ matrix $X = \big(x_{ij} : i, j = 1, \ldots, n\big)$ is called *doubly stochastic* provided it is non-negative $x_{ij} \geq 0$ and the sum of entries in every row and every column is 1:

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } \ j = 1, \ldots, n \qquad \text{and} \qquad \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } \ i = 1, \ldots, n.$$

The polyhedron B_n of all $n \times n$ doubly stochastic matrices is called the *Birkhoff Polytope* or the *Assignment Polytope.*

PROBLEMS.

1°. Prove that the set of integer doubly stochastic matrices is the set of permutation matrices.

2° Let $A = (a_{ij})$ be an $n \times n$ orthogonal matrix and let $X = (x_{ij})$, where $x_{ij} = (a_{ij})^2$. Then X is a doubly stochastic matrix.

3°. Prove that the permutation matrices X^σ are extreme points of the Birkhoff Polytope.

(13.2) Birkhoff - von Neumann Theorem. *The vertices of the Birkhoff Polytope B_n are exactly the $n \times n$ permutation matrices.*

Proof. Because of Problem 3, Section 13.1, it suffices to prove that if X is an extreme point of B_n, then $X = X^\sigma$ for some permutation σ. We prove this by induction on n. The case $n = 1$ is obvious. Suppose that $n > 1$. Let us consider the affine subspace $L \subset \mathbb{R}^{n^2}$ consisting of the matrices $X = (x_{ij})$ such that

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for} \quad j = 1, \ldots, n \qquad \text{and} \qquad \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1, \ldots, n.$$

We claim dim $L = (n-1)^2$. Indeed, if we arbitrarily fill the upper left $(n-1) \times (n-1)$ corner by numbers $x_{ij} : i, j = 1, \ldots, n-1$, there is a unique way to append it to a matrix from L: $x_{in} = 1 - \sum_{j=1}^{n-1} x_{ij}$ for $i = 1, \ldots, n-1$, $x_{nj} = 1 - \sum_{i=1}^{n-1} x_{ij}$ for $j = 1, \ldots, n-1$, and $x_{nn} = (2-n) + \sum_{i,j=1}^{n-1} x_{ij}$. In the space L, the polytope B_n is defined by n^2 linear inequalities $x_{ij} \geq 0$. If X is an extreme point of B_n, by Theorem 12.2, some $(n-1)^2$ of these inequalities must be active on X. In other words, $x_{ij} = 0$ for some $(n-1)^2$ entries of X. In particular, there must be a row, say, i_0 with $x_{i_0 j} = 0$ for all but one $j = j_0$. Clearly, there can not be a row containing zeros alone, and if every row contains at least two non-zero entries, the total number of zero entries would have been at most $n(n-2) < (n-1)^2$. Now it is clear that $x_{i_0 j_0} = 1$ and that all other entries in the i_0-th row and in the j_0-th column must be zero. Crossing out the i_0-th row and the j_0-th column we get an $(n-1) \times (n-1)$ doubly stochastic matrix, which must be an extreme point of B_{n-1}, so we may apply the induction conjecture. □

PROBLEMS.

1. Prove that dim $B_n = (n-1)^2$, so B_n has an interior point in the subspace L, constructed in the proof.

2. Find the radius of the ball in L, centered at $x_{ij} = 1/n$ and touching the boundary of the polytope B_n.

3. Prove that $x_{ij} = 0$ is a support hyperplane of a face of B_n of dimension $(n-1)^2 - 1$ and that $x_{ij} = 1$ is a support hyperplane of a face of B_n of dimension $(n-2)^2$.

4. Let $A = (a_{ij})$ be an $n \times n$ matrix. The *permanent* of A is $\operatorname{per}(A) = \sum_\sigma \prod_{i=1}^n a_{i\sigma(i)}$, where the sum is taken over all $n!$ permutations. Prove that the permanent of a doubly stochastic matrix is strictly positive.

5*. Prove that the smallest value of the permanent of a doubly stochastic matrix is $n!/n^n$ (the Van der Waerden Conjecture).

6. Let us fix a vector $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. For a permutation σ, let $\sigma(x) = (\sigma(\xi_1), \ldots, \sigma(\xi_n))$. The polytope $P(x)$ that is the convex hull of the vectors $\sigma(x)$, where σ ranges over the set of all $n!$ permutations, is called a *permutation polytope.* Construct a linear transformation $T : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^n$, such that $T(B_n) = P(x)$.

7. Let us fix n real numbers $\lambda_1, \ldots, \lambda_n$. Prove that the diagonal $(a_{11}, \ldots, a_{nn})$ of an $n \times n$ symmetric matrix A with the eigenvalues $\lambda_1, \ldots, \lambda_n$ lies in the permutation polytope $P(l)$, where $l = (\lambda_1, \ldots, \lambda_n)$ (Schur's Theorem).

Hint: Use Problem 6 and Problem 2, Section 13.1.

(13.3) The Assignment Problem

The *Assignment Problem* is formulated as follows: given an $n \times n$ matrix $C = (c_{ij})$, find a permutation σ of the set $\{1, \ldots, n\}$ such that $\sum_{i=1}^n c_{i\sigma(i)}$ is the maximum (or minimum) possible. Because of Theorem 13.2, the Assignment Problem can be posed as a *linear programming problem*:

$$\text{Find } \max \sum_{ij=1}^{n} c_{ij} x_{ij}$$

$$\text{Subject to: } \sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1, \ldots, n; \qquad \sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, n;$$

$$x_{ij} \geq 0 \text{ for } i, j = 1, \ldots, n.$$

In particular, methods of linear programming can be applied to solve the Assignment Problem.

Friday, January 30

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14. The Transportation Problem

(14.1) The problem

Let $G = (V, E)$ be a directed (finite) graph with the set of vertices $V = \{1, \ldots, n\}$ and set of edges E. Any two vertices $i, j \in V$ can be either connected by an edge $i \to j$, or two edges $i \to j$ and $j \to i$ going in the opposite directions or not connected at all. We assume that the graph has no loops $i \to i$.

Suppose further, that to each vertex i, a real number a_i is assigned, which can be positive ("demand"), or negative ("supply"), or zero, and that to every edge $i \to j$ a real number c_{ij} is assigned, which is interpreted as a "cost" of shipping one unit of a commodity.

We want to find a flow, satisfying the supply-demand constraints and minimizing the cost. In other words, we want to assign a real number x_{ij} to every edge $i \to j$ such that

$$\text{The total cost } \sum_{i,j:\ (i \to j) \in E} c_{ij} x_{ij} \text{ is minimized subject to the constraints:}$$

(14.1.1) For every vertex i,

$$\sum_{j:\ (j\to i)\in E} x_{ji} - \sum_{j:\ (i\to j)\in E} x_{ij} = a_i \text{ (the balance requirement);}$$

(14.1.2) For every edge $(i \to j) \in E$,

$x_{ij} \geq 0$ (no wrong way shipment).

(14.2) Definition. A flow in G can be identified with a point in $\mathbb{R}^m$, where $m = |E|$ is the number of edges of the graph. The set of all feasible flows $\mathbf{x} \in \mathbb{R}^m$, that is the set of all flows $\mathbf{x} = (x_{ij})$, satisfying (14.1.1) and (14.1.2) is called the *transportation polyhedron* and denoted $T(G, \mathbf{a})$, where $\mathbf{a} = (a_1, \ldots, a_n)$ is the vector of demands/supplies.

PROBLEMS.

1. Prove that if the transportation polyhedron $T(G, \mathbf{a})$ is non-empty, then $\sum_{i=1}^n a_i = 0$.

2°. Construct an example of a transportation problem, such that $\sum_{i=1}^n a_i = 0$ but the transportation polyhedron $T(G, \mathbf{a})$ is empty.

3°. Construct an example of the transportation problem where $T(G, \mathbf{a})$ is an unbounded polyhedron.

4. Suppose that G does not contain any directed cycle $i_1 \to i_2 \to \ldots \to i_l \to i_1$ for $l \geq 2$. Prove that $T(G, \mathbf{a})$ is a bounded polyhedron (polytope). It is called the *transportation polytope.*

5. Suppose that there exists a feasible solution x_{ij} and that the costs a_{ij} are strictly positive. Prove that there exists an optimal solution and that the set of all optimal solutions is a compact polyhedron (polytope).

Hint: Prove that an optimal solution can not have very large flows x_{ij}.

(14.3) Proposition. *Let $X = \big(x_{ij} : (i \to j) \in E\big)$ be an extreme point of the transportation polyhedron $T(G, \mathbf{a})$. Let $S \subset E$ be the set of edges $i \to j$ such that $x_{ij} > 0$. Then S does not contain any cycle $v_1 - v_2 - \ldots - v_l - v_1 : l \geq 2$, where the vertices $v_k, v_{k+1} : k = 1, \ldots, l-1$ and v_l, v_1 are connected by an edge (of whatever orientation: $v_k \to v_{k+1}$ or $v_{k+1} \to v_k$).*

Proof. Suppose that there is such a cycle C consisting of edges with a strictly positive flow. Let us choose an $\epsilon > 0$ and let us construct two flows $Y = (y_{ij})$ and $Z = (z_{ij})$ as follows:

$$y_{ij} = \begin{cases} x_{ij} & \text{if } (i \to j) \notin C \\ x_{ij} + \epsilon & \text{if } i = v_k, j = v_{k+1} \text{ or } i = v_l, j = v_1 \\ x_{ij} - \epsilon & \text{if } i = v_{k+1}, j = v_k \text{ or } i = v_1, j = v_l \end{cases}$$

and

$$z_{ij} = \begin{cases} x_{ij} & \text{if } (i \to j) \notin C \\ x_{ij} - \epsilon & \text{if } i = v_k, j = v_{k+1} \text{ or } i = v_l, j = v_1 \\ x_{ij} + \epsilon & \text{if } i = v_{k+1}, j = v_k \text{ or } i = v_1, j = v_l. \end{cases}$$

Then the flows Y and Z satisfy the balance condition (14.1.1). Furthermore, if $\epsilon > 0$ is small enough, Y and Z are non-negative flows. Finally, $X = (Y + Z)/2$, which proves that X can not be an extreme point of $T(G, \mathbf{a})$. □

(14.4) Definition. A graph without cycles is called a *forest.* A connected graph without cycles is called a *tree.*

PROBLEMS.

1°. Prove that a forest is a union of non-intersecting trees.

2. Prove that every (finite) forest has a vertex which is connected with at most one vertex of the forest.

Hint: Starting with a vertex of the forest, follow some path while there is still a way to go, without turning back. Since there are no cycles and the number of vertices is finite, the path will end somewhere. The last vertex of the path is the vertex with the required property.

(14.5) Corollary. *Suppose that all demands/supplies a_i, $i = 1, \ldots, n$ are integer numbers. Then every extreme point of the transportation polyhedron $T(G, \mathbf{a})$ is an integer flow x_{ij}.*

Proof. Let $X = (x_{ij})$ be an extreme point. Proposition 14.3 asserts that the set $S \subset E$ with non-zero flows x_{ij} is a forest. We claim, that once we know S, we can compute the flows x_{ij} from $a_1, \ldots, a_n$ by using addition and subtraction only. This, of course, would imply that all x_{ij} are integers.

If S is a forest with at least one edge, by Problem 2, Section 14.4, there is a vertex i such that there is only one edge with a non-zero flow incident to i. Let j be the other end of that edge. Clearly, if this edge is $i \to j$, we must have $a_i < 0$ and $x_{ij} = -a_i$. If this edge is $j \to i$, we must have $a_i > 0$ and $x_{ji} = a_i > 0$. Now, we delete the vertex i with all edges of G incident to it, modify forest S accordingly, and adjust the demand/supply vector: if we had $x_{ij} > 0$, we let new $a_j := a_j + a_i$, if we had $x_{ji} > 0$ we let new $a_j = a_j - a_i$. Hence we get a new graph G' with $n - 1$ vertices and integer demands/supplies, a new forest S' of G' and a new balanced non-negative flow x'_{ij} in G' such that S' is the set of edges, where x'_{ij} are strictly positive. We proceed as above, until there are no edges in the forest. At that moment the flow x_{ij} is determined completely. □

PROBLEM.

1. An alternative proof of the Birkhoff - von Neumann Theorem (Theorem 13.2): Consider the graph G with $2n$ vertices $1, \ldots, n$ and $1', \ldots, n'$. Let $a_i = 1$ be the demand for $i = 1, \ldots, n$ and let $a_{i'} = -1$ be the supply for $i' = 1', \ldots, n'$. Let $i' \to j$, $i' = 1', \ldots, n'$, $j = 1, \ldots, n$ be the edges of G. Prove that the feasible flows $x_{i'j}$ are the doubly stochastic matrices and that the integral feasible flows are the permutation matrices.

Monday, February 2

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15. The Moment Curve and the Cyclic Polytope

(15.1) The Moment Curve

The curve $g(\tau) = (\tau, \tau^2, \ldots, \tau^d) \in \mathbb{R}^d$ for $a \leq \tau \leq b$ is called the *moment curve.* Let $f(x) = \langle c, x \rangle$ be a linear function on $\mathbb{R}^d$, where $c = (\gamma_1, \ldots, \gamma_d)$. Then

$$\langle c, g(\tau) \rangle = \gamma_d \tau^d + \gamma_{d-1} \tau^{d-1} + \ldots + \gamma_1 \tau$$

is a polynomial in τ with zero constant term.

PROBLEMS.

1. Prove that each affine hyperplane $H \subset \mathbb{R}^d$ intersects the moment curve $g(\tau)$ in at most d points.

2. Let $\delta_1 < \delta_2 < \ldots < \delta_d$ be real numbers. Consider the curve in $\mathbb{R}^d$

$$h(\tau) = \big(\exp\{\delta_1 \tau\}, \exp\{\delta_2 \tau\}, \ldots, \exp\{\delta_d \tau\}\big) \quad \text{for } a \leq \tau \leq b.$$

Prove that each affine hyperplane $H \subset \mathbb{R}^d$ intersects $h(\tau)$ in at most d points.

3. Let S^1 be the circle in the natural parameterization $S^1 = \{0 \leq \tau \leq 2\pi\}$. Suppose that $d = 2k$ is even and let $h: S^1 \longrightarrow \mathbb{R}^d$ be the closed curve

$$h(\tau) = \big(\sin \tau, \cos \tau, \sin 2\tau, \cos 2\tau, \ldots, \sin k\tau, \cos k\tau\big), \quad 0 \leq \tau \leq 2\pi.$$

Prove that each affine hyperplane $H \subset \mathbb{R}^d$ intersects the curve $h(\tau)$ in at most d points.

Remark. A set of continuous functions $f_1, \ldots, f_d : X \longrightarrow \mathbb{R}$ is called a *Chebyshev system* on X, if $x \longmapsto \big(f_1(x), \ldots, f_d(x)\big)$ is an embedding of X in $\mathbb{R}^d$ (so that distinct points are mapped into distinct points) and every affine hyperplane H intersects the image of X in not more than d points.

4. Prove that there is no Chebyshev system on the circle S^1, consisting of an odd number of functions.

5. Prove there is no Chebyshev system on Y (three intervals having one common point).

(15.2) Definition. Let us choose n distinct points $v_1 = g(\tau_1), \ldots, v_n = g(\tau_n)$: $a < \tau_1 < \tau_2 < \ldots < \tau_n < b$ on the moment curve. The convex hull $C(d,n) = \operatorname{conv}(v_1, \ldots, v_n)$ is called the *cyclic polytope*.

PROBLEM.

1°. Prove that dim $C(d,n) = d$ for $n \geq d+1$.

(15.3) Proposition. *Let $k \leq d/2$. Then the convex hull* $\operatorname{conv}(v_{i_1}, \ldots, v_{i_k})$ *of any k vertices of $C(d,n)$ is a face of $C(d,n)$.*

Proof. We will demonstrate a linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, $f(x) = \langle c, x \rangle$ and a number α such that $f(v_{i_1}) = \ldots = f(v_{i_k}) = \alpha$ and $f(v_j) < \alpha$ for $j \neq i_1, \ldots, i_k$. Clearly, the affine hyperplane $H = \big\{x : f(x) = \alpha\big\}$ will be a support hyperplane of the face $\operatorname{conv}(v_{i_1}, \ldots, v_{i_k})$. To construct f, let us consider the polynomial

$$p(\tau) = -(\tau - a)^{d-2k}(\tau - \tau_{i_1})^2 \cdots (\tau - \tau_{i_k})^2.$$

This is a polynomial of degree d in τ, such that $p(\tau_j) = 0$ for $j = i_1, \ldots, i_k$ and $p(\tau_j) < 0$ for $j \neq i_1, \ldots, i_k$. Let us write

$$p(\tau) = \gamma_d \tau^d + \gamma_{d-1} \tau^{d-1} + \ldots + \gamma_1 \tau_1 - \alpha$$

and let $c = (\gamma_1, \ldots, \gamma_d)$. Then $\langle c, g(\tau) \rangle = \gamma_1 \tau + \gamma_2 \tau^2 + \ldots + \gamma_d \tau^d$, $\langle c, v_j \rangle = \alpha$ for $j = i_1, \ldots, i_k$ and $\langle c, v_j \rangle < \alpha$ for $j \neq i_1, \ldots, i_k$. □

PROBLEMS.

1. Let $P = \operatorname{conv}(v_1, \ldots, v_n) \subset \mathbb{R}^d$ be a d-dimensional polytope with n vertices and let $k > d/2$. Prove that if every k vertices $v_{i_1}, \ldots, v_{i_k}$ are the vertices of a face of P, then $n = d+1$.

Hint: Use Radon's Theorem (see Theorem 5.3).

2. Prove that a set W of d vertices $v_{i_1} = g(\tau_{i_1}), \ldots, v_{i_d} = g(\tau_{i_d})$ is the set of vertices of a face of $C(d,n)$ if and only if the following *Gale's evenness condition* is satisfied: every two vertices $v_i = g(\tau_i)$ and $v_j = g(\tau_j)$ of $C(d,n)$ that are not in W are separated on the moment curve by an even number of points from W.

3. Describe the faces of the polytope $C(4,n)$.

4*. For $1 \leq k \leq d-1$, the number f_k of k-dimensional faces of $C(d,n)$ is

$$f_k = \begin{cases} \sum_{j=1}^{(d+1)/2} \frac{n}{n-j}\binom{n-j}{j}\binom{j}{k+1-j} & \text{if } d \text{ is odd} \\ \sum_{j=0}^{d/2} \frac{k+2}{n-j}\binom{n-j}{j+1}\binom{j+1}{k+1-j} & \text{if } d \text{ is even.} \end{cases}$$

Remark. The cyclic polytope $C(d,n)$ has the following important property: among all polytopes in $\mathbb{R}^d$ with n vertices, the polytope $C(d,n)$ has the largest number of faces of each dimension $k = 1, \ldots, d-1$, see P. McMullen and G.C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge University Press, 1971.

16. The Moment Cone

Let us consider the space $\mathbb{R}^{d+1}$ with the coordinates $x = (\xi_0, \xi_1, \ldots, \xi_d)$ (we start with the zeroth coordinate). Let us modify the moment curve: we let

$$\widehat{g}(\tau) = (1, \tau, \tau^2, \ldots, \tau^d) \in \mathbb{R}^{d+1} \quad \text{for} \quad a \leq \tau \leq b.$$

Hence $\widehat{g}(\tau)$ lies in the hyperplane $\xi_0 = 1$ in $\mathbb{R}^{d+1}$.

(16.1) Definition. The *moment cone*

$$M_{d+1} = \operatorname{cone}\big(\widehat{g}(\tau) : a \leq \tau \leq b\big) \subset \mathbb{R}^{d+1}$$

is the conic hull of the curve $\widehat{g}(\tau)$.

PROBLEMS.

1. Prove that M_{d+1} is a closed set.

Hint: Consider a sequence of points $x_n \in M_{d+1}$ such that $x_n \longrightarrow x$. Let $B_{d+1} = \operatorname{conv}\big(\widehat{g}(\tau) : a \leq \tau \leq b\big)$ be the convex hull of the curve $\widehat{g}$. Write $x_n = \alpha_n y_n$, where $y_n \in B_{d+1}$. Using Corollary 5.2, prove that there is a subsequence $y_{n_i} \longrightarrow y \in B_{d+1}$ and there is a subsequence $\alpha_{n_i} \longrightarrow \alpha$. Prove that $x = \alpha y \in M_{d+1}$.

2. Prove that $\dim\ M_{d+1} = d+1$.

We will need a simple lemma.

(16.2) Lemma. *Let V be a vector space, let $K \subset V$ be a cone and let $H \subset \mathbb{R}^d$ be a support hyperplane of K. Then $0 \in H$.*

Proof. Let $H = \big\{x : f(x) = \alpha\big\}$, where $f : V \longrightarrow \mathbb{R}$ is a linear function and $\alpha \in \mathbb{R}$ is a number. In other words, $f(x) \leq \alpha$ for any $x \in K$ (H isolates K) and $f(x_0) = \alpha$ for some $x_0 \in K$ ($H \cap K \neq \emptyset$). Since K is a cone, $0 \in K$, so $0 = f(0) \leq \alpha$. Suppose that $\alpha > 0$. Since $f(x_0) = \alpha$ and K is a cone, we conclude that $2x_0 \in K$. But then $f(2x_0) = 2\alpha < \alpha$, which is a contradiction. Hence $\alpha = 0$ and the proof follows. $\square$

(16.3) Proposition. *Let $u \in \partial M_{d+1}$ be a point on the boundary of M_{d+1}. Let us write u as a conic combination of points on the curve $\widehat{g}(\tau)$:*

$$u = \sum_{j=1}^{m} \lambda_j \widehat{g}(\tau_j), \quad \textit{where } \lambda_j > 0 \textit{ for } j = 1, \ldots, m \quad \textit{and} \quad a \leq \tau_1 < \ldots < \tau_m \leq b.$$

Then $m \leq (d+2)/2$. Furthermore, if $m = (d+2)/2$, then d is even and $\tau_1 = a$, $\tau_m = b$.

Proof. Let H be a support hyperplane of M_{d+1} at u (it exists by Theorem 10.6). By Lemma 16.2, H should contain the origin, so $H = \big\{x : \langle c, x \rangle = 0\big\}$ for some $c = (\gamma_0, \gamma_1, \ldots, \gamma_d) \neq 0$. We have $\langle c, x \rangle \geq 0$ for $x \in M_{d+1}$ and $\langle c, u \rangle = 0$. In particular, $\langle c, \widehat{g}(\tau) \rangle \geq 0$ for all $a \leq \tau \leq b$. Let

$$p(\tau) = \langle c, \widehat{g}(\tau) \rangle = \gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d.$$

Hence $p(\tau)$ is a polynomial of degree d which is non-negative on the interval $[a, b]$. Furthermore, since $\langle c, u \rangle = 0$ and $\lambda_j > 0$ we must have $p(\tau_j) = 0$ for $j = 1, \ldots, m$. Suppose that $t \in (a, b)$ is a root of p, which lies strictly inside the interval (a, b). Then the multiplicity of the root must be an even number, since otherwise $p(\tau)$ would change sign in a neighborhood of t. The only roots of multiplicity one can be $\tau = a$ and $\tau = b$. Since the total number of roots of p, counting multiplicities, is at most d, we get $m \leq 2 + (d-2)/2 = (d+2)/2$, where "2" stands for the possible endpoint roots $\tau = a$ and $\tau = b$ and $(d-2)/2$ is the maximum number of roots inside (a, b) (not counting multiplicities). $\square$

(16.4) Corollary. *Let*

$$m = \begin{cases} (d+1)/2 & \textit{if } d \textit{ is odd;} \\ (d+2)/2 & \textit{if } d \textit{ is even.} \end{cases}$$

Let $u \in M_{d+1}$ be a point. Then u can be represented as a conic combination of m (or fewer) points on the curve $\widehat{g}(\tau)$:

$$u = \sum_{i=1}^{m} \lambda_i \widehat{g}(\tau_i)$$

where $\lambda_i \geq 0$ for $i = 1, \ldots, m$ and $a \leq \tau_1 < \ldots < \tau_m \leq b$.

Proof. If $u \in \partial M_{d+1}$, the result follows by Proposition 16.3. Suppose that $u \in \operatorname{int}(M_{d+1})$. Let $t \geq 0$ be a parameter, and let us "shrink" the interval $[a, b] \longrightarrow [a+t, b-t]$ as t grows. Let $M_{d+1}(t)$ be the conic hull of $\widehat{g}(\tau)$: $a+t \leq \tau \leq b-t$. When $t = (b-a)/2$, the curve $\widehat{g}$ consists of a single point, so the cone $M_{d+1}(t)$ consists of a single ray. If the point u is still in the cone, the proof follows. Otherwise, there is a value t^* such that $u \in \partial M_{d+1}(t^*)$. Now we use Proposition 16.3. $\square$

Wednesday, February 4

17. Application: "Double Precision" Formulas for Numerical Integration

(17.1) Proposition. *Let us fix an interval $[a,b]$ and a non-negative continuous density $\rho(\tau)$ on $[a,b]$. Then, for any positive integer m, there exist m points $\tau_1^*, \ldots, \tau_m^*$ on the interval $[a,b]$ and m non-negative numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$\int_a^b f(\tau)\rho(\tau)\ d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for any polynomial f of degree at most $2m-1$.

Proof. Let $d = 2m-1$. Let $u = (\xi_0, \ldots, \xi_d) \in \mathbb{R}^{d+1}$, where $\xi_i = \int_a^b \tau^i \rho(\tau)\ d\tau$ for $i = 0, \ldots, d$. Let us prove that $u \in M_{d+1}$, where M_{d+1} is the moment cone $M_{d+1} = \operatorname{cone}\bigl(\widehat{g}(\tau) : a \leq \tau \leq b\bigr\}$. Indeed, since ρ is continuous, u can be written as a limit of Riemann sums:

$$u = \lim_{N \longrightarrow +\infty} \frac{b-a}{N} \sum_{i=1}^N \rho(\tau_i)\widehat{g}(\tau_i),$$

where $\tau_1, \ldots, \tau_N$ are equally spaced points on $[a,b]$. Since every Riemann sum is in M_{d+1}, and the moment cone is closed (see Problem 1, Section 16.1), we get that $u \in M_{d+1}$. Therefore, by Corollary 16.4, we can write u as a conic combination of m points on $\widehat{g}$:

$$u = \sum_{i=1}^m \lambda_j \widehat{g}(\tau_i^*),$$

where $\lambda_i \geq 0$ and $a \leq \tau_1^* < \ldots < \tau_m^* \leq b$. Now, let $f(\tau) = \gamma_d \tau^d + \ldots + \gamma_0$ be a polynomial of degree at most d. Let $c = (\gamma_0, \ldots, \gamma_d) \in \mathbb{R}^{d+1}$. Then

$$f(\tau_i^*) = \langle c, \widehat{g}(\tau_i^*)\rangle \text{ for } \ i = 1, \ldots, m \quad \text{and} \int_a^b f(\tau)\rho(\tau)\ d\tau = \langle c, u\rangle.$$

The proof follows. □

Formulas for numerical integration

$$\int_a^b f(\tau)\rho(\tau)\ d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

that are exact for polynomials f of degree up to $d = 2m-1$ are often called the *double precision* integration formulas (for obvious reasons).

PROBLEMS.

1. Prove, that there are no m points $\tau_1^*, \ldots, \tau_m^*$ on the interval $[0,1]$ and m real numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\int_0^1 f(\tau)\, d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for any polynomial of degree $2m$.

2. Prove, that for every interval $[a,b]$, every strictly positive continuous density $\rho(\tau)$ on $[a,b]$, and every positive integer m there is only one set of points $\tau_1^*, \ldots, \tau_m^*$ on the interval $[a,b]$ and only one set of non-negative numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\int_a^b f(\tau)\rho(\tau)\, d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for every polynomial of degree at most $2m-1$.

3*. A function

$$f(\tau) = \gamma_0 + \sum_{k=1}^d (\alpha_k \sin k\tau + \beta_k \cos k\tau) \quad \text{for} \quad 0 \leq \tau \leq 2\pi$$

is called a *trigonometric polynomial* of degree at most d. Let ρ be a non-negative continuous function on $[0, 2\pi]$ such that $\rho(0) = \rho(2\pi)$. Prove that there exist $d+1$ points $0 \leq \tau_0^* < \ldots < \tau_d^* < 2\pi$ and $d+1$ non-negative numbers $\lambda_0, \ldots, \lambda_d$, such that the formula

$$\int_0^{2\pi} f(\tau)\rho(\tau)\, d\tau = \sum_{i=0}^d \lambda_i f(\tau_i^*)$$

is exact for any trigonometric polynomial of degree at most d.

Hint: Use Problem 3, Section 15.1.

4*. Let us fix $d = 2m$ distinct real numbers $\delta_1, \ldots, \delta_d$. A function

$$f(\tau) = \sum_{i=1}^d \alpha_i \exp\{\delta_i \tau\}$$

is called an *exponential polynomial* with exponents $\delta_1, \ldots, \delta_d$. Let ρ be a non-negative continuous density on the interval $[a,b]$. Prove that, there exist m points $\tau_1^*, \ldots, \tau_m^*$ on the interval $[a,b]$ and m non-negative numbers $\lambda_1, \ldots, \lambda_m$, such that

$$\int_a^b f(\tau)\rho(\tau)\, d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for any exponential polynomial with exponents $\delta_1, \ldots, \delta_d$.

Hint: Use Problem 2, Section 15.1.

Let us consider small examples of double precision formulas for integrating $\displaystyle\int_0^1 f(\tau)\, d\tau$.

(17.2) Example. The formula that uses one node and is exact for polynomials of degree at most 1 is

$$\int_0^1 f(\tau)\ d\tau = f\Big(\frac{1}{2}\Big).$$

PROBLEM.

1°. Prove that the formula of Example 17.1 is indeed exact on polynomials of degree at most 1 and that this is the only such a formula.

(17.3) Example. Let us find formulas that use two nodes and and are exact on polynomials of degree at most 2. In the xy plane, let us consider the parabola arc $\big\{g(\tau) = (\tau, \tau^2) : 0 \leq \tau \leq 1\big\}$. Let $u = (1/2, 1/3)$. There are infinitely many formulas

$$\int_0^1 f(\tau)\ d\tau = \lambda_1 f(\tau_1^*) + \lambda_2 f(\tau_2^*)$$

that are exact on polynomials of degree 2. The necessary and sufficient condition for $0 \leq \tau_1^* < \tau_2^* \leq 1$ is that the interval $[g(\tau_1^*), g(\tau_2^*)]$ on the xy plane contains u. Then λ_1 and λ_2 are found from $u = \lambda_1 g(\tau_1^*) + \lambda_2 g(\tau_2^*)$. For example, if we choose $\tau_1^* = 0$ then $\tau_2^* = 2/3$ and we get a formula

$$\int_0^1 f(\tau)\ d\tau = \frac{1}{4} f(0) + \frac{3}{4} f\Big(\frac{2}{3}\Big).$$

PROBLEM.

1. Find the one-parametric family of formulas with two nodes which are exact on polynomials of degree 2.

(17.4) Example. Here is a formula with two nodes which is exact for polynomials of degree at most 3:

$$\int_0^1 f(\tau)\ d\tau = \frac{1}{2} f\Big(\frac{1}{2} - \frac{1}{2\sqrt{3}}\Big) + \frac{1}{2} f\Big(\frac{1}{2} + \frac{1}{2\sqrt{3}}\Big).$$

PROBLEM.

1. Prove that the above formula is indeed exact on polynomials of degree at most 3 and that this is the only one such a formula which uses two nodes.

Friday, February 6

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18. The Cone of Non-Negative Polynomials

(18.1) Cones, extreme rays, and bases

Let V be a vector space and let $K \subset V$ be a convex cone. We say that a convex set $B \subset V$ is a *base* of K provided $0 \notin B$ and for every point $u \in K, u \neq 0$ there is a unique representation $u = \lambda v$ with $v \in B$ and $\lambda > 0$. We say that $u \in K, u \neq 0$ spans an *extreme ray* of K if, whenever $u = (u_1 + u_2)/2$ for $u_1, u_2 \in K$, we must have $u_1 = \lambda_1 u$, $u_2 = \lambda_2 u$ for some $\lambda_1, \lambda_2 \geq 0$.

Suppose that $K \subset \mathbb{R}^d$ is a cone. We say that K has a *compact base* provided there is a convex compact set $B \subset \mathbb{R}^d$ which is a base of K.

PROBLEMS.

1. Prove that the cone with a compact base is closed.

Hint: See Problem 1, Section 16.1 and the Hint there.

2. Prove, that if K is a cone with a compact base, then 0 is an extreme point of K.

(18.2) Lemma. *Let K be a cone with a base B. A point $u \in K$ spans an extreme ray of K if and only if $u = \lambda v$, where $\lambda > 0$ and v is an extreme point of B.*

Proof. Suppose that u spans an extreme ray of K. Let $v \in B$ be a point such that $u = \lambda v$ for $\lambda > 0$. Suppose that $v = (v_1 + v_2)/2$. Then $u = (u_1 + u_2)/2$, where $u_1 = \lambda v_1$ and $u_2 = \lambda v_2$. Since u spans an extreme ray, we must have $u_1 = \mu_1 u$ and $u_2 = \mu_2 u$ for some $\mu_1, \mu_2 \geq 0$. Then $u_1 = (\mu_1 \lambda) v$ and $u_2 = (\mu_2 \lambda) v$. Since B is a base, we must have $v_1 = v_2 = v$, so v is an extreme point.

Suppose that $u = \lambda v$, where $\lambda > 0$ and v is an extreme point of B. Let us show that u spans an extreme ray of K. Suppose that $u = (u_1 + u_2)/2$. Then $u_1 = \lambda_1 v_1$ and $u_2 = \lambda_2 v_2$ for some $v_1, v_2 \in B$ and some non-negative λ_1, λ_2. Without loss of generality, we can assume that $\lambda_1, \lambda_2 > 0$. Then we can write $u = (\lambda_1 v_1 + \lambda_2 v_2)/2 = \beta(\alpha_1 v_1 + \alpha_2 v_2)$, where $\beta = (\lambda_1 + \lambda_2)/2$, $\alpha_1 = \lambda_1/(\lambda_1 + \lambda_2)$, and $\alpha_2 = \lambda_2/(\lambda_1 + \lambda_2)$. Since B is a base, we must have $\alpha_1 v_1 + \alpha_2 v_2 = v$. We note, that $\alpha_1, \alpha_2 > 0$ and that $\alpha_1 + \alpha_2 = 1$. If $v_1 \neq v_2$, it follows that v lies inside the interval $[v_1, v_2]$, which contradicts the assumption that v is an extreme point of B. Therefore, we must have $v_1 = v_2$, so u spans an extreme ray of K. □

(18.3) Corollary. *Let $K \subset \mathbb{R}^d$ be a cone with a compact base. Then every point $u \in K$ can be written as a conic combination*

$$u = \sum_{i=1}^{m} \lambda_i u_i, \quad \lambda_i \geq 0 : \ i = 1, \ldots, m,$$

where u_i span extreme rays of K.

Proof. Let B be a base of K. Let us write $u = \lambda v$, where $v \in B$. By the Krein-Milman Theorem (Theorem 11.3), we can express v as a convex combination of extreme points $v_1, \ldots, v_m$ of B. Then u is a conic combination of $u_1 = \lambda_1 v_1, \ldots, u_m = \lambda_m v_m$. By Lemma 18.2, it follows that $u_1, \ldots, u_m$ span extreme rays of K. □

(18.4) Definition. Let us interpret the space $\mathbb{R}^{d+1}$ as the space of all polynomials in one variable τ of degree at most d: a polynomial $\gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d$ is represented by the vector $(\gamma_0, \ldots, \gamma_d)$. Let $K_+ \subset \mathbb{R}^{d+1}$ be the set of all non-negative polynomials p: $p(\tau) \geq 0$ for all $\tau \in \mathbb{R}$.

PROBLEMS.

1°. Prove that K_+ is a convex closed cone in $\mathbb{R}^{d+1}$.

2. Prove that K_+ has a non-empty interior if and only if d is even.

3. Suppose that d is even. Prove, that $p \in \partial K_+$ if and only if $p(\tau) \geq 0$ for all $\tau \in \mathbb{R}$ and either deg $p < d$ or $p(\tau_0) = 0$ for some $\tau_0 \in \mathbb{R}$.

(18.5) Proposition. *The cone K_+ has a compact base. A polynomial $p \in K_+$ spans an extreme ray if and only if*

$$p(\tau) = \alpha \prod_{i=1}^{k} (\tau - \tau_i)^2$$

for some $\alpha > 0$ and some (not necessarily distinct) $\tau_1, \ldots, \tau_k$: $2k \leq d$.

Proof. Let

$$B = \Big\{ q \in K_+ : \int_{-\infty}^{+\infty} q(\tau) e^{-\tau^2} \ d\tau = 1 \Big\}.$$

Obviously, B is non-empty convex closed set. It is also clear that for any $p \in K_+$, $p \neq 0$ there is a unique representation $p = \lambda q$ for $q \in B$ and $\lambda > 0$. Take

$$\lambda = \int_{-\infty}^{+\infty} p(\tau) e^{-\tau^2} \ d\tau.$$

It remains to show that B is compact. Let us define a norm $\| \cdot \|_1$ on $\mathbb{R}^{d+1}$:

$$\|p\|_1 = \int_{-\infty}^{+\infty} |p(\tau)| e^{-\tau^2} \ d\tau.$$

Let $\delta > 0$ be the minimum value of the continuous function $\|p\|_1$ on the unit sphere $S = \big\{ p : \|p\| = 1 \big\}$, where $\| \cdot \|$ is the usual Euclidean norm. Then $\|p\| \leq 1/\delta$ for all $p \in B$, so B is compact. Hence B is a compact base of K_+.

We note that $p \in K_+$ if and only if

the degree of p is even,

the highest coefficient of p is positive,

the multiplicity of every root is even (otherwise, p changes its sign in a neighborhood of the root), and

at every root τ_i, the first non-zero derivative $p^{(k)}(\tau_i)$ is positive.

Suppose that $p \in K_+$ spans an extreme ray of K_+. Let $\deg \ p = 2k$ and let $(\tau_1, m_1), \ldots, (\tau_s, m_s)$ be the roots of p with their multiplicities. Suppose that $m_1 + \ldots + m_s < 2k$. Let $q(\tau) = (\tau - \tau_1)^{m_1} \cdots (\tau - \tau_s)^{m_s}$. Then $\deg \ q < \deg \ p$. Let $p_1 = p + \epsilon q$ and $p_2 = p - \epsilon q$. Then $p = (p_1 + p_2)/2$. If $\epsilon > 0$ is sufficiently small, we will have $p_1, p_2 \in K_+$, so p can not span an extreme ray of K_+. The contradiction shows that that if p spans an extreme ray of K_+, we must have $p(\tau) = \alpha(\tau - \tau_1)^{m_1} \cdots (\tau - \tau_s)^{m_s}$ for some $\alpha > 0$.

Let us prove that polynomials $p(\tau) = \alpha(\tau - \tau_1)^2 \cdots (\tau - \tau_k)^2$ span extreme rays of K_+. Let us rewrite $p(\tau) = \alpha(\tau - \tau_1)^{m_1} \cdots (\tau - \tau_s)^{m_s}$, where $\tau_1, \ldots, \tau_s$ are distinct roots and $m_1, \ldots, m_s$ are the multiplicities. Suppose that $p = (p_1 + p_2)/2$, where $p_1, p_2 \in K_+$. Clearly, $\deg \ p_1, \deg \ p_2 \leq 2k$, since otherwise the highest term of p_1 or p_2 would have been negative, which contradicts the assumption that $p_1, p_2 \in K_+$. Since $p_1, p_2 \in K_+$, each root τ_i of p must be a root of both p_1 and p_2. Furthermore, the multiplicity of τ_i in p_1 and p_2 must be at least m_i (otherwise, the first non-zero derivative at τ_i of one of the polynomials p_1 and p_2 would have been negative).

We conclude that

$$p_1(\tau) = \beta_1(\tau - \tau_1)^{m_1} \cdots (\tau - \tau_s)^{m_s} \quad \text{and} \quad p_2(\tau) = \beta_2(\tau - \tau_1)^{m_1} \cdots (\tau - \tau_s)^{m_s}$$

for non-negative $\beta_1, \beta_2 \geq 0$. Therefore, p spans an extreme ray of K_+. □

(18.6) Corollary. *Let $p(\tau)$ be a polynomial such that $p(\tau) \geq 0$ for all $\tau \in \mathbb{R}$. Then*

$$p = \sum_{i=1}^{m} q_i^2$$

for some polynomials $q_i : i = 1, \ldots, m$. Furthermore, all roots of all polynomials q_i are real.

Proof. Corollary 18.3 states that p is a conic combination of polynomials p_i which span extreme rays of K_+. Proposition 18.5 implies that $p_i = q_i^2$ for some polynomial q_i, where all roots of q_i are real. $\square$

PROBLEMS.

1. Let $d = 2$. Draw a nice picture of the cone of non-negative quadratic polynomials $a\tau^2 + b\tau + c$ in $\mathbb{R}^3 = \big\{(a, b, c) : a, b, c \in \mathbb{R}\big\}$.

2*. Corollary 18.6 can be strengthened: prove that every polynomial $p(\tau)$ of degree $2k$ such that $p(\tau) \geq 0$ for all $\tau \in \mathbb{R}$, admits a unique representation of the form

$$p(\tau) = \alpha \prod_{i=1}^{k} (\tau - \tau_i)^2 + \beta \prod_{i=1}^{k-1} (\tau - \tau_i')^2,$$

where $\alpha, \beta > 0$ and $\tau_1 < \tau_1' < \tau_2 < \tau_2' < \ldots < \tau_{k-1} < \tau_{k-1}' < \tau_k$, see S. Karlin and W.J. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Wiley, Interscience, 1966.

In particular, every non-negative polynomial is the sum of only two squares of polynomials.

3. Let $K_+[0, +\infty) \subset \mathbb{R}^{d+1}$ be the set of all polynomials $p(\tau)$ of degree at most d, such that $p(\tau) \geq 0$ for all $\tau \geq 0$. Prove that $K_+[0, +\infty)$ is a convex closed cone with a compact base and that the polynomials that span extreme rays of K are

$$p(\tau) = \alpha \prod_{i=1}^{k} (\tau - \tau_i)^2 \quad (2k \leq d) \quad \text{and} \quad p(\tau) = \alpha\tau \prod_{i=1}^{k} (\tau - \tau_i)^2 \quad (2k + 1 \leq d),$$

where $\alpha > 0$ and $\tau_i \geq 0$ for $i = 1, \ldots, k$.

4. Let $[a, b]$ be an interval and let $K_+[a, b] \subset \mathbb{R}^{d+1}$ be the cone of all polynomials of degree at most d that are non-negative on $[a, b]$. Prove that $K_+[a, b]$ is a convex cone with a compact base and that the polynomials that span extreme rays of $K_+[a, b]$ are

$$p(\tau) = \alpha \prod_{i=1}^{k} (\tau - \tau_i)^2, \text{ where } \alpha > 0,\ 2k \leq d, \text{ and } \tau_1, \ldots, \tau_k \in [a, b];$$

$$p(\tau) = \alpha(\tau - a) \prod_{i=1}^{k} (\tau - \tau_i)^2, \text{ where } \alpha > 0,\ 2k + 1 \leq d, \text{ and } \tau_1, \ldots, \tau_k \in [a, b];$$

$$p(\tau) = \alpha(b - \tau) \prod_{i=1}^{k} (\tau - \tau_i)^2, \text{ where } \alpha > 0,\ 2k + 1 \leq d, \text{ and } \tau_1, \ldots, \tau_k \in [a, b];$$

$$p(\tau) = \alpha(\tau - a)(b - \tau) \prod_{i=1}^{k} (\tau - \tau_i)^2, \text{ where } \alpha > 0,\ 2k + 2 \leq d, \text{ and } \tau_1, \ldots, \tau_k \in [a, b].$$

5*. Prove, that a polynomial $p(\tau)$ of degree d, which is non-negative on $[a, b]$, has a unique representation

$$p(\tau) = \begin{cases} \alpha \prod_{i=1}^{k} (\tau - \tau_{2i-1})^2 + \beta(\tau - a)(b - \tau) \prod_{i=1}^{k-1} (\tau - \tau_{2i})^2 & \text{if } \ d = 2k \\ \alpha(\tau - a) \prod_{i=1}^{k} (\tau - \tau_{2i})^2 + \beta(b - \tau) \prod_{i=1}^{k} (\tau - \tau_{2i-1})^2 & \text{if } \ d = 2k+1, \end{cases}$$

where $\alpha, \beta > 0$ and $a \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_{d-1} \leq b$,
see S. Karlin and W.J. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Wiley, Interscience, 1966.

Monday, February 9

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19. The Cone of Positive Semidefinite Matrices

(19.1) The space of symmetric matrices

An $n \times n$ matrix $A = (a_{ij})$ is called *symmetric*, provided $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, n$. The vector space of all $n \times n$ symmetric matrices A can be identified with the Euclidean space $\mathbb{R}^d$, where $d = n(n+1)/2$. Let A be an $n \times n$ symmetric matrix and let U be an $n \times n$ orthogonal matrix (that is, $U^t = U^{-1}$). Then $U^{-1}AU$ is a symmetric matrix.

The scalar product of two symmetric matrices A and B is defined as

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

An important formula for the scalar product is

$$\langle A, B \rangle = \operatorname{tr}(AB) = \operatorname{tr}(BA),$$

where tr is the trace, that is the sum of all diagonal entries. In particular, it follows that if U is $n \times n$ orthogonal matrix, then

$$\langle A, B \rangle = \langle U^{-1}AU, U^{-1}BU \rangle,$$

since $\langle U^{-1}AU, U^{-1}BU \rangle = \operatorname{tr}(U^{-1}AUU^{-1}BU) = \operatorname{tr}(U^{-1}ABU) = \operatorname{tr}(AB) = \langle A, B \rangle$ (we use that $\operatorname{tr}(C) = \operatorname{tr}(U^{-1}CU)$ for any invertible U). We often identify a symmetric matrix A with the quadratic form $q_A : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$$q_A(x) = \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j,$$

where $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. If $x = (\xi_1, \ldots, \xi_n)$, we denote by $x \otimes x$ the $n \times n$ symmetric matrix $X = (x_{ij})$ where $x_{ij} = \xi_i \xi_j$. Then one can write

$$q_A(x) = \langle A, x \otimes x \rangle.$$

PROBLEMS.

1°. Check that the dimension of the space of symmetric $n \times n$ matrices is indeed $n(n+1)/2$.

2°. Check that $\sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{tr}(AB)$, where $A = (a_{ij})$ and $B = (b_{ij})$ are symmetric $n \times n$ matrices.

3°. Check that $U^t AU$ is a symmetric matrix, provided U is any $n \times n$ matrix and A is an $n \times n$ symmetric matrix.

(19.2) Positive semidefinite matrices

An $n \times n$ symmetric matrix A is called *positive semidefinite* provided $q_A(x) \geq 0$ for all $x \in \mathbb{R}^n$. An $n \times n$ symmetric matrix A is called *positive definite* provided A is positive semidefinite and $q_A(x) = 0$ only if $x = 0$. The set of all positive semidefinite $n \times n$ symmetric matrices we denote $\mathcal{S}_+$.

Some linear algebra. For every symmetric $n \times n$ matrix A, there is an orthogonal matrix U such that $U^{-1}AU$ is a diagonal matrix, having the eigenvalues of A on the diagonal. The number of non-zero eigenvalues is equal to the rank of A. A symmetric matrix is positive semidefinite if and only if all eigenvalues of A are non-negative, and positive definite if and only if all eigenvalues are positive. In particular, if A is a positive (semi)definite matrix and U is an orthogonal matrix, then $U^{-1}AU$ is a positive (semi)definite matrix. In other words, for any orthogonal matrix U, the linear transformation $X \longmapsto U^{-1}XU$ of $\mathbb{R}^d$: $d = n(n+1)/2$ maps the cone $\mathcal{S}_+$ onto itself. If A is positive semidefinite, then all diagonal entries are non-negative and $|a_{ij}| \leq (a_{ii} + a_{jj})/2$ for every pair $1 \leq i \neq j \leq n$. Sylvester's criterion states that a symmetric $n \times n$ matrix A is positive semidefinite if and only if the determinant of each upper left corner submatrix $(a_{ij}) : i, j = 1, \ldots, k$ is non-negative.

PROBLEMS.

1°. Prove that $\mathcal{S}_+ \subset \mathbb{R}^d$, $d = n(n+1)/2$ is a closed convex cone.

2. Prove that A is an interior point of $\mathcal{S}_+$ if and only if A is positive definite.

3. Prove that $B = \{A \in \mathcal{S}_+ : \operatorname{tr}(A) = 1\}$ is a compact base of $\mathcal{S}_+$.

4. Draw a nice picture of the cone in $\mathbb{R}^3$ of positive semidefinite 2×2 matrices.

5°. Check, that if A is a positive semidefinite 2×2 matrix then $a_{11}, a_{22} \geq 0$ and $|a_{12}| \leq (a_{11} + a_{22})/2$.

(19.3) Proposition. *Let A be an $n \times n$ positive semidefinite matrix. Suppose that* rank $A = r$. *If $r = n$, then A is an interior point of $\mathcal{S}_+$. If $r < n$, then A is an interior point of a face $\mathcal{F}$ of $\mathcal{S}_+$, where* dim $\mathcal{F} = r(r+1)/2$.

Proof. If rank $A = n$, then A is positive definite, so the result follows by Problem 2, Section 19.2.

Suppose that rank $A = r < n$. We will construct a support hyperplane $\mathcal{H}$ to $\mathcal{S}_+$ at A. Let $\lambda_1, \ldots, \lambda_r > 0$ be the non-zero eigenvalues of A. Let us find an orthogonal matrix U, such that $U^{-1}AU = D$, where D is the diagonal matrix such that $\lambda_1, \ldots, \lambda_r$ are the first r diagonal entries of D and all other entries of D are 0:

$$D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_r, 0, \ldots, 0\}.$$

Let

$$C = \operatorname{diag}\{0, \ldots, 0, 1, \ldots, 1\}$$

be the diagonal matrix whose first r diagonal entries are 0 and the last $n-r$ diagonal entries are 1's, and let $Q = UCU^{-1}$. Then Q is a non-zero positive semidefinite matrix and

$$\langle Q, A \rangle = \langle UCU^{-1}, UDU^{-1} \rangle = \langle C, D \rangle = 0.$$

Furthermore, for any positive semidefinite $n \times n$ matrix X, the matrix $Y = U^{-1}XU$ is positive semidefinite and

$$\langle Q, X \rangle = \langle UCU^{-1}, UYU^{-1} \rangle = \langle C, Y \rangle \geq 0,$$

since diagonal entries of Y must be non-negative. Therefore, the hyperplane $\mathcal{H} = \{X : \langle Q, X\rangle = 0\}$ is a support hyperplane of $\mathcal{S}_+$ at A. Let us describe the corresponding face

$$\mathcal{F} = \{X \in \mathcal{S}_+ : \langle Q, X\rangle = 0\}.$$

The map $X \longmapsto Y = U^{-1}XU$ is a non-degenerate linear transformation which maps $\mathcal{S}_+$ onto itself, maps Q to C and A to D. Then the face $\mathcal{F}$ is mapped onto a face $\mathcal{F}'$, containing D and consisting of all positive semidefinite matrices Y such that $\langle Y, C\rangle = 0$:

$$\mathcal{F}' = \{Y \in \mathcal{S}_+ : \langle Y, C\rangle = 0\}.$$

Clearly, Y must have the last $n-r$ diagonal entries equal to zero. Since Y positive semidefinite, all entries in the last $n-r$ rows and last $n-r$ columns must be 0 (see Section 19.2). The upper left $r \times r$ submatrix of Y can be an arbitrary positive semidefinite matrix. Thus the face $\mathcal{F}'$ may be identified with the cone of all $r \times r$ positive semidefinite matrices (in particular, $\dim \ F' = (r+1)r/2)$ and it is seen that $\mathcal{F}'$ contains D in its interior. Since $Y \longmapsto X = UYU^{-1}$ is a non-degenerate linear transformation, which maps D to A and $\mathcal{F}'$ onto $\mathcal{F}$, we conclude that $\dim \ \mathcal{F} = r(r+1)/2$ and $\mathcal{F}$ contains A in its interior. □

PROBLEM.

1. Prove that $A \in \mathcal{S}_+$ spans an extreme ray of $\mathcal{S}_+$ if and only if rank $A = 1$.

Wednesday, February 11

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20. The Idea of Semidefinite Relaxation

Often, to express that X is positive semidefinite, instead of writing $X \in \mathcal{S}_+$, we write $X \succeq 0$. To express that X is positive definite, we write $X \succ 0$.

(20.1) Proposition. *Let us fix k symmetric $n \times n$ matrices $A_1, \ldots, A_k$ and k real numbers $\alpha_1, \ldots, \alpha_k$. Suppose that the set*

$$\mathcal{K} = \Big\{X \succeq 0 : \langle A_i, X\rangle = \alpha_i, \ i = 1, \ldots, k\Big\}$$

is non-empty. Then $\mathcal{K}$ has an extreme point, and for any extreme point Y of $\mathcal{K}$,

$$\operatorname{rank} \ Y \leq \left[\frac{\sqrt{8k+1}-1}{2}\right],$$

where $[\cdot]$ is the integer part of a number, that is, the smallest integer not exceeding the number.

Proof. First, we prove that $\mathcal{K}$ has an extreme point. Let us consider a linear function $f(X) = \operatorname{tr}(X)$ on $\mathcal{K}$. Let $X_0 \in \mathcal{K}$ be a point and let $\alpha_0 = \operatorname{tr}(X_0)$. We note that the set $\{X \succeq 0 : \ \operatorname{tr}(X) \leq \alpha_0\}$ is compact (since every entry of a positive semidefinite matrix X with trace α_0 must not exceed α_0, see Section 19.2). Therefore, $f(X)$ attains its minimum α on $\mathcal{K}$ and $\mathcal{F} = \{X \in \mathcal{K} : f(X) = \alpha\}$ is a compact face of $\mathcal{K}$. By the Krein-Milman Theorem (Theorem 11.3) $\mathcal{F}$ must have an extreme point Y, which by Theorem 11.2 must be an extreme point of $\mathcal{K}$.

Now, let Y be an extreme point of $\mathcal{K}$. Let $r = \text{rank }\ Y$. Then, by Proposition 19.3, Y must be an interior point of a face $\mathcal{F}$ of $\mathcal{S}_+$ of dimension $r(r+1)/2$. We have $\mathcal{K} = \mathcal{S}_+ \cap \mathcal{L}$, where $\mathcal{L}$ is the affine subspace in the space of symmetric matrices defined by the constraints $\langle A_i, X\rangle = \alpha_i$ for $i = 1, \ldots, k$. Hence the codimension of $\mathcal{L}$ is at most k. We observe that Y is an interior point of the intersection $\mathcal{F} \cap \mathcal{L}$. Since Y is an extreme point, we must have dim $(\mathcal{F} \cap \mathcal{L}) = 0$. Therefore, dim $\mathcal{F} \le k$, that is $r(r+1)/2 \le k$, and the proof follows. □

(20.2) Corollary. *Let us fix k symmetric $n \times n$ matrices, $A_1, \ldots, A_k$, and k real numbers $\alpha_1, \ldots, \alpha_k$. If there is a matrix $X \succeq 0$ such that*

$$\langle A_i, X\rangle = \alpha_i \quad \textit{for} \ \ i = 1, \ldots, k,$$

then there is a matrix $X_0 \succeq 0$ which satisfies the constraints and such that additionally

$$\text{rank }\ X_0 \le \left[\frac{\sqrt{8k+1} - 1}{2}\right].$$

Proof. Choose X_0 to be an extreme point of the set

$$\mathcal{K} = \big\{X \succeq 0 : \langle A_i, X\rangle = \alpha_i, \ i = 1, \ldots, k\big\}$$

and use Proposition 20.1. □

(20.3) Corollary. *Let us fix two symmetric $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ and two real numbers α and β. The system of two quadratic equations*

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j = \alpha \quad \textit{and} \quad \sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j = \beta$$

has a real solution $x = (\xi_1, \ldots, \xi_n)$ if and only if the system of two linear matrix equations

$$\langle A, X\rangle = \alpha \quad \textit{and} \quad \langle B, X\rangle = \beta$$

has a positive semidefinite solution $X \succeq 0$.

Proof. Let $x = (\xi_1, \ldots, \xi_n)$ be a solution to the system of quadratic equations. Let $X = x \otimes x$ (that is, $X = (x_{ij})$, where $x_{ij} = \xi_i\xi_j$). Then X is a positive semidefinite matrix and $\langle A, X\rangle = \alpha$ and $\langle B, X\rangle = \beta$.

On the other hand, suppose there is a solution $X \succeq 0$ to the system of equations $\langle A, X\rangle = \alpha$ and $\langle B, X\rangle = \beta$. Corollary 20.2 states that we can assume that rank $X \le 1$ (substitute $k = 2$ in the formula). Then $X = (x_{ij})$ can be written as $x_{ij} = \xi_i\xi_j$ for some set of numbers $\xi_1, \ldots, \xi_n$. Then $x = (\xi_1, \ldots, \xi_n)$ is a solution to the system of quadratic equations. □

PROBLEMS.

1. Construct an example of a system of three quadratic equations

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j = \alpha, \quad \sum_{i,j=1}^{n} b_{ij}\xi_i\xi{=}\beta, \quad \sum_{i,j=1}^{n} c_{ij}\xi_i\xi_j = \gamma$$

which does not have a solution $(\xi_1, \ldots, \xi_n)$, but the corresponding system of linear matrix equations

$$\langle A, X\rangle = \alpha, \quad \langle B, X\rangle = \beta, \quad \langle C, X\rangle = \gamma$$

has a positive semidefinite solution $X \succeq 0$.

2°. Check that $X = (x_{ij})$ has the form $x_{ij} = \xi_i \xi_j$ if and only if rank $X \leq 1$ and X is positive semidefinite.

3. Consider the function $f(X) = \ln \det X$ on the set of positive definite matrices. Prove that f is *concave*, that is $f\big((1/2)X + (1/2)Y\big) \geq (1/2)f(X) + (1/2)f(Y)$. Furthermore, $f(X) \longrightarrow -\infty$ as X approaches the boundary of the cone $\mathcal{S}_+$ of positive semidefinite matrices.

4. Let $A_1, \ldots, A_k$ be $n \times n$ symmetric matrices and let $\alpha_1, \ldots, \alpha_k$ be real numbers. Let

$$\mathcal{K} = \Big\{X \succeq 0 : \langle A_i, X\rangle = \alpha_i, \ i = 1, \ldots, k\Big\}.$$

Suppose that $X \in \mathcal{K}$ and that rank $X = r$. Let us decompose $X = QQ^t$, where Q is an $n \times r$ matrix of rank r. Then the dimension of the smallest face of $\mathcal{K}$ containing X is equal to the codimension of $\operatorname{Span}\big\{Q^t A_1 Q, \ldots, Q^t A_k Q\big\}$ in the space of $r \times r$ symmetric matrices.

For this and related results, see M.M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer, 1997.

5^*. Let us fix two $n \times n$ symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$. Let $S^{n-1} \subset \mathbb{R}^n$ be the unit sphere $S^{n-1} = \big\{x \in \mathbb{R}^n : \|x\| = 1\big\}$. Consider the map $q : S^{n-1} \longrightarrow \mathbb{R}^2$, $q(x) = \big(\alpha(x), \beta(x)\big)$, where

$$\alpha(x) = \sum_{i,j=1}^n a_{ij}\xi_i\xi_j, \quad \beta(x) = \sum_{i,j=1}^n b_{ij}\xi_i\xi_j, \quad \text{and} \quad x = (\xi_1, \ldots, \xi_n).$$

Prove that the image of the map q is a convex set in $\mathbb{R}^2$ provided $n > 2$ (a version of the Toeplitz-Hausdorff Convexity Theorem). Show that the result is not true for $n = 2$.

(20.4) The idea of semidefinite relaxation

Quite often, we have to solve a problem of optimizing a quadratic form under quadratic constraints:

Find $\min q_0(x)$

Subject to: $q_i(x) = \alpha_i : \ i = 1, \ldots, k$,

where $x \in \mathbb{R}^n$ and $q_i, i = 0, \ldots, k$ are quadratic forms $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. Such problems are generally very difficult to solve. Let $Q_i, i = 0, \ldots, k$ be the matrices of the forms (we say that $A = (a_{ij})$ is the matrix of the quadratic form $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$). It makes sense to consider a *semidefinite relaxation*

Find $\min\langle Q_0, X\rangle$

Subject to: $\langle Q_i, X\rangle = \alpha_i : \ i = 1, \ldots, k$,

where $X \succeq 0$ is a positive semidefinite matrix. This problem is *convex* and easy to handle. If $x = (\xi_1, \ldots, \xi_n)$ is a feasible solution of the original problem, then

$X = (x_{ij})$: $x_{ij} = \xi_i \xi_j$ is a feasible solution of the relaxed problem. Of course, it may happen that there is a feasible solution X of the relaxed problem, but there are no feasible solutions of the first problem. Generally speaking, the optimal value in the relaxed problem is smaller. However, if X is a feasible solution *and* rank $X \leq 1$, then $X = x \otimes x$ for some $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and this x is a feasible solution to the original problem.

(20.5) Corollary. *Let us fix 3 symmetric $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. Consider two optimization problems:*

Find $\min\langle C, X\rangle$

Subject to: $\langle A, X\rangle = \alpha, \quad \langle B, X\rangle = \beta,$ where

$X \succeq 0.$

and

Find $\min \sum_{i,j=1}^{n} c_{ij}\xi_i\xi_j$

Subject to: $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j = \alpha, \quad \sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j = \beta,$ where

$x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$

Suppose that there is a unique optimal solution X to the first problem. Then rank $X \leq 1$, $X = x \otimes x$ *and* $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ *is the optimal solution to the second problem. Suppose that the set of optimal solutions X to the first problem is compact. Then the optimal values of the objective functions in the two problems are equal.*

Proof. If X is the unique solution, then X must be an extreme point of the set of feasible solutions $\mathcal{K} = \big\{X \succeq 0 : \langle A, X\rangle = \alpha, \ \langle B, X\rangle = \beta\big\}$ (see Theorem 11.1, Part 1). Therefore, by Proposition 20.1, we must have rank $X \leq 1$.

If the set of optimal solutions is compact, it must have an extreme point X by the Krein-Milman Theorem (Theorem 11.3). Part 2 of Theorem 11.2 implies that X must be an extreme point of K, so rank $X \leq 1$. The rest follows as in Section 20.4. □

Friday, February 13

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21. Semidefinite Relaxation: Applications to "Linkages" and "Molecules"

(21.1) Some Linear Algebra: Gram matrices

Let $v_1, \ldots, v_n$ be some vectors in $\mathbb{R}^d$. Let us define an $n \times n$ matrix $X = (x_{ij})$ by $x_{ij} = \langle v_i, v_j\rangle$ (we consider the usual scalar product in $\mathbb{R}^d$). Matrix X is called the *Gram matrix* of vectors $v_1, \ldots, v_n$. It is known that X is positive semidefinite and that rank $X \leq d$; in fact, rank X is the dimension of $\operatorname{Span}(v_1, \ldots, v_n)$. Conversely, if X is a positive definite matrix, such that rank $X \leq d$, then X is the Gram matrix of some n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$: $x_{ij} = \langle v_i, v_j\rangle$.

(21.2) Graph realization problem

Suppose we are given an (undirected) weighted graph $G = (V, E; \rho)$, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices, E is the set of edges and $\rho : E \longrightarrow \mathbb{R}_+$ is a function, which assigns to every edge $(i, j) \in E$ a non-negative number ("length") ρ_{ij}. We say that G is *d-realizable*, if one can place the vertices $v_1, \ldots, v_n$ in $\mathbb{R}^d$ in such a way that $\|v_i - v_j\| = \rho_{ij}$ for any edge $(i, j) \in E$. We say that G is *realizable*, if it is d-realizable for some d. The most important case is 3-realizability: some problems of robotics ("linkages") and computational chemistry ("molecules") lead to problems of 3-realizability of certain graphs, see also G.M. Crippen and T.F. Havel, *Distance Geometry and Molecular Conformation*, Wiley, New York, 1988.

PROBLEMS.

Some classical problems of the field:

1*. Prove, that one can place 7 points $v_1, \ldots, v_7$ in $\mathbb{R}^3$ in such a way that $\|v_1 - v_2\| = \|v_2 - v_3\| = \ldots = \|v_6 - v_7\| = \|v_7 - v_1\| = 1$ and $\|v_1 - v_3\| = \|v_2 - v_4\| = \ldots = \|v_5 - v_7\| = \|v_1 - v_6\| = \|v_2 - v_7\| = \sqrt{8/3}$.

The constants 1 and $\sqrt{8/3}$ are chosen in such a way that the angles between consecutive intervals v_{i-1}, v_i and v_i, v_{i+1} are equal to $\arccos -1/3$, that is, to the angle between two intervals connecting vertices of a regular tetrahedron with its center.

2*. Prove, that every such a configuration of 7 points in $\mathbb{R}^3$ has one degree of freedom (modulo rigid motions).

3*. Prove, that one can place 6 points $v_1, \ldots, v_6$ in $\mathbb{R}^3$ in such a way that $\|v_1 - v_2\| = \|v_2 - v_3\| = \ldots = \|v_5 - v_6\| = \|v_6 - v_1\| = 1$ and $\|v_1 - v_3\| = \|v_2 - v_4\| = \ldots = \|v_4 - v_6\| = \|v_1 - v_5\| = \|v_2 - v_6\| = \sqrt{8/3}$.

4*. Prove, that there are two such configurations of 6 points, one of which has one degree of freedom (modulo rigid motions) and the other is rigid.

An easy problem.

5°. Prove, that if a weighted graph with n vertices is realizable, it is $(n-1)$-realizable.

For every edge $e = (i, j)$ of G, let us introduce an $n \times n$ symmetric matrix $A_{ij} = (a^{ij}_{st})$, where

$$a^{ij}_{st} = \begin{cases} 1 & \text{if } s = i, t = i \text{ or } \ s = j, t = j, \\ -1 & \text{if } s = i, t = j \text{ or } \ s = j, t = i \\ 0 & \text{elsewhere.} \end{cases}$$

Let $v_1, \ldots, v_d$ be a realization of the graph in $\mathbb{R}^d$, and let $X = (x_{ij})$, $x_{ij} = \langle v_i, v_j \rangle$ be the Gram matrix of $v_1, \ldots, v_n$. Then $X \succeq 0$ and for any edge $e = (i, j)$ of G, we have

$$\langle A_{ij}, X \rangle = \langle v_i, v_i \rangle + \langle v_j, v_j \rangle - 2\langle v_i, v_j \rangle = \|v_i - v_j\|^2 = \rho^2_{ij}.$$

Hence we conclude that the problem of realizability of G is equivalent to the following convex (*semidefinite*) problem:

Is there a matrix $X \succeq 0$ such that

$$\langle A_{ij}, X \rangle = \rho^2_{ij}$$

for every edge $(i,j) \in E$?

The problem of d-realizability is equivalent to the above problem with one additional non-convex constraint:

$$\operatorname{rank}\ X \leq d.$$

It turns out, that if d is large enough, realizability is equivalent to d-realizability.

(21.3) Proposition. *Let k be the number of edges of G. Suppose that*

$$d \geq \left[\frac{\sqrt{8k+1}-1}{2}\right].$$

Then G is d-realizable, if and only if it is realizable. In particular, if $k \leq 9$, the graph is realizable if and only if it is 3-realizable.

Proof. By Corollary 20.2, if the system of linear matrix equations

$$\langle A_{ij}, X\rangle = \rho_{ij}^2 : \quad (i,j) \in E$$

has a solution $X \succeq 0$, it has a solution $X \succeq 0$, such that additionally, $\operatorname{rank}\ X \leq [(\sqrt{8k+1}-1)/2]$. □

PROBLEMS.

1. For any d, construct an example of a weighted graph G with $k = \binom{d+2}{2}$ edges, such that G is realizable, but not d-realizable.

2. Suppose that G is a cycle $v_1 - v_2 - \ldots - v_n - v_1$ with some weights on the edges. Prove that G is realizable, if and only if it is 2-realizable.

3*. Suppose that G has n vertices $v_1, \ldots, v_n$ and $2n$ edges: $(v_1,v_2), (v_2,v_3), \ldots, (v_{n-1},v_n), (v_n,v_1)$ and $(v_1,v_3), (v_2,v_4), \ldots, (v_{n-2},v_n), (v_{n-1},v_1), (v_n,v_2)$ with some weights on the edges. Is it true, that if G is realizable, it is 4-realizable?

4*. A graph $G=(V,E)$ is called *chordal*, provided for any cycle $v_1 - v_2 - \ldots - v_k - v_1$, where $k \geq 4$ there is a chord $v_i - v_j$, where $1 \leq i < j \leq k$. A subset $K \subset V$ is called a *clique*, if every two vertices from K are connected by an edge. Prove, that G is chordal if and only if it has the following property:

for any choice of weights on the edges, the graph is realizable, provided every clique is realizable.

See M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer, 1997.

5. Let us fix an undirected graph $G=(V,E)$ with n vertices $v_1, \ldots, v_n$ and k edges, and a space $\mathbb{R}^d$. The set of all n-tuples $(v_1, \ldots, v_n)$, where $v_i \in \mathbb{R}^d$ is identified with the space $\mathbb{R}^d \times \ldots \times \mathbb{R}^d = \mathbb{R}^{dn}$. Graph G defines a *rigidity map* $\mathbb{R}^{dn} \longrightarrow \mathbb{R}^k$; $(v_1, \ldots, v_n) \longmapsto (\ldots, \|v_i - v_j\|^2, \ldots) : (i,j) \in E$: we place arbitrarily the vertices $v_1, \ldots, v_n$ in $\mathbb{R}^d$ and list all squared lengths of edges of G. Prove, that if $d \geq [(\sqrt{8k+1}-1)/2]$, the image of the rigidity map is a convex set in $\mathbb{R}^k$.

Hint: Prove, that the image coincides with the image of the cone of positive semidefinite matrices under a linear transformation $X \longmapsto (\ldots, \langle A_{ij}, X\rangle, \ldots)$.

6. Let $A=(a_{ij}), B=(b_{ij}), C=(c_{ij}), D=(d_{ij})$, and $E=(e_{ij})$ be $n \times n$ real symmetric matrices. Consider the quadratic map $\phi : \mathbb{C}^n \longrightarrow \mathbb{R}^5$, where

$$(z_1, \ldots, z_n) \longmapsto \left(\sum_{i,j=1}^n a_{ij} z_i \overline{z_j},\ \sum_{i,j=1}^n b_{ij} z_i \overline{z_j},\ \sum_{i,j=1}^n c_{ij} z_i \overline{z_j},\ \sum_{i,j=1}^n d_{ij} z_i \overline{z_j},\ \sum_{i,j=1}^n e_{ij} z_i \overline{z_j}\right).$$

Prove, that the image $\phi(\mathbb{C}^n)$ is a convex set in $\mathbb{R}^5$.

Hint: Use Corollary 20.2 with $k = 5$.

See A.I. Barvinok, Problems of distance geometry and convex properties of quadratic maps, *Discrete Comput. Geom.* **13** (1995), no. 2, 189–202.

Monday, February 16

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22. Separation Theorems in Euclidean Space

Let V be a vector space. Recall, that sets $A, B \subset V$ are separated by a hyperplane H if A and B belong to different closed halfspaces $\overline{H_+}$ and $\overline{H_-}$. Equivalently, if there exist a non-zero linear function $f : V \longrightarrow \mathbb{R}$ and a number α such that $f(x) \leq \alpha$ for each $x \in A$ and $f(x) \geq \alpha$ for each $x \in B$. Sets $A, B \subset V$ are strictly separated by a hyperplane H if A and B belong to different open halfspaces H_+ and H_-. Equivalently, if there exist a non-zero linear function $f : V \longrightarrow \mathbb{R}$ and a number α such that $f(x) < \alpha$ for any $x \in A$ and $f(x) > \alpha$ for any $x \in B$ (see Section 9.2).

(22.1) Definition. Let V be a vector space and let $A, B \subset V$ be sets. The *Minkowski sum* $A + B$ is defined as

$$A + B = \{x + y : \ x \in A, \ y \in B\}.$$

PROBLEM.

1°. Prove, that if sets A and B are convex, their Minkowski sum $A+B$ is convex as well.

(22.2) Theorem. *Let A and B be non-empty convex sets in $\mathbb{R}^d$ such that $A \cap B = \emptyset$. Then there is a hyperplane H that separates A and B.*

Proof. Consider the set

$$C = A - B = \{x - y : x \in A, y \in B\}.$$

We claim that C is convex. Let $u = x_1 - y_1$ and $v = x_2 - y_2$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$. Then, for any $0 \leq \alpha \leq 1$, we have $x = \alpha x_1 + (1 - \alpha)x_2 \in A$, $y = \alpha y_1 + (1 - \alpha)y_2 \in B$ and

$$\alpha u + (1 - \alpha)v = \alpha(x_1 - y_1) + (1 - \alpha)(x_2 - y_2) = x - y \in C.$$

Since $A \cap B = \emptyset$, we have $0 \notin C$. Therefore, by Theorem 10.8, there exists a hyperplane H such that $0 \in H$ and H isolates C. In other words, there exists a non-zero linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, such that $f(x - y) \leq 0$ for all $x \in A$, $y \in B$. We have $f(x) \leq f(y)$ for each $x \in A$ and each $y \in B$. Therefore, there exists an $\alpha \in \mathbb{R}$, such that $f(x) \leq \alpha$ for all $x \in A$ and $f(y) \geq \alpha$ for all $y \in B$ (we can choose $\alpha = \sup\{f(x) : x \in A\}$, for example). The hyperplane $H = \{x \in \mathbb{R}^d : f(x) = \alpha\}$ separates A and B. □

PROBLEMS.

1°. Suppose that one of the sets $A, B \subset \mathbb{R}^d$ is open. Prove that $C = A - B$ is open.

2. Construct an example of two closed sets $A, B \subset \mathbb{R}^2$ such that $C = A - B$ is not closed.

3. Let $A, B \subset \mathbb{R}^d$ be sets. Suppose, that A is closed and B is compact. Prove that the set $C = A - B$ is closed.

Hint: Let $x_n \in C$ be a sequence of points and suppose that $x_n \longrightarrow x$. We have to prove that $x \in C$. We may write $x_n = y_n - z_n$, where $y_n \in A$ and $z_n \in B$. Since B is compact, there is a subsequence z_{n_k} which converges to a point of B: $z_{n_k} \longrightarrow z \in B$. Then we must have $y_{n_k} \longrightarrow y \in A$ (since A is closed). Therefore, $x = y - z \in C$.

4. Prove, that if $A, B \subset \mathbb{R}^d$ are compact sets, then $C = A - B$ is a compact set.

5°. Let $A, B \subset \mathbb{R}^d$ be open sets. Suppose that a hyperplane H separates A and B. Prove, that H strictly separates A and B.

(22.3) Theorem. *Let $A \subset \mathbb{R}^d$ be a closed convex set and let $u \notin A$ be a point. Then there exists a hyperplane H that strictly separates A and u.*

Proof. Since A is a closed set and $u \notin A$, there exists an $r > 0$ such that the open ball $B(u; r) = \{x : \|x - u\| < r\}$ does not intersect A. Let $B = B(0; r/2) = \{x : \|x\| < r/2\}$. We claim that $A + B$ and $u + B$ are open non-intersecting convex sets. Indeed,

$$A + B = \bigcup_{x \in A} B(x; r/2)$$

is a union of open sets, so it is open. Similarly, $u + B = B(u; r/2)$ is an open ball centered at u. It follows by Problem 1, Section 22.1, that $A + B$ and $u + B$ are convex. Suppose that $(A + B) \cap (B + u) \neq \emptyset$. For a point $x \in (A + B) \cap (B + u)$, we have $\|u - x\| < r/2$ and $\|y - x\| < r/2$ for some $y \in A$. Therefore, $\|u - y\| < r$, which is a contradiction. Therefore, the sets $A + B$ and $B + u$ do not intersect.

By Theorem 22.2, there is a hyperplane H that separates $A + B$ and $u + B$. Since the sets $A + B$ and $u + B$ are open, H must strictly separate the sets (see Problem 5, Section 22.2). $\square$

PROBLEMS.

1. Construct an example of two non-empty convex closed non-intersecting sets A and B in $\mathbb{R}^2$ that can not be strictly separated by a hyperplane.

2. Let $A \subset \mathbb{R}^d$ be a non-empty convex closed set and let $u \notin A$ be a point. Prove, that there exists a unique point $v \in A$, such that $\|u - v\| \leq \|u - x\|$ for all $x \in A$. Furthermore, prove that the hyperplane H orthogonal to $u - v$ and passing through the point $(u + v)/2$ strictly separates A and u.

3. Using Problem 3, Section 22.2, prove that if $A, B \subset \mathbb{R}^d$ are non-empty non-intersecting convex sets, where A is closed and B is compact, then A and B can be strictly separated by a hyperplane H.

4. Let $A, B \subset \mathbb{R}^d$ be non-empty non-intersecting convex sets. Suppose that B is compact and A is closed. Prove, that there is a pair of poits $u \in A$ and $v \in B$ such that $\|u - v\| \leq \|x - y\|$ for all $x \in A$ and all $y \in B$. Furthermore, prove that the hyperplane H orthogonal to $u - v$ and passing through $(u + v)/2$ strictly separates A and B.

(22.4) What can go wrong in infinite dimension?

PROBLEM.

1. Let V be the space of all polynomials in one variable. Let $A \subset V$ be the set of polynomials whose highest coefficient (coefficient of the monomial of the highest degree) is positive and let $B \subset V$ be the set of polynomials whose highest coefficient is negative. Prove, that A and B are convex, that $A \cap B = \emptyset$ and that A and B can not be separated by a hyperplane.

We need the notion of algebraically open set (see Definition 9.3).

(22.5) Theorem. *Let V be a vector space and let $A, B \subset V$ be non-empty convex sets such that $A \cap B = \emptyset$. Suppose that A is algebraically open. Then A and B can be separated by a hyperplane.*

Proof. Let $C = A - B$. We can write C as a union of algebraically open sets:

$$C = \bigcup_{b \in B} (A - b).$$

Therefore, C is algebraically open (see Problem 6, Section 9.3). Furthermore, C is convex and $0 \notin C$. Therefore, by Theorem 9.4, there exists a hyperplane H, such that $0 \in H$ and H isolates C. The proof follows as in Theorem 22.2. □

(22.6) Definition. Let V be a vector space and let $A \subset V$ be a convex set. We say that $v \in A$ *lies in the algebraic interior* of A, provided for any straight line $L = \{v + \tau u : \tau \in \mathbb{R}\}$, passing through v, the point v lies in the interior of the intersection $A \cap L$. The set of all points v that lie in the algebraic interior of A is called the *algebraic interior* of A.

PROBLEM.

1. Let V be a vector space and let $A \subset V$ be a convex set. Let u_0 be a point in the algebraic interior of A. Prove, that for any point $u_1 \in A$ and any $0 \leq \alpha < 1$, the point $u_\alpha = (1-\alpha)u_0 + \alpha u_1$ lies in the algebraic interior of A.

   Hint: See Lemma 10.2.

2. Prove, that the algebraic interior of a convex set is a convex algebraically open set.

   Hint: Use Problem 1.

3. Let V be a vector space, let $A, B \subset V$ be convex sets, and let $H \subset V$ be a hyperplane. Suppose that H separates B and the algebraic interior of A. Prove, that H separates A and B.

   Hint: Use Problem 1.

(22.7) Corollary. *Let V be a vector space and let $A, B \subset V$ be non-empty convex sets such that $A \cap B = \emptyset$. Suppose that A has a non-empty algebraic interior. Then A and B can be separated by a hyperplane.*

Proof. Let A_1 be the algebraic interior of A. Then, by Problem 2, Section 22.6, A_1 is a non-empty algebraically open convex set. By Theorem 22.5, there is a hyperplane H that separates A_1 and B. Then, by Problem 3, Section 22.6, H separates A and B. □

23. Topological Vector Spaces, Convex Sets, and Hyperplanes

(23.1) Topological vector spaces

Let V be a vector space. We make it a *topological vector space* by declaring some subsets $U \subset V$ *open*, so that some axioms are satisfied. A set $A \subset V$ is called *closed*, provided its complement $V \setminus A$ is open. An open set containing a given point is called a *neighborhood* of the point.

AXIOMS:

$\emptyset$ and V are open sets;

union of open sets is an open set;

if U_1 and U_2 are open sets, then $U_1 \cap U_2$ is an open set;

every point $u \in V$ is a closed set;

for every $w_1, w_2 \in V$, for every neighborhood U of $u = w_1 + w_2$, there is a neighborhood W_1 of w_1 and a neighborhood W_2 of w_2, such that $W_1 + W_2 \subset U$, where $W_1 + W_2 = \{x + y : x \in W_1,\ y \in W_2\}$;

for every $w \in V$ and every $\alpha \in \mathbb{R}$, for every neighborhood U of $u = \alpha w$, there is a neighborhood W of w and a number $r > 0$ such that $\beta x \in U$, provided $x \in W$ and $|\alpha - \beta| < r$.

If V is a set (not necessarily a vector space) with a collection of subsets that satisfy the first three axioms, V is called a *topological space*. A map $\phi : X \longrightarrow Y$, where X and Y are topological spaces, is called *continuous* provided for every open set $U \subset Y$, the *preimage* $\phi^{-1}(U) = \{x \in X : \phi(x) \in U\}$ is open. Equivalently, the map ϕ is continuous, provided for each point $w \in X$ and each neighborhood U of $u = \phi(w)$, there is a neighborhood W of w, such that $\phi(x) \in U$ for each $x \in W$. The last two axioms relate the topological and linear structure. They say that the addition $V \times V \longrightarrow V$, $(x, y) \longmapsto x + y$ and multiplication $\mathbb{R} \times V \longrightarrow V$, $(\alpha, x) \longmapsto \alpha x$ are continuous operations.

In particular, for any given $u \in V$, the *translation* $x \longmapsto u + x$ is a continuous transformation, and for any given $\alpha \in \mathbb{R}$, the *scaling* $x \longmapsto \alpha x$ is a continuous transformation.

PROBLEMS.

1. Let V be a topological vector space and let $x \in V$ be a vector. Prove, that if $U \subset V$ is an open (closed) set, then the translation $U + x$ is an open (closed) set.
2. Let V be a topological vector space and let $\alpha \neq 0$ be a number. Prove, that if $U \subset V$ is an open (closed) set, then $\alpha U = \{\alpha x : x \in U\}$ is an open (closed) set in V.
3. Let V be a topological vector space and let $w_1 \neq w_2$ be two distinct points in V. Prove, that there exist neighborhoods W_1 of w_1 and W_2 of w_2, such that $W_1 \cap W_2 = \emptyset$.

(23.2) Definition. Let V be a topological vector space and let $u \in V$ be a point. A family of neighborhoods $\{U_i\}$ of u is called a *local base* at u, provided for every neighborhood U of u, there is a neighborhood $U_i \subset U$ from the local base.

To define a topological vector space, it suffices to define its local base at the origin (see Problem 1 below).

PROBLEMS.

1. Let V be a topological vector space and let $\{U_i\}$ be a local base at the origin. Then every non-empty open set $U \subset V$ is a union of translations $U_i + x_i$ of some sets U_i from the base and some $x_i \in V$.

2. A set A in a vector space V is called *balanced*, provided $\alpha x \in A$ for each $x \in A$ and each α, such that $|\alpha| \leq 1$. Prove, that every neighborhood of the origin in a topological vector space contains a balanced neighborhood of the origin.

(23.3) Definitions. Let V be a topological vector space, let $A \subset V$ be a set, and let $u \in V$ be a point. We say that u lies in the interior of A, provided there is a neighborhood $U \subset A$ of u. We say that u lies in the closure of A, provided for every neighborhood U of u, we have $U \cap A \neq \emptyset$. The set of all points in the interior of A is called the *interior* of A and denoted $\operatorname{int}(A)$. The set of all points in the *closure* of A is called the *closure* of A and denoted $\operatorname{cl}(A)$.

The following result is very similar to Lemma 10.2.

(23.4) Lemma. *Let V be a topological vector space and let $A \subset V$ be a convex set. Let $u_0 \in \operatorname{int}(A)$ and $u_1 \in A$. Then, for every $0 \leq \alpha < 1$ and $u_\alpha = (1-\alpha)u_0 + \alpha u_1$, we have $u_\alpha \in \operatorname{int}(A)$.*

Proof. Let $U_0 \subset A$ be a neighborhood of u_0. Let us consider a map $T : V \longrightarrow V$:

$$x \longmapsto \frac{1}{1-\alpha}(x - u_1) + u_1.$$

Then T is continuous and $T(u_\alpha) = u_0$. Therefore, the preimage $U_\alpha = T^{-1}(U_0)$ is a neighborhood of u_α. Let us show that $U_\alpha \subset A$. Indeed, if $x \in U_\alpha$, then $T(x) = y \in A$. Solving for x, we get $x = (1-\alpha)y + \alpha u_1$. Since A is convex, the proof follows. □

(23.5) Theorem. *Let V be a topological vector space and let $A \subset V$ be a convex set. Then $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ are convex sets.*

Proof. The proof that $\operatorname{int}(A)$ is a convex set follows the proof of Corollary 10.3. Let us prove that $\operatorname{cl}(A)$ is a convex set. Let $u_0, u_1 \in \operatorname{cl}(A)$ and let $u_\alpha = \alpha u_0 + (1-\alpha)u_1$ for $0 < \alpha < 1$. We know that every neighborhood of u_0 or u_1 intersects A and we must prove that every neighborhood of u_α intersects A. Let U be a neighborhood of u_α. Since the operation $(x, y) \longmapsto \alpha x + (1-\alpha)y$ is continuous, there must be a neighborhood U_0 of u_0 and U_1 of u_1, such that $\alpha x + (1-\alpha)y \in U$ for every $x \in U_0$ and every $y \in U_1$. Since $u_0 \in \operatorname{cl}(A)$, there is a point $x \in U_0 \cap A$ and since $u_1 \in \operatorname{cl}(A)$, there is a point $y \in U_1 \cap A$. Since A is convex, for $z = \alpha x + (1-\alpha)y$ we have $z \in A$. Therefore, $U \cap A \neq \emptyset$. □

PROBLEMS.

1°. Prove that $\operatorname{int}(A+x) = \operatorname{int}(A) + x$ and $\operatorname{cl}(A+x) = \operatorname{cl}(A) + x$ for each $A \subset V$ and each $x \in V$.

2°. Prove that $\operatorname{int}(\alpha A) = \alpha \operatorname{int}(A)$ and $\operatorname{cl}(\alpha A) = \alpha \operatorname{cl}(A)$ for each $A \subset V$ and each $\alpha \neq 0$.

(23.6) Theorem. *Let V be a topological vector space and let $H \subset V$ be an affine hyperplane. Then either* $\mathrm{cl}(H) = H$ *(that is, H is closed), or* $\mathrm{cl}(H) = V$ *(that is, H is dense in V).*

Proof. It suffices to prove the result for linear hyperplanes. Let $f : V \longrightarrow \mathbb{R}$ be a linear function such that

$$H = \{x \in V : f(x) = 0\}.$$

Suppose that H is not closed. Then there is a point $u \in V$, such that $u \notin H$, but every neighborhood of u contains a point from H. Let $\alpha = f(u) \neq 0$. Let us choose any $w \in V$. Suppose that $\beta = f(w)$. Let $\gamma = \beta/\alpha$ and let $h = w - \gamma u$. Then $f(h) = f(w) - \gamma f(u) = 0$, so $h \in H$. In other words, $w = h + \gamma u$. Since multiplication and addition are continuous operations, for every neighborhood W of w, there is a neighborhood U of u, such that $h + \gamma U \subset W$. Since $u \in \mathrm{cl}(H)$, there must be a point $h_1 \in U \cap H$. Then the point $h + \gamma h_1$ lies in W. Hence $W \cap H \neq \emptyset$ and $w \in \mathrm{cl}(H)$. Thus we proved that if H is not closed, then for each point $w \in V$, we have $w \in \mathrm{cl}(H)$. □

PROBLEMS.

1. Consider the space $C[0,1]$ of all continuous functions on the interval $[0,1]$. We make $C[0,1]$ a topological vector space by choosing a local base at the origin consisting of the sets

$$U_\epsilon = \Big\{ f \in C[0,1] : |f(\tau)| < \epsilon \quad \text{for all} \quad \tau \in [0,1] \Big\}$$

for any $\epsilon > 0$. Let $L \subset C[0,1]$ be a subspace consisting of all smooth functions (a function is called smooth if its differentiable at every point and the derivative is continuous). Prove that L has infinite codimension and that $\mathrm{cl}(L) = C[0,1]$.

2^*. Using Zorn's Lemma and Problem 1, show that there exists a dense hyperplane in $C[0,1]$.

3. Every hyperplane in $\mathbb{R}^d$ is closed.

4. Consider the space $C[0,1]$ of all continuous functions on the interval $[0,1]$. We make $C[0,1]$ a topological vector space by choosing a local base at the origin consisting of the sets

$$U_\epsilon = \Big\{ f \in C[0,1] : \int_0^1 \sqrt{|f(\tau)|}\, d\tau < \epsilon \Big\}$$

for any $\epsilon > 0$. Prove, that for every hyperplane $H \subset C[0,1]$, we have $\mathrm{cl}(H) = C[0,1]$.

As a general reference, see W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

Friday, February 20

24. Separation Theorems in Topological Vector Spaces

(24.1) Lemma. *Let V be a topological vector space and let $A \subset V$ be a convex open set. Then A is an algebraically open set.*

Proof. Let $L = \{u + \tau v : \tau \in \mathbb{R}\}$ be a straight line in V $(v \neq 0)$. We must prove that the intersection $A \cap L$ is an open interval in L, possibly empty. If $A \cap L$ is not empty, let $w \in A \cap L$, $w = u + \tau_0 v$. Since A is open, there is a neighborhood W of w, such that $W \subset A$. Since addition and scalar multiplication are continuous operations, there exists an $\epsilon > 0$, such that if $|\tau - \tau_0| < \epsilon$, then $u + \tau v \in W$. This implies that w is an interior point of the intersection $A \cap L$. Therefore, A is an algebraically open set. $\square$

PROBLEMS.

1. Let E be a vector space of all infinite sequences of real numbers $x = (\xi_1, \xi_2, \dots)$, such that for any $x \in E$ only finitely many terms ξ_i are non-zero (E can be considered as a space of polynomials in one variable). We make E a topological vector space by choosing the local base at the origin

$$U_\epsilon = \left\{ x : \sum_{i=1}^{\infty} \xi_i^2 < \epsilon \right\}$$

for $\epsilon > 0$. Let $A = \{x \in E : |\xi_k| < 1/k \text{ for } k = 1, \dots\}$. Prove, that A is convex and algebraically open, but not open.

2. Let E be a vector space. Let us declare a set $A \subset E$ open if and only if it is algebraically open. Prove, that this converts E into a topological vector space (cf. Problem 9, Section 9.3).

3. Let E be a topological vector space of Problem 2. Prove, that every hyperplane in E is closed.

4. Let E be a topological vector space and let $A, B \subset E$ be non-empty open sets. Suppose that a hyperplane $H \subset E$ separates A and B. Prove, that H strictly separates A and B and that H is closed.

(24.2) Theorem. *Let E be a topological vector space and let $A, B \subset E$ be convex sets. Suppose that $A \cap B = \emptyset$ and that* $\operatorname{int}(A) \neq \emptyset$. *Then there is a closed hyperplane $H \subset E$ that separates A and B.*

Proof. By Lemma 24.1, the algebraic interior of A is not empty. Therefore, there is a hyperplane $H \subset E$ that separates A and B. By Theorem 23.6, either H is closed or H is dense in E. In the latter case, H must have a non-empty intersection with any open subset, in particular, with the interior of A. Therefore, H has a non-empty intersection with the algebraic interior of A. Then H can not isolate A and thus can not separate A and B. The contradiction shows that H must be a closed hyperplane. $\square$

(24.3) Definition. A topological vector space E is called *locally convex*, provided for every point $u \in E$ and every neighborhood U of u, there is a convex neighborhood $W \subset U$ of u.

PROBLEMS.

1. Let E be a vector space. A function $\|\cdot\| : E \longrightarrow \mathbb{R}$ is called a *norm*, provided

$\|u\| \geq 0$ for each $u \in E$ and $\|u\| = 0$ only if $u = 0$,
$\|\alpha u\| = |\alpha|\|u\|$ for each $u \in E$ and each $\alpha \in \mathbb{R}$,
$\|u + v\| \leq \|u\| + \|v\|$ for each $u \in E$ and each $v \in E$.

Let us call a subset $U \subset E$ open, if for any point $u \in U$, there is an $r > 0$ such that the ball $B(u, r) = \{x : \|x - u\| < r\}$ is contained in U: $B(u, r) \subset U$. Prove, that this makes E a locally convex topological vector space. Such a space E is called a *normed space.*

2. Let E be a topological vector space and let $A \subset E$ be an open set. Prove that $\operatorname{conv}(A)$ is an open set.

3. Let us fix p, $0 < p < 1$. Let us consider the set $L^p[0, 1]$ of measurable functions such that $\int_0^1 |f(\tau)|^p \, d\tau < +\infty$ (we identify functions that coincide almost everywhere). Prove, that $L^p[0, 1]$ is a vector space. Furthermore, let us choose a local base at 0, consisting of the sets

$$U_\epsilon = \left\{ f \in L^p : \int_0^1 |f(\tau)|^p \, d\tau < \epsilon \right\}$$

for all $\epsilon > 0$. Prove, that this makes $L^p[0, 1]$ a topological vector space. Prove, that the only convex open sets in $L^p[0, 1]$ are $\emptyset$ and the whole space $L^p[0, 1]$.

(24.4) Theorem. *Let E be a locally convex topological vector space. Let $A \subset E$ be a closed convex set and let $u \notin A$ be a point. Then there exists a closed hyperplane H that strictly separates A and u.*

Proof. Since A is closed, the complement $U = E \setminus A$ is a neighborhood of u. Let $U_0 = U - u$ be a translation of U, so U_0 is a neighborhood of the origin. Because the transformation $(x, y) \longmapsto x - y$ is continuous, there are neighborhoods V_1 and V_2 of the origin, such that $x - y \in U_0$ for each $x \in V_1$ and each $y \in V_2$. Since E is locally convex, we can choose V_1 and V_2 to be convex.

Consider the Minkowski sums $A + V_2$ and $u + V_1$. The sets are convex (see Problem 1, Section 22.1) and open, since $A + V_2 = \bigcup_{x \in A} (V_2 + x)$ is a union of open sets. Furthermore, $(A + V_2) \cap (u + V_1) = \emptyset$. Indeed, if $(A + V_2) \cap (u + V_1) \neq \emptyset$, then for a point $a \in A$, a point $y \in V_2$, and a point $x \in V_1$, we will have $a + y = u + x$, that is $a = u + (x - y)$. Since $x - y \in U_0$, this would imply that A and U intersect, which is a contradiction. Theorem 24.2 implies that there is a closed hyperplane that separates $A + V_2$ and $u + V_1$. Since $A + V_1$ and $u + V_1$ are open, the hyperplane H must strictly separate $A + V_1$ and $u + V_1$. $\square$

PROBLEMS.

1°. Prove, that in a locally convex topological vector space E, any two distinct points $x \neq y$ can be strictly separated by a closed hyperplane.

2. Let E be a locally convex topological vector space and let $A, B \subset E$ be convex sets, such that A is closed, B is compact, and $A \cap B = \emptyset$. Prove, that A and B can be separated by a closed hyperplane.

Hint: It suffices to construct convex neighborhoods V_1 and V_2 of the origin, such that $(A + V_1) \cap (B + V_2) = \emptyset$ and then use Theorem 24.2. To achieve this goal, for every point $x \in B$, construct convex neighborhoods $V_1(x)$ and $V_2(x)$ of the origin, such that $(A + V_1(x)) \cap (x + V_2(x)) = \emptyset$. Since B is compact, there is a finite set of points $x_1, \ldots, x_n \in B$, such that the sets $x_i + V_2(x_i)$, $i = 1, \ldots, n$ cover B. Let $V_1 = \bigcap_{i=1}^{n} V_1(x)$ and $V_2 = \bigcap_{i=1}^{n} V_2(x)$.

(24.5) Theorem. *Let E be a topological vector space, let $f : E \longrightarrow \mathbb{R}$ be a linear function, and let $\alpha \in \mathbb{R}$ be a number. The hyperplane $H^\alpha = \big\{x \in E : f(x) = \alpha\big\}$ is closed if and only if f is continuous.*

Proof. If f is continuous, then $H = f^{-1}(\alpha)$ is a closed set, as the preimage of a closed set $\{\alpha\}$.

Let us prove that if H is closed, then f is continuous. Let us choose a $u \in E$. Suppose that $f(u) = \beta$. To prove that f is continuous in u, we have to prove that for any $\epsilon > 0$, there is a neighborhood U of u, such that $|f(x) - \beta| < \epsilon$ for any $x \in U$. Hence it suffices to prove that the set $\big\{x \in E : \beta - \epsilon < f(x) < \beta + \epsilon\big\}$ is open. This set can be represented as the intersection of two halfspaces $H_+^{\beta-\epsilon}$ and $H_-^{\beta+\epsilon}$, where

$$H_+^\gamma = \big\{x \in E : f(x) > \gamma\big\} \quad \text{and} \quad H_-^\gamma = \big\{x \in E : f(x) < \gamma\big\}$$

for a $\gamma \in \mathbb{R}$. Hence it suffices to prove that for any hyperplane $H^\gamma = \big\{x \in E : f(x) = \gamma\big\}$ both halfspaces H_+^γ and H_-^γ are open. Since H^γ is a translation of H^α, we conclude that H^γ is closed. Furthermore, since all hyperplanes H^γ are translations of each other, all halfspaces H_+^γ are translations of each other and all halfspaces H_-^γ are translations of each other, it suffices to prove that if some hyperplane $H = \big\{x : f(x) = \gamma\big\}$ is closed, then the halfspaces $H_+ = \big\{x : f(x) > \gamma\big\}$ and $H_- =\big\{x : f(x) < \gamma\big\}$ are open.

Suppose, for example, that $H_+ = \big\{x : f(x) > \gamma\big\}$ is not open. Then there is a point $u \in H_+$ such that every neighborhood U of u intersects $\overline{H_-} = \big\{x : f(x) \leq \gamma\big\}$. If we prove that U must intersect H, this would contradict the assumption that H is closed.

Finally, it boils down to proving that if $H \subset E$ is an affine hyperplane, and $u \in H_+$ is a point such that every neighborhood of u intersects $\overline{H_-}$, then every neighborhood of u intersects H. Without loss of generality (applying a translation, if necessary), we may assume that $u = \mathbf{0}$ is the origin. Since $0 \cdot \mathbf{0} = \mathbf{0}$ and the scalar multiplication is continuous, for each neighborhood U of the origin, there is a neighborhood V of the origin, and a number $\delta > 0$ with the property that $\tau x \in U$ for any $|\tau| < \delta$ and any $x \in V$. Let $W = \bigcup_{0<\tau<\delta} \tau V$. Then W is a neighborhood of the origin, $W \subset U$ and $[0, x] \subset W$ for any $x \in W$ (cf. Problem 2, Section 23.2). Since W is a neighborhood of the origin, it must intersect $\overline{H_-}$. If $x \in W \cap H_-$, then the interval $[0, x]$ intersects H (linear algebra) and W intersects H. Since $W \subset U$, U must intersect H as well. $\square$

PROBLEM.

1. Let E be a topological vector space and let $L \subset E$ be a subspace. Prove, that $\mathrm{cl}(L)$ is a subspace.

(24.6) The Dual Space

Let E be a topological vector space. The dual space E^*, as a vector space, consists of all continuous linear functions $f : E \longrightarrow \mathbb{R}$ with the addition: $g = f_1 + f_2$ provided $g(x) = f_1(x) + f_2(x)$ for all $x \in E$ and scalar multiplication: $g = \alpha f$ provided $g(x) = \alpha f(x)$ for all $x \in E$. There is a remarkable topology, called weak* topology, on E^*. Open sets in E^* are unions of elementary open sets of the type

$$U(x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n)$$

$$= \Big\{ f \in E^* : \alpha_i < f(x_i) < \beta_i \quad \text{for} \quad i = 1, \dots, n \Big\},$$

where $x_1, \dots, x_n \in E$ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n \in \mathbb{R}$.

PROBLEMS.

1. Prove, that E^* is a locally convex topological vector space.

2. Prove, that for any x, the function $\phi_x : E^* \longrightarrow \mathbb{R}$, $\phi(f) = f(x)$ is a continuous linear function on E^*.

3*. Prove, that every continuous linear function $\phi : E^* \longrightarrow \mathbb{R}$ has the form $\phi(f) = f(x)$ for some $x \in E$.

4. Prove, that for any two distinct points $f, g \in E^*$, there is a continuous linear function $\phi : E^* \longrightarrow \mathbb{R}$, such that $\phi(f) \neq \phi(g)$.

5*. (The Banach-Alaoglu Theorem). Let $U \subset E$ be a neighborhood of the origin. Let

$$K = \Big\{ f \in E^* : \ |f(x)| \leq 1 \quad \text{for} \quad \text{all } x \in U \Big\}.$$

Prove that K is compact.

Monday, February 23

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25. The Krein-Milman Theorem in Topological Vector Spaces

(25.1) Krein-Milman Theorem. *Let E be a locally convex topological vector space and let $K \subset E$ be a convex compact set. Then K is the closure of the convex hull of the set of its extreme points, $K = \operatorname{cl}\Big(\operatorname{conv}\big(\operatorname{ex}(K)\big)\Big)$.*

Proof. First, we establish that every non-empty compact set K in E has an extreme point. Let us call a non-empty convex compact subset $A \subset K$ *extreme* provided for any two points $x, y \in K$ and $z = (x+y)/2$, whenever $z \in A$, we must have $x, y \in A$. Clearly, K is an extreme set. Let $X \subset K$ be the smallest extreme subset (that is, not containing any extreme subset of K, other than itself). The existence of X is established via Zorn's Lemma or the axiom of choice. Let us prove that X is a point. Suppose that X contains two different points, say x_1 and x_2. Let us choose a closed hyperplane H that strictly separates x_1 and x_2, see Theorem 24.4. In other words, there is a continuous linear function $f : E \longrightarrow \mathbb{R}$ such that $f(x_1) < f(x_2)$. Let $\alpha = \min\big\{f(y) : y \in X\big\}$. Since X is compact (a closed subset of a compact) the minimum is attained. Let $Y = \big\{x \in X : f(x) = \alpha\big\}$ be a face of X. Clearly, Y is a convex compact subset of K and Y is an extreme set (cf. Theorem 11.2). On the other hand, Y does not contain x_2, so Y is strictly smaller than X. The contradiction shows that X must be a point, that is, an extreme point.

Now, we prove that K is the closure of the convex hull of the set of its extreme points. Let $A = \mathrm{cl}\Big(\mathrm{conv}\big(\mathrm{ex}(K)\big)\Big)$. Then A is a closed convex subset of K. We must prove that $A = K$. Suppose, there is a point $u \in K \setminus A$. Let us choose a closed hyperplane H that strictly separates u from A. In other words, there is a continuous linear function $f : E \longrightarrow \mathbb{R}$ such that $f(x) > f(u)$ for any $x \in A$. Let $\alpha = \min\big\{f(x) : x \in K\big\}$ (the minimum is attained since f is continuous and K is compact) and let $F = \big\{x \in K : f(x) = \alpha\big\}$ be the corresponding face of K. So, F is a compact convex set and, as we proved, must contain an extreme point v, which will be an extreme point of K (cf. Theorem 11.2). On the other hand, $v \notin A$, which is a contradiction. $\square$

PROBLEMS.

1. Let $C[0,1]$ be the vector space of all continuous functions on the interval $[0,1]$. Let $B = \big\{f : |f(\tau)| \leq 1 \text{ for all } x \in [0,1]\big\}$. Find the extreme points of B.

2. Let A be the following subset of $C[0,1]$:

$$A = \left\{f : \int_0^1 f(\tau)\, d\tau = 0 \quad \text{and} \quad |f(\tau)| \leq 1 \text{ for all } \tau \in [0,1]\right\}.$$

Prove, that A has no extreme points.

3. Let E be a topological vector space and let A be a closed convex set, such that $\mathrm{int}(A) \neq \emptyset$. Prove, that for each point $u \in \partial A$, there is a closed support hyperplane at u, that is, a closed hyperplane H that contains u and isolates A.

4. Let $A \subset E$ be a convex set in a vector space E and let F be a face of A. Prove, that F is an extreme set of A, that is, for any two $x, y \in A$, whenever $z = (x+y)/2 \in F$, we must have $x, y \in F$.

5. Let $L \subset C[0,1]$ be the vector space of all smooth functions on the interval $[0,1]$. We make L a topological vector space by using the $C[0,1]$ metric: $\mathrm{dist}(f,g) = \max\big\{|f(x) - g(x)| : x \in [0,1]\big\}$. Let $A = \big\{f : f'(1/2) > 0\big\}$ and $B = \big\{f : f'(1/2) < 0\big\}$ be subsets of L.

Prove, that A and B are convex, non-intersecting, algebraically open subsets of L, so there is a hyperplane $H \subset L$ strictly separating them. Prove that A and B are dense in L. Deduce, that there is no closed hyperplane $H \subset L$ strictly separating A and B.

26. The Spaces L^1 and L^∞

Let $L^1[0,1]$ be a vector space of all integrable functions on the interval $[0,1]$, that is, Lebesgue measurable functions f such that

$$\int_0^1 |f(\tau)|\, d\tau < +\infty.$$

As usual, we don't distinguish functions that differ on a set of measure 0. We make $L^1[0,1]$ a topological vector space by introducing the metric

$$\mathrm{dist}(f,g) = \int_0^1 |f(\tau) - g(\tau)|\, d\tau.$$

The local base at the origin consists of the neighborhoods

$$U_\epsilon = \left\{ f : \int_0^1 |f(\tau)| \, d\tau < \epsilon \right\}$$

for $\epsilon > 0$.

Let $L^\infty[0,1]$ be the set of all Lebesgue measurable functions f on the interval $[0,1]$ such that $|f(\tau)| \leq C$ for some constant C and for almost all (all, except from a set of zero measure) $\tau \in [0,1]$. As usual, we don't distinguish functions that differ on a set of measure 0. It is known, that every continuous linear function $\phi : L^1[0,1] \longrightarrow \mathbb{R}$ has the form

$$\phi(f) = \int_0^1 f(\tau) g(\tau) \, d\tau$$

for some $g \in L^\infty[0,1]$. We make L^∞ a topological vector space, by endowing it with the weak* topology.

(26.1) Important fact

Let $B \subset L^\infty[0,1]$ be the set

$$B = \left\{ f : 0 \leq f(\tau) \leq 1 \quad \text{for almost all} \quad \tau \in [0,1] \right\}.$$

The set B is compact in the weak* topology (see, for example, W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

PROBLEMS.

1. Prove, that if f is an extreme point of B, then $f(\tau) \in \{0,1\}$ for almost all $\tau \in [0,1]$.

2*. Prove, that the set B is compact in the weak* topology.

Hint: Consider a neighborhood U of the origin in $L^1[0,1]$:

$$U = \left\{ g : \int_0^1 |g(\tau)| \, d\tau < 1 \right\}.$$

Let $K = \left\{ f \in L^\infty : \int_0^1 (fg) \, d\tau \leq 1 \text{ for all } g \in U \right\}$. Prove, that K consists of the functions $f \in L^\infty[0,1]$ such that $|f(\tau)| \leq 1$ for almost all $\tau \in [0,1]$. Use the Banach-Alaoglu Theorem, see Problem 5, Section 24.6, to claim that K is compact. Show, that $B \subset K$ is a closed subset.

3. Prove that $\operatorname{conv}\bigl(\operatorname{ex}(B)\bigr)$ is not closed.

4. Deduce from (26.1), that the set

$$K = \left\{ f \in L^\infty[0,1] : |f(\tau)| \leq 1 \quad \text{for almost all} \quad \tau \in [0,1] \right\}$$

is weak* compact.

(26.2) Proposition. *Let us fix m functions $a_1(\tau), \ldots, a_m(\tau) \in L^1[0,1]$ and m numbers $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Let $B \subset L^\infty$ be the set*

$$B = \Big\{ f : 0 \leq f(\tau) \leq 1 \quad \textit{for almost all} \quad \tau \in [0,1] \Big\},$$

and let

$$A = \Big\{ f \in B : \int_0^1 a_i(\tau) f(\tau) = \alpha_i \quad \textit{for } i = 1, \ldots, m \Big\}.$$

Then A is a convex compact (in weak topology) subset of L^∞. If u is an extreme point of A, then $u(\tau) \in \{0,1\}$ for almost all τ.*

Proof. It is obvious that A is convex. Furthermore, A is a closed subset of B and so A is compact as follows from 26.1.

Let u be an extreme point of A. Suppose that measure$\big\{\tau : u(\tau) \notin \{0,1\}\big\} > 0$. Then for some $\delta > 0$ and $X = \big\{\tau : \delta \leq |u(\tau)| \leq 1 - \delta\big\}$, the measure of X is positive. Let us find $m+1$ pairwise disjoint subsets $X_1, \ldots, X_{m+1}$ of X of positive measure and let χ_j be the indicator function of X_i:

$$\chi_j(\tau) = \begin{cases} 1 & \text{if } \tau \in X_j \\ 0 & \text{if } \tau \notin X_j. \end{cases}$$

Let $\epsilon_1, \ldots, \epsilon_{m+1}$ be real numbers and let $v = \epsilon_1 \chi_1 + \ldots + \epsilon_{m+1} \chi_m$. Furthermore, let $u_+ = u + v$ and let $u_- = u - v$. Then $u = (u_+ + u_-)/2$. Obviously, if $|\epsilon_1| < \delta, \ldots, |\epsilon_m| < \delta$, we will have $0 \leq u_+(\tau), u_-(\tau) \leq 1$ for almost all τ. Now we show that we can choose a non-zero v so that $u_+, u_- \in A$. Indeed, it suffices to choose $\epsilon_1, \ldots, \epsilon_{m+1}$ in such a way, that the system of m homogeneous equations

$$\sum_{j=1}^{m+1} \epsilon_j \int_0^1 a_i(\tau) \chi_j(\tau) \, d\tau = 0, \quad i = 1, \ldots, m$$

in $m+1$ variables $\epsilon_1, \ldots, \epsilon_{m+1}$ is satisfied. Since the number of variables exceeds the number of equations, there must be a solution $\epsilon_1, \ldots, \epsilon_{m+1}$, where not all ϵ's are zero. Scaling, if necessary, we make $\epsilon_1, \ldots, \epsilon_{m+1} < \delta$. Then $u_-, u_+ \in A$. That contradicts the assumption that u is an extreme point of A. $\square$

PROBLEM.

1. Let us consider the spaces $L^\infty([0,1], \mathbb{R}^d)$ and $L^1([0,1], \mathbb{R}^d)$ of vector-valued functions $f : [0,1] \longrightarrow \mathbb{R}^d$. Let us fix a polytope $P \subset \mathbb{R}^d$, m functions $a_1, \ldots, a_m \in L^1([0,1], \mathbb{R}^d)$, and m numbers $\alpha_1, \ldots, \alpha_m$. Let

$$A = \Big\{ f \in L^\infty([0,1], \mathbb{R}^d) :$$

$$f(\tau) \in P \quad \text{and} \quad \int_0^1 \langle c_i, f(\tau) \rangle \, d\tau = \alpha_i \quad \text{for } i = 1, \ldots, m \Big\}.$$

Prove, that A is a convex set and that if f is an extreme point of A, then $f(\tau)$ is a vertex of P for almost all τ.

27. Example: Problems of Linear Optimal Control

(27.1) Solving linear systems of differential equations with control

Let $A(\tau)$ and $B(\tau)$ be $n \times n$ matrices, where $\tau \in [0,1]$ is a real parameter. We assume that $A(\tau)$ and $B(\tau)$ are smooth functions of τ. We consider a system of linear differential equations

$$\frac{d}{d\tau}\mathbf{x}(\tau) = A(\tau)\mathbf{x}(\tau) + B(\tau)\mathbf{u}(\tau), \tag{27.1.1}$$

where $\mathbf{x}(\tau) = (x_1(\tau), \dots, x_n(\tau))$ and $\mathbf{u}(\tau) = (u_1(\tau), \dots, u_n(\tau))$ are vectors from $\mathbb{R}^n$. The function $\mathbf{x}(\tau)$ is a *solution* of the system (27.1.1), whereas $\mathbf{u}(\tau)$ is a *control.* If the control is chosen, under mild assumptions the solution $\mathbf{x}$ is determined by the *initial condition*:

$$\mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x}_0 \in \mathbb{R}^n$ is a given vector. Under mild assumptions, the solution $\mathbf{x}$ can be found as follows. Let $X(\tau)$ be an $n \times n$ matrix, which is a solution to the matrix system

$$\frac{d}{d\tau}X(\tau) = A(\tau)X(\tau) \quad \text{with the intial condition} \quad X(0) = I,$$

where I is an $n \times n$ matrix. Then

$$\mathbf{x}(\tau) = X(\tau)\Big(\mathbf{x}_0 + \int_0^\tau X^{-1}(t)B(t)\mathbf{u}(t)\ dt\Big). \tag{27.1.2}$$

In general, we want to choose the control $\mathbf{u} \in L^\infty\big([0,1], \mathbb{R}^n\big)$. Then, if $X(\tau)$ and $B(\tau)$ are continuous, the integral is well-defined. Let us consider $\mathbf{x}(\tau)$ as a solution to the original system of differential equations, even though it may be, say, non-differentiable at some points. In particular, if we are to choose the control $\mathbf{u}(\tau)$ in such a way that some *terminal* condition $\mathbf{x}(1) = \mathbf{x}_1$, is satisfied, we get the following integral constraint:

$$\int_0^1 X^{-1}(\tau)B(\tau)\mathbf{u}(\tau)\ d\tau = X^{-1}(1)\mathbf{x}_1 - \mathbf{x}_0. \tag{27.1.3}$$

PROBLEM.

1°. Check formula (27.1.2).

(27.2) Example. A problem of linear control. Let us fix smooth real functions $a_0(\tau), a_1(\tau), b(\tau), c_0(\tau), c_1(\tau)$, and $d(\tau)$: $\tau \in [0,1]$. Consider a differential equation for a function $x(\tau)$

$$\ddot{x}(\tau) = a_0(\tau)x(\tau) + a_1(\tau)\dot{x}(\tau) + b(\tau)u(\tau),$$

where $u(\tau)$ is a control, which we would like to choose in such a way, that the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

and the terminal conditions

$$x(1) = x_1, \quad \dot{x}(1) = v_1$$

are satisfied and the functional

$$\int_0^1 \Big(c_0(\tau)x(\tau) + c_1(\tau)\dot{x}(\tau) + d(\tau)u(\tau)\Big)\, d\tau$$

is minimized. In addition, we have a restriction for $u(\tau)$:

$$0 \leq u(\tau) \leq 1 \quad \text{for all} \quad \tau \in [0, 1].$$

This is a typical problem of linear optimal control. The differential equations describes the movement of an "object", thus relating the coordinate $x(\tau)$, the velocity $\dot{x}(\tau)$ and the acceleration $\ddot{x}(\tau)$. The control u is a "force" we may apply. We want to transfer the "object" from the initial position (coordinate, velocity) to the final position (coordinate, velocity), so that the total cost we pay for the coordinate ("gravity"), for the velocity ("friction"), and for the control ("fuel") is minimized.

Our aim is to show that this optimization problem can be considered as a *linear programming problem* of the type:

Find $u \in L^\infty[0, 1]$, minimizing $\displaystyle\int_0^1 g(\tau)u(\tau)\, d\tau$

Subject to:

$$\int_0^1 f_1(\tau)u(\tau)\, d\tau = \alpha_1, \quad \int_0^1 f_2(\tau)u(\tau)\, d\tau = \alpha_2, \quad \text{and} \quad 0 \leq u(\tau) \leq 1,$$

where the functions $f_1, f_2, g \in L^1[0, 1]$ and the numbers α_1, α_2 can be computed explicitly.

To see this, let us write the differential equation in the form (27.1.1) by introducing a new variable $v(\tau) = \dot{x}(\tau)$. We get

$$\frac{d}{d\tau}x(\tau) = v(\tau) \quad \text{and} \quad \frac{d}{d\tau}v(\tau) = a_0(\tau)x(\tau) + a_1(\tau)v(\tau) + b(\tau)u(\tau)$$

with the conditions

$$\begin{pmatrix} x(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(1) \\ v(1) \end{pmatrix} = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}.$$

Thus, for vector

$$\mathbf{x}(\tau) = \begin{pmatrix} x(\tau) \\ v(\tau) \end{pmatrix} \quad \text{and} \quad \mathbf{u}(\tau) = \begin{pmatrix} 0 \\ u(\tau) \end{pmatrix}$$

we have

$$A(\tau) = \begin{pmatrix} 0 & 1 \\ a_0(\tau) & a_1(\tau) \end{pmatrix} \quad \text{and} \quad B(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & b(\tau) \end{pmatrix}.$$

Now, the formulas for f_1, f_2 and α_1, α_2 are obtained from (27.1.3). The formula for g can be obtained from (27.1.2).

(27.3) Corollary. *If the problem of optimal linear control is feasible, there exists an optimal solution* $u \in L^\infty[0,1]$ *such that* $u(\tau) \in \{0,1\}$ *for almost all* $\tau \in [0,1]$. *If the optimal solution* $u \in L^\infty[0,1]$ *is unique, then* $u(\tau) \in \{0,1\}$ *for almost all* $\tau \in [0,1]$.

Proof. Follows by Proposition 11.2 and Proposition 26.2.

PROBLEMS.

1°. Consider the equation

$$\ddot{x}(\tau) = a_0(\tau)x(\tau) + a_1(\tau)\dot{x}(\tau) + b(\tau)u(\tau)$$

with the initial conditions

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 0.$$

Show, that the solution $x(\tau)$ depends linearly on the control $u(\tau)$: if $x_1(\tau)$ is the solution for $u_1(\tau)$, $x_2(\tau)$ is the solution for $u_2(\tau)$, then $x(\tau) = \alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)$ is the solution for $u(\tau) = \alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)$, where α_1 and α_2 are real numbers.

2°. Let $y(\tau)$ be a solution to the equation

$$\ddot{y}(\tau) = a_0(\tau)y(\tau) + a_1(\tau)\dot{y}(\tau)$$

with some initial conditions

$$y(0) = x_0 \quad \text{and} \quad \dot{y}(0) = v_0$$

and let $z(\tau)$ be a solution of the equation

$$\ddot{z}(\tau) = a_0(\tau)z(\tau) + a_1(\tau)\dot{z}(\tau) + b(\tau)u(\tau)$$

with zero initial conditions. Prove that $x(\tau) = y(\tau) + z(\tau)$ is a solution to the equation

$$\ddot{x}(\tau) = a_0(\tau)x(\tau) + a_1(\tau)\dot{x}(\tau) + b(\tau)u(\tau)$$

with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Problems 1 and 2 provide some intuition why problems of linear optimal control can be written as linear programs of optimizing $\int_0^1 g(\tau)u(\tau)\,d\tau$ subject to integral constraints $\int_0^1 f_i(\tau)u(\tau)\,d\tau = \alpha_i$ and the "domain" constraint $0 \leq u(\tau) \leq 1$. First, Problem 2 allows us to reduce (in "nice" cases) the general case to the case of zero initial conditions. Next, if the initial conditions $x(0)$, $\dot{x}(0)$ are zero, the terminal values $x(1)$ and $\dot{x}(1)$ are linear functions of the control u. A "reasonable" linear function (say, weak* continuous) has the form $\int_0^1 f(\tau)u(\tau)\,d\tau$ for some $f \in L^1[0,1]$. Thus fixing the terminal values of the solution x amounts to fixing some integral constraints on u.

Friday, February 27

Solutions to Some Problems from the First Set

Monday, March 9

28. The "Dual Carathéodory" Theorem

(28.1) Lemma. *Let V be a vector space and let $K \subset V$ be a convex set, such that for every straight line $L \subset V$, the intersection $K \cap L$ is a closed bounded interval, possibly empty or a point. Let $H \subset V$ be an affine hyperplane. Then, every extreme point of $K \cap H$ is a convex combination of at most two extreme points of K.*

Proof. Let u be an extreme point of $K \cap H$. If u is an extreme point of K, the proof follows. Otherwise, there are two distinct points $u_+, u_- \in K$, such that $u = (u_+ + u_-)/2$. Let L be the straight line passing through u_+ and u_-. The intersection $L \cap K = [u_1, u_2]$ is a closed interval, containing u in its interior. We claim that u_1 and u_2 are extreme points of K. Suppose, for example, that u_1 is not an extreme point of K. Then there are two distinct points $v_+, v_- \in K$, such that $u_1 = (v_+ + v_-)/2$. Clearly, $v_+ \notin L$ and $v_- \notin L$. Consider the 2-dimensional plane A passing through the points v_+, v_-, and u_2 and the triangle $\Delta = \operatorname{conv}(v_+, v_-, u_2)$ in the plane A. The point u is an interior point of the triangle. Since H is a hyperplane, the intersection $A \cap H$ is a straight line, passing through u. Since u is an interior point of Δ, the intersection $\Delta \cap H$ is an interval, containing u as its interior point. Since K is convex, $\Delta \cap H \subset K \cap H$, so the intersection $K \cap H$ contains an interval, containing u as its interior point, which contradicts the assumption that u is an extreme point of $K \cap H$.

The contradiction shows that u_1 and u_2 are extreme points of $K \cap H$. Since u is a convex combination of u_1 and u_2, the proof follows. □

(28.2) Theorem. *Let V be a vector space and let $K \subset V$ be a convex set, such that for every straight line $L \subset V$, the intersection $K \cap L$ is a closed bounded interval, possibly empty or a point. Let $H_1, \ldots, H_m \subset V$ be a finite set of affine hyperplanes in V. Then, every extreme point of the intersection $K \cap H_1 \cap \ldots \cap H_m$ is a convex combination of at most $m+1$ extreme points of K.*

Proof. First, we prove that every extreme point of the intersection $K \cap H_1 \cap \ldots \cap H_m$ is a convex combination of extreme points of K. Let us define $K_0 = K$, $K_1 = K \cap H_1$, $K_i = K_{i-1} \cap H_i$, $i = 1, \ldots, m$. Applying Lemma 28.1, we conclude that every extreme point of K_i is a convex combination of at most 2 extreme points of K_{i-1}. Therefore, every extreme point of $K_m = K \cap H_1 \cap \ldots \cap H_m$ is a convex combination of at most 2^m extreme points of $K_0 = K$.

Let u be an extreme point of $K \cap H_1 \cap \ldots \cap H_m$. Let us choose a representation

$$u = \alpha_1 u_1 + \ldots + \alpha_n u_n, \quad \text{where} \quad \alpha_1 + \ldots + \alpha_n = 1, \quad \alpha_i \geq 0 \quad \text{and} \quad u_i \in \operatorname{ex}(K)$$

for $i = 1, \ldots, n$ with the smallest possible n (we know that we can choose $n \leq 2^m$). Clearly, $\alpha_i > 0$ for all $i = 1, \ldots, n$. Furthermore, the points $u_1, \ldots, u_n$ are affinely independent, since otherwise we could have reduced n as in the proof of Carathéodory's Theorem (see Theorem 5.1). Therefore, u is an interior point of the $(n-1)$-dimensional simplex $\Delta = \operatorname{conv}(u_1, \ldots, u_n) \subset K$ (cf. Problem 1, Section 10.3) If $n > m+1$, then the intersection $\Delta \cap H_1 \ldots \cap H_m$ contains an interval containing u as its interior point, which contradicts the assumption that u is an extreme point of $K \cap H_1 \cap \ldots \cap H_m$. Hence $n \leq m+1$ and the proof follows. □

PROBLEMS.

1. Show by example, that, in general, the constant $m+1$ in Theorem 28.2 can not be made smaller.

2°. Let $K = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be a ball. Prove, that every extreme point of $K \cap H_1 \cap \ldots \cap H_m$ is an extreme point of K.

3. Let $\mathbb{R}^d : d = n(n+1)/2$ be the space of $n \times n$ symmetric matrices and let $K = \{X \succeq 0 \text{ and } \operatorname{trace}(X) = 1\}$. Prove, that every extreme point of $K \cap H_1 \cap \ldots \cap H_m$ is a convex combination of not more than $[(\sqrt{8m+1}-1)/2]+1$ extreme points of K.

Hint: Use Proposition 20.1.

4. Let $\mathbb{R}^d : d = n^2$ be the space of all $n \times n$ matrices and let $K \subset \mathbb{R}^d$ be the Birkhoff Polytope (the polytope of doubly stochastic matrices, see Section 13). Let $L \subset \mathbb{R}^d$ be the subspace of all $n \times n$ symmetric matrices. Prove, that every extreme point of $K \cap L$ is a convex combination of not more than 2 extreme points of K.

5. A generalization of Problem 4. Let G be a finite group of linear transformations of $\mathbb{R}^d$. In other words, if $g, h \in G$ are linear transformations of $\mathbb{R}^d$, then $g^{-1}h \in G$ is a linear transformation of $\mathbb{R}^d$. Let $K \subset \mathbb{R}^d$ be a convex compact set, which is G-invariant: $g(x) \in K$ for all $g \in G$ and all $x \in K$. Let $L = \{x \in \mathbb{R}^d : g(x) = x \text{ for all } g \in G\}$ be the subspace of G-invariant vectors (check, that L is indeed a subspace). Prove, that every extreme point of $K \cap L$ is a convex combination of not more than $|G|$ (the cardinality of G) extreme points of K.

29. The Space of Measures

(29.1) The space of continuous functions and its dual

Let $C[0,1]$ be the space of all real-valued continuous functions on the interval $[0,1]$. There is a natural metric on $C[0,1]$: $\operatorname{dist}(f,g) = \max\{|f(\tau) - g(\tau)| : \tau \in [0,1]\}$. This makes $C[0,1]$ a topological vectors space. The local base at the origin consists of the sets

$$U_\epsilon = \Big\{ f \in C[0,1] : |f(\tau)| < \epsilon \quad \text{for all} \quad \tau \in [0,1] \Big\}$$

for $\epsilon > 0$. Let $V[0,1]$ be the space of all continuous linear functionals $\phi : C[0,1] \longrightarrow \mathbb{R}$. The space $V[0,1]$ is often called the *space of signed Borel measures* on [0,1]. The reason for such a name is that every continuous linear functional $\phi : C[0,1] \longrightarrow \mathbb{R}$ can be represented in the form

$$\phi(f) = \int_0^1 f \, d\mu,$$

where μ is a signed Borel measure. For example, μ may be a regular measure, like $\mu = \tau^2 \ d\tau$,

$$\phi(f) = \int_0^1 f(\tau)\tau^2 \ d\tau$$

or a δ-measure, like $\mu = \delta_{1/2}$:

$$\phi(f) = f(1/2).$$

We make $V[0,1]$ a topological vector space by introducing the weak* topology. In other words, every continuous linear functional on $V[0,1]$ has the form $\mu \longmapsto \int_0^1 f \ d\mu$ for some fixed $f \in C[0,1]$.

PROBLEMS.

1. A linear functional $\phi : C[0,1] \longrightarrow \mathbb{R}$ is called *positive*, provided $\phi(f) \geq 0$ for any $f \in C[0,1]$ such that $f(\tau) \geq 0$ for any $\tau \in [0,1]$. Prove, that every positive linear functional is continuous.

Hint: Use that $f = \big((|f| + f) - (|f| - f)\big)/2$.

2. Prove, that every continuous linear functional $\phi : C[0,1] \longrightarrow \mathbb{R}$ can be represented in the form $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are positive functionals.

Hint: Let $f \in C[0,1]$ be a non-negative function, $f(\tau) \geq 0$ for all $\tau \in [0,1]$. Let $\phi_1(f) = \sup\big\{\phi(g) : 0 \leq g(\tau) \leq f(\tau) \text{ for all } \tau \in [0,1]\big\}$.

3. Let us define the *norm* of a function $f \in C[0,1]$ by $\|f\| = \max\{|f(\tau)| : 0 \leq \tau \leq 1\}$ and the norm of a linear functional $\phi : C[0,1] \longrightarrow \mathbb{R}$ by $\phi(f) = \sup\big\{|\phi(f)| : f \in C[0,1], \ \|f\| = 1\big\}$. Find the norms of the linear functionals:

$$\phi(f) = \int_0^1 f(\tau) \ d\tau, \quad \phi(f) = \int_0^1 \tau f(\tau) \ d\tau, \quad \text{and} \quad \phi_\epsilon(f) = \frac{f(1/2 + \epsilon) - f(1/2 - \epsilon)}{2\epsilon},$$

where $\epsilon > 0$ is a parameter.

(29.2) Definitions. Let $\Delta \subset V[0,1]$ be the set of all linear functionals $\phi : C[0,1] \longrightarrow \mathbb{R}$ that are positive (that is, $\phi(f) \geq 0$ for all f such that $f(\tau) \geq 0$ for all $\tau \in [0,1]$) and such that $\phi(\mathbf{1}) = 1$, where $\mathbf{1}$ is the function that is identically 1 on $[0,1]$. The set Δ is called the set of all *Borel probability measures on* $[0,1]$. Let us fix a $\tau \in [0,1]$. The linear functional $\delta_\tau : C[0,1] \longrightarrow \mathbb{R}$, where $\delta_\tau(f) = f(\tau)$, is called the *delta measure*.

PROBLEMS.

1°. Prove, that Δ is a convex set.

2°. Prove, that $\delta_\tau \in \Delta$ for any $\tau \in [0,1]$.

3°. Prove, that $\delta_\tau(fg) = \delta_\tau(f)\delta_\tau(g)$ for any $\tau \in [0,1]$ and any two functions $f, g \in C[0,1]$.

4. Let $\phi : C[0,1] \longrightarrow \mathbb{R}$ be a linear functional such that $\phi(fg) = \phi(f)\phi(g)$ for any two functions $f, g \in C[0,1]$. Prove, that either $\phi = 0$ or $\phi = \delta_\tau$ for some $\tau \in [0,1]$.

5. Prove, that δ_τ is an extreme point of Δ for any $\tau \in [0,1]$.

6*. Prove, that if ϕ is an extreme point of Δ, then $\phi = \delta_\tau$ for some $\tau \in [0,1]$.

(29.3) Important fact: The set Δ is compact in the weak* topology (see, for example, W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

PROBLEM.

1*. Prove that Δ is weak* compact.

Hint: Use the Banach-Alaoglu Theorem, see Problem 5 of Section 24.6.

Wednesday, March 11

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30. The "Simplex" of Probability Measures

(30.1) Proposition. *The extreme points of the set Δ of Borel probability measures on the interval $[0,1]$ are the delta-measures δ_τ for $\tau \in [0,1]$.*

Proof. Let us choose a $\tau^* \in [0,1]$. First, we prove that $\delta_* = \delta_{\tau^*}$ is indeed an extreme point of Δ. Clearly, $\delta_* \in \Delta$ (see Problem 2, Section 29.2). Suppose that $\delta_* = (\phi_1 + \phi_2)/2$ for some $\phi_1, \phi_2 \in \Delta$. Let f be any continuous function such that $f(\tau^*) = 1$ and $f(\tau) \leq 1$ for all $\tau \in [0,1]$. Since ϕ_1 and ϕ_2 are positive linear functionals, we have

$$\phi_i(f) = \phi_i\big(\mathbf{1} - (\mathbf{1} - f)\big) = \phi_i(\mathbf{1}) - \phi_i(\mathbf{1} - f) \leq 1.$$

On the other hand $\delta_*(f) = 1$. Therefore, we must have $\phi_1(f) = \phi_2(f) = 1$. By linearity, it follows then that $\phi_1(f) = \phi_2(f) = f(\tau^*)$ for any function $f \in C[0,1]$ such that $f(\tau^*) \geq f(\tau)$ for all $\tau \in [0,1]$. Now, for any $f \in C[0,1]$, we have $f = g_1 - g_2$, where $g_1(\tau) = \min\{f(\tau^*), f(\tau)\}$ and $g_2(\tau) = \min\{0, f(\tau^*) - f(\tau)\}$. Hence g_1 and g_2 are continuous functions on the interval $[0,1]$ attaining their maximum value at τ^*. Therefore, $\phi_i(g_j) = g_j(\tau^*)$ for $i,j = 1,2$. By linearity, we conclude that $\phi_i(f) = \phi_i(g_1) - \phi_i(g_2) = f(\tau^*)$. Therefore, $\phi_1 = \phi_2 = \delta_*$, so δ_* is an extreme point.

Suppose that ϕ is an extreme point of Δ. First, we establish that $\phi(fg) = \phi(f)\phi(g)$ for any two functions $f, g \in C[0,1]$ and then we deduce that $\phi = \delta_\tau$ for some $\tau \in [0,1]$. Let us fix a function $h \in C[0,1]$. Then $\psi : C[0,1] \longrightarrow \mathbb{R}$ defined by $\psi(f) = \phi(hf)$ is a continuous linear functional on $C[0,1]$. Let us choose h such that $0 < h(\tau) < 1$ for all $\tau \in [0,1]$. Since ϕ is positive, we have $1 > \phi(h) > 0$. We define $\psi_1, \psi_2 : C[0,1] \longrightarrow \mathbb{R}$ by

$$\psi_1(f) = \frac{\phi(hf)}{\phi(h)} \quad \text{and} \quad \psi_2(f) = \frac{\phi\big((\mathbf{1} - h)f\big)}{\phi(\mathbf{1} - h)}.$$

It is easy to see that $\psi_i \in \Delta$. We can write ϕ as a convex combination

$$\phi = \phi(h)\psi_1 + \phi(\mathbf{1} - h)\psi_2.$$

Since ϕ is an extreme point of Δ, we must have $\psi_1 = \psi_2 = \phi$. In particular, $\phi(fh) = \phi(h)\phi(f)$ for any $f \in C[0,1]$ and any h such that $0 < h(\tau) < 1$ for each $\tau \in [0,1]$. By linearity, it follows that $\phi(fg) = \phi(f)\phi(g)$ for any two $f, g \in C[0,1]$.

Let $H = \big\{f \in C[0,1] : \phi(f) = 0\big\}$ be the kernel of ϕ. Hence H is a (closed) hyperplane in $C[0,1]$. We observe that for every $f \in H$, there is a $\tau \in [0,1]$ such that $f(\tau) = 0$. Indeed, if $f \in H$ is a function which is nowhere 0, then $1/f$ is a continuous function and we would have had $1 = \phi(\mathbf{1}) = \phi(f(1/f)) = \phi(f)\phi(1/f) = 0$. Next, we observe that any set of finitely many functions $f_1, \ldots, f_m \in H$ has a common zero τ^*: $f_1(\tau^*) = \ldots = f_m(\tau^*) = 0$. Otherwise, the function $f = f_1^2 + \ldots + f_m^2$ is everywhere positive and $f \in H$ since $\phi(f) = \phi(f_1^2) + \ldots + \phi(f_m^2) = \phi^2(f_1) + \ldots + \phi^2(f_m) = 0$. But we already proved that every function $f \in H$ must have a zero. Finally, we conclude that there is a point $\tau^* \in [0,1]$ such that $f(\tau^*) = 0$ for any $f \in H$. Indeed, for any $f \in C[0,1]$, the set $X_f = \big\{\tau \in [0,1] : f(\tau) = 0\big\}$ is a closed set and any finite intersection $X_{f_1} \cap \ldots \cap X_{f_m}$ for $f_1, \ldots, f_m \in H$ is non-empty. Since $[0,1]$ is a compact interval, the intersection $\bigcap_{f \in H} X_f$ is non-empty. Now we see that H is a subset of the kernel of δ_{τ^*}, which implies that $\phi = \alpha\delta_{\tau^*}$ for some $\alpha \in \mathbb{R}$. Hence $\phi = \delta_{\tau^*}$ and the proof follows. □

Remark. The set Δ can be considered as an (uncountably) infinite-dimensional analogue of the standard d-dimensional simplex $\Delta_d \subset \mathbb{R}^{d+1}$:

$$\Delta_d = \Big\{(\gamma_0, \ldots, \gamma_d) : \gamma_0 + \ldots + \gamma_d = 1 \quad \text{and} \quad \gamma_i \geq 0 \quad \text{for} \quad i = 0, \ldots, d\Big\}.$$

The analogy is based on the following observation: the "discretization" of the space $C[0,1]$ is the space $C[\tau_0, \ldots, \tau_d]$ of all continuous functions on a discrete set of $d+1$ points $0 \leq \tau_0 < \tau_1 < \ldots < \tau_d \leq 1$. The space $C[\tau_0, \ldots, \tau_d]$ can be identified with $\mathbb{R}^{d+1}$, the space of all $(d+1)$-tuples $(\xi_0, \ldots, \xi_d)$. The "discretization" of $V[0,1]$ is the space of all continuous linear functionals $\phi : C[\tau_0, \ldots, \tau_d] \longrightarrow \mathbb{R}$, which is identified with the space $\mathbb{R}^{d+1}$ of all $(d+1)$-tuples $(\gamma_0, \ldots, \gamma_d)$: $\phi(\xi_0, \ldots, \xi_d) = \gamma_0\xi_0 + \ldots + \gamma_d\xi_d$. Then the "discretization" of $\Delta \subset V[0,1]$ is the simplex Δ_d.

Notation. For $\mu \in V[0,1]$ and $f \in C[0,1]$, instead of $\mu(f)$ we often write $\displaystyle\int_0^1 f \, d\mu$.

PROBLEM.

1. Let $O \subset V[0,1]$ be the set of functions ϕ such that $|\phi(f)| \leq 1$ for any $f \in C[0,1]$ such that $|f(\tau)| \leq 1$ for each $\tau \in [0,1]$. Prove, that the extreme points of O are δ_τ and $-\delta_\tau$ for $\tau \in [0,1]$.

(30.2) Application: optimization over the set of probability measures

Let μ be a Borel probability measure on the interval $[0,1]$ and let $f_1, \ldots, f_m \in C[0,1]$ be continuous functions on the interval $[0,1]$. Suppose that the measure μ is not known, but we know the expectations $\displaystyle\int_0^1 f_i \, d\mu = a_i$, $i = 1, \ldots, m$ (and the functions $f_1, \ldots, f_m$ themselves). We want to estimate the expectation $a_0 = \displaystyle\int_0^1 f_0 \, d\mu$ of yet another (known) function $f_0 \in C[0,1]$. Quite often, we have $f_i(\tau) = \tau^i$, $i = 1, \ldots, m$, so a_i are *moments* of an unknown probability distribution, and f_0 is some other function of interest. We can interpret μ as a point in $\Delta \subset V[0,1]$. Then $a_0^- \leq a_0 \leq a_0^+$, where a_0^- (resp., a_0^+) is the solution to the following *linear programming problem*:

Find $a_0^- = \inf \displaystyle\int_0^1 f_0 \, d\mu$ (resp. $a_0^+ = \sup \displaystyle\int_0^1 f_0 \, d\mu$),

where $\mu \in \Delta$ is a probability measure subject to the constraints $\displaystyle\int_0^1 f_i \, d\mu = a_i$, $i = 1, \ldots, m$.

(30.3) Proposition. *Suppose that the set of feasible measures, that is the set of $\mu \in \Delta$, such that*

$$\int_0^1 f_i \, d\mu = a_i : \quad i = 1, \dots, m$$

is non-empty. Then, in the optimization problem above, there is an optimal solution μ, which is a convex combination of at most $m+1$ delta-measures:

$$\mu = \sum_{i=1}^{m+1} \lambda_i \delta_{\tau_i},$$

where $\sum_{i=1}^{m+1} \lambda_i = 1$, $\lambda_i \geq 0$, and $\tau_i \in [0,1]$ for $i = 1, \dots, m$.

Proof. The set of feasible measures μ is the intersection $\Delta \cap H_1 \cap \ldots \cap H_m$, where H_i is the affine hyperplane

$$H_i = \Big\{ \phi \in V[0,1] : \int_0^1 f_i \, d\phi = a_i \Big\}.$$

The set Δ is convex and weak* compact (see (29.3)). Furthermore, the hyperplanes H_i are weak* closed. Therefore, the intersection $\Delta \cap H_1 \cap \ldots \cap H_m$ is convex and weak* compact. Therefore, the (continuous) linear functional $\mu \longmapsto \int_0^1 f_0 \, d\mu$ attains its maximum and minimum on the set of feasible probability measures. Furthermore, there is an optimal solution μ which is an extreme point of the intersection $\Delta \cap H_1 \cap \ldots \cap H_m$ (see Theorem 11.2). By Theorem 28.2, each extreme point of the intersection is a convex combination of at most $m+1$ extreme points of Δ. By Proposition 30.1, the extreme points of Δ are the measures δ_τ: $\tau \in [0,1]$. The proof now follows. □

By Proposition 30.3, to find a_0^- (resp. a_0^+) it suffices to solve the following finite-dimensional optimization problem:

Find $a_0^- = \min \sum_{i=1}^{m+1} \lambda_i f_0(\tau_i)$, resp. $a_0^+ = \max \sum_{i=1}^{m+1} \lambda_i f_0(\tau_i)$,

where $\sum_{i=1}^{m+1} \lambda_i = 1$, $\lambda_i \geq 0$, $\tau_i \in [0,1]$ and

$\sum_{i=1}^{m+1} \lambda_i f_j(\tau_i) = a_j$ for $j = 1, \dots, m$.

PROBLEMS.

1. Let us fix an $a \in [0,1]$. Find a probability measure μ on $[0,1]$ such that $\int_0^1 \tau \, d\mu = a$ and $\int_0^1 (\tau - a)^2 \, d\mu$ is maximized.

2. Let μ be a probability measure on $[0,1]$. Prove, that for the *variance*

$$D(\mu) = \int_0^1 \tau^2 \, d\mu - \Big(\int_0^1 \tau \, d\mu \Big)^2$$

one has $0 \leq D(\mu) \leq 1/4$.

Hint: Use Problem 1.

3. Let us consider the set S_{d+1} of all $(d+1)$-tuples $(\xi_0, \ldots, \xi_d)$, where $\xi_i = \int_0^1 \tau^i \, d\mu$ for some $\mu \in \Delta$ and $i = 0, \ldots, d$. Then S_{d+1} is the section of the moment cone M_{d+1} by the hyperplane $\xi_0 = 1$ (see Section 16).

Friday, March 13

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31. Duality

(31.1) Definition. Let E and F be vector spaces. A non-degenerate bilinear form $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ is called a *pairing* of E and F. In other words, for each $e \in E$ and each $f \in F$, a number $\langle e, f \rangle \in \mathbb{R}$ is defined, in such a way that

$$\langle \alpha_1 e_1 + \alpha_2 e_2, f \rangle = \alpha_1 \langle e_1, f \rangle + \alpha_2 \langle e_2, f \rangle \quad \text{and} \quad \langle e, \alpha_1 f_1 + \alpha_2 f_2 \rangle = \alpha_1 \langle e, f_1 \rangle + \alpha_2 \langle e, f_2 \rangle$$

for any $e, e_1, e_2 \in E$, for any $f, f_1, f_2 \in F$, and for any $\alpha_1, \alpha_2 \in \mathbb{R}$ (that is, $\langle\rangle$ is a bilinear form).

$$\text{If} \quad \langle e, f \rangle = 0 \quad \text{for any} \quad e \in E, \quad \text{then} \quad f = 0;$$

and

$$\text{If} \quad \langle e, f \rangle = 0 \quad \text{for any} \quad f \in F, \quad \text{then} \quad e = 0$$

(that is, $\langle\rangle$ is non-degenerate).

(31.2) Examples. Our main examples are:

(31.2.1) Let $E = \mathbb{R}^d$, let $F = \mathbb{R}^d$, and let

$$\langle x, y \rangle = \sum_{i=1}^{d} \xi_i \eta_i, \quad \text{where} \quad x = (\xi_1, \ldots, \xi_d) \quad \text{and} \quad y = (\eta_1, \ldots, \eta_d).$$

(31.2.2) Let $d = n(n+1)/2$, let $E = F = \mathbb{R}^d$ be the space of $n \times n$ symmetric matrices, and let

$$\langle A, B \rangle = \operatorname{trace}(AB) = \sum_{i,j=1}^{n} a_{ij} b_{ij}, \quad \text{where} \quad A = (a_{ij}) \quad \text{and} \quad B = (b_{ij}).$$

(31.2.3) Let $E = L^1[0,1]$, let $F = L^\infty[0,1]$, and let

$$\langle f, g \rangle = \int_0^1 f(\tau) g(\tau) \, d\tau, \quad \text{where} \quad f \in L^1[0,1] \quad \text{and} \quad g \in L^\infty[0,1].$$

(31.2.4) Let $E = C[0,1]$, let $F = V[0,1]$, and let

$$\langle f, \mu \rangle = \int_0^1 f \, d\mu, \quad \text{where} \quad f \in C[0,1] \quad \text{and} \quad \mu \in V[0,1].$$

PROBLEM.

1°. Check, that (31.2.1)–(31.2.4) are indeed pairings.

2. Prove, that there is no pairing with $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$, where $n \neq m$.

3°. Let $\mathbb{R}_\infty$ be the vector space of all sequences $x = (\xi_i : i = 1, \ldots)$ such that all but finitely many ξ_i's are 0 and let $\mathbb{R}^\infty$ be the space of all sequences $x = (\xi_i : i = 1, \ldots)$. The addition is "pointwise": $z = x + y$, where $z = (\zeta_i)$, $x = (\xi_i)$ and $y = (\eta_i)$, provided $\zeta_i = \xi_i + \eta_i$ for $i = 1, \ldots$. Similarly, the scalar multiplication is "pointwise": $y = \alpha x$, where $\alpha \in \mathbb{R}$, $y = (\eta_i)$, and $x = (\xi_i)$, provided $\eta_i = \alpha \xi_i$ for $i = 1, \ldots$. Let

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \eta_i, \quad \text{where} \quad x = (\xi_i) \in \mathbb{R}_\infty \quad \text{and} \quad y = (\eta_i) \in \mathbb{R}^\infty.$$

Note, that the sum is well-defined, since it contains only finitely many non-zero terms. Prove, that $\langle\rangle$ is a pairing.

This example is of interest, since the spaces have different "dimensions" (the space $\mathbb{R}_\infty$ has a countable basis, whereas $\mathbb{R}^\infty$ does not have a countable basis).

(31.3) Definition. Let $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ be a pairing. We make E and F topological vector spaces in the following way. A *basic open set* $U \subset E$ is a set of the type

$$U = \Big\{ e \in E : \ \alpha_i < \langle e, f_i \rangle < \beta_i : \ i = 1, \ldots, m \Big\},$$

where $f_1, \ldots, f_m \in F$ are some vectors and $\alpha_1, \beta_1, \ldots, \alpha_m, \beta_m \in \mathbb{R}$ are some numbers. An open set in E is the union of some basic open sets. Similarly, a *basic open set* $W \subset F$ is a set of the type

$$W = \Big\{ f \in F : \ \alpha_i < \langle f, e_i \rangle < \beta_i : \ i = 1, \ldots, m \Big\},$$

where $e_1, \ldots, e_m \in E$ are some vectors and $\alpha_1, \beta_1, \ldots, \alpha_m, \beta_m \in \mathbb{R}$ are some numbers. An open set in F is a union of some basic open sets.

PROBLEMS.

1°. Check, that E and F are indeed locally convex topological vector spaces.

2. a) Prove, that in Examples (31.2.1) and (31.2.2), the topology of the pairing coincides with the standard topology in $\mathbb{R}^d$.

b) Prove, that in Examples (31.2.2) and (31.2.3), the topology of the pairing in the space F is the weak* topology.

3. Let V be a vector space and let $\phi, \phi_1, \ldots, \phi_m : V \longrightarrow \mathbb{R}$ be linear functionals such that $\phi(x) = 0$ whenever $\phi_1(x) = \ldots = \phi_m(x) = 0$. Prove, that $\phi = \alpha_1 \phi_1 + \ldots + \alpha_m \phi_m$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

Hint: Consider a linear transformation $\Phi : V \longrightarrow \mathbb{R}^m$:
$\Phi(x) = \big(\phi_1(x), \ldots, \phi_m(x)\big)$. Prove, that there is a linear functional $\psi : \mathbb{R}^m \longrightarrow \mathbb{R}$, such that $\phi(x) = \psi\big(\Phi(x)\big)$ for all $x \in V$. Now use that $\psi(y)$ is a linear combination of the coordinates of y.

If we have a pairing $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ and we make E and F topological vector spaces as prescribed by Definition 31.3, we say that $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ is a *duality.*

(31.4) Theorem. *Let $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ be a duality. Then, for every $f \in F$, the function $\phi(e) = \langle e, f\rangle$ is a continuous linear function on E and for every $e \in E$, the function $\psi(f) = \langle e, f\rangle$ is a continuous linear function on F. Furthermore, every continuous linear function $\phi : E \longrightarrow \mathbb{R}$ has the form $\phi(e) = \langle e, f\rangle$ for some fixed $f \in F$, and every continuous linear function $\psi : F \longrightarrow \mathbb{R}$ has the form $\psi(f) = \langle e, f\rangle$ for some fixed $e \in E$.*

Proof. Let us prove that $\phi(e) = \langle e, f\rangle$ is a continuous linear function on F. Clearly, ϕ is linear. Let us choose an $e_0 \in E$ and an $\epsilon > 0$. Let $\alpha = \langle e_0, f\rangle$ and let

$$U = \{e \in E : \ \alpha - \epsilon < \langle e, f\rangle < \alpha + \epsilon\}.$$

Then U is a (basic) neighborhood of e_0 and for every $e \in U$ we have $|\phi(e) - \phi(e_0)| < \epsilon$. Hence ϕ is continuous in e_0. Since e_0 was arbitrary, ϕ is continuous.

Let $\phi : E \longrightarrow \mathbb{R}$ be a continuous linear function. In particular, ϕ is continuous at $x = 0$. This implies, in particular, that there exists a neighborhood U of $x = 0$ such that $|\phi(x)| < 1$ for any $x \in U$. We can choose U to be a basic open set

$$U = \{x : \ \alpha_i < \langle x, f_i\rangle < \beta_i : i = 1, \ldots , m\},$$

where $\alpha_i < 0 < \beta_i$ and $f_i \in F$ for $i = 1, \ldots , m$. Since ϕ is linear, for any $\epsilon > 0$, as long as

$$x \in \epsilon U = \{x : \ \epsilon\alpha_i < \langle x, f_i\rangle < \epsilon\beta_i : i = 1, \ldots , m\},$$

we have $|\phi(x)| < \epsilon$. Let

$$L = \{x : \ \langle x, f_i\rangle = 0 : i = 1, \ldots , m\}$$

be a linear subspace in F. We have $L \subset \epsilon U$ for any $\epsilon > 0$, therefore, $|\phi(x)| < \epsilon$ for all $x \in L$ and any $\epsilon > 0$. Hence $\phi(x) = 0$ for any $x \in L$, or, in other words, $\phi(x) = 0$ provided $\phi_i(x) = \langle x, f_i\rangle = 0$ for $i = 1, \ldots , m$. Problem 3, Section 31.3 implies that ϕ can be written as a linear combination $\phi = \alpha_1\phi_1 + \ldots + \alpha_m\phi_m$. In other words, $\phi(e) = \langle e, f\rangle$, where $f = \alpha_1 f_1 + \ldots + \alpha_m f_m$. □

PROBLEM.

1°. Prove, that for every continuous linear function $\phi : E \longrightarrow \mathbb{R}$, a vector $f \in F$ such that $\phi(e) = \langle e, f\rangle$ for all $e \in E$ is unique.

Hint: Use that the pairing is non-degenerate.

(31.5) Definition. Let $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$ be dualities. Let $T : E_1 \longrightarrow E_2$ be a continuous linear transformation. We define the *dual transformation* $T^* : F_2 \longrightarrow F_1$ as follows: For $f_2 \in F_2$, the function

$$\phi(e_1) = \langle T(e_1), f_2\rangle_2$$

is a continuous linear function $\phi : E_1 \longrightarrow \mathbb{R}$. Therefore, by Theorem 31.4 and Problem 1, Section 31.4, it can be written as

$$\phi(e_1) = \langle e_1, f_1\rangle$$

for a unique $f_1 \in F_1$. We let $T^*(f_2) = f_1$.

Note, that

$$\langle T(e_1), f_2\rangle_2 = \langle e_1, T^*(f_2)\rangle_1$$

for any $e \in E_1$ and any $f_2 \in F_2$.

PROBLEMS.

1. Let $\langle\rangle_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be the dualities of Example 31.2.1. Let us choose the standard orthonormal bases in $\mathbb{R}^n$ and $\mathbb{R}^m$. The matrix of the linear transformation $T^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is the transpose of the matrix of the transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

2. Let $\langle\rangle_1 : L^1[0,1] \times L^\infty \longrightarrow \mathbb{R}$ be the duality of Example 31.2.3 and let $\langle\rangle_2 : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be the duality of Example 31.2.1. Consider a linear transformation $T : L^1[0,1] \longrightarrow \mathbb{R}^m$:

$$T(f) = \Big(\int_0^1 f(\tau)g_1(\tau)\ d\tau, \dots , \int_0^1 f(\tau)g_m(\tau)\ d\tau\Big),$$

where $g_1, \dots , g_m \in L^\infty$ are some fixed function. Prove, that $T^* : \mathbb{R}^m \longrightarrow L^\infty[0,1]$ has the form

$$T(x) = \xi_1 g_1 + \dots + \xi_m g_m, \quad \text{where} \quad x = (\xi_1, \dots , \xi_m).$$

3. Let $\langle\rangle_1 : E \times F \longrightarrow \mathbb{R}$ be a duality and let $\langle\rangle_2 : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be the duality of Example 31.2.1. Let $T : E_1 \longrightarrow \mathbb{R}^m$ be a continuous linear transformation. Prove, that

$$T(e) = \big(\langle e, f_1\rangle_1, \dots , \langle e, f_m\rangle_1\big)$$

for some $f_1, \dots , f_m \in F$ and all $e \in E$. Prove, that the dual transformation T^* has the form:

$$T(x) = \xi_1 f_1 + \dots + \xi_m f_m, \quad \text{where} \quad x = (\xi_1, \dots , \xi_m).$$

4. Let $T : E_1 \longrightarrow E_2$ be a continuous linear transformation. Prove, that $T^* : F_2 \longrightarrow F_1$ is a continuous linear transformation.

5. Prove, that $(T^*)^* = T$.

Monday, March 16

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32. Polars

(32.1) Definition. Let $\langle\rangle : E \times F \longrightarrow \mathbb{R}$ be a duality. Let $A \subset E$ be a set. The set $A^\circ \subset F$,

$$A^\circ = \Big\{f \in F : \langle e, f\rangle \leq 1 \quad \text{for any} \quad e \in A\Big\}$$

is called the *polar* of A. Similarly, if $A \subset F$ is a set, the set $A^\circ \subset E$,

$$A^\circ = \Big\{e \in E : \langle e, f\rangle \leq 1 \quad \text{for any} \quad f \in A\Big\}$$

is called the polar of A.

PROBLEMS.

Some important properties of polars.

1°. Prove, that if $A = \{0\}$ is the origin in E, then $A^\circ = F$. Similarly, if $A = \{0\}$ is the origin in F, then $A^\circ = E$. Furthermore, $E^\circ = \{0\}$ is the origin in F and $F^\circ = \{0\}$ is the origin in E.

2°. Prove, that $\left(\bigcup_{i \in I} A_i\right)^\circ = \bigcap_{i \in I} A_i^\circ$, where $\{A_i : i \in I\}$ is a family of subsets.

3°. Prove, that $A^\circ \subset B^\circ$ if $B \subset A$.

4°. Let A be a set and let $\alpha \neq 0$ be a number. Let $\alpha A = \big\{\alpha x : x \in A\big\}$. Prove, that $(\alpha A)^\circ = \alpha^{-1} A^\circ$.

5°. Prove, that $A \subset (A^\circ)^\circ$.

6. Let $L \subset E$ be a (linear) subspace. Prove, that

$$L^\circ = \Big\{f \in F : \langle e, f \rangle = 0 \quad \text{for any} \quad e \in E\Big\}.$$

Similarly, for subspaces $L \subset F$.

7. Let $K \subset E$ be a cone, that is for any $x \in K$ and any $\alpha \geq 0$, $\alpha x \in K$. Prove that,

$$K^\circ = \Big\{f \in F : \langle e, f \rangle \leq 0 \quad \text{for any} \quad e \in E\Big\}.$$

Similarly, for cones $K \subset F$.

8. Let $A = \operatorname{conv}(e_1, \ldots, e_m)$, where $e_i \in E$ for $i = 1, \ldots, m$. Prove, that

$$A^\circ = \Big\{f \in F : \langle f, e_i \rangle \leq 1, \ i = 1, \ldots, m\Big\}.$$

(32.2) Bipolar Theorem. *The polar A° is a closed convex set containing the origin. If A is a closed convex set containing the origin, then $(A^\circ)^\circ = A$.*

Proof. Suppose that $A \subset E$. Then

$$A^\circ = \bigcap_{e \in A} \{f \in F : \langle f, e \rangle \leq 1\}$$

is the intersection of closed convex halfspaces containing the origin. Therefore, A° is closed, convex, and contains the origin. Suppose that $A \subset E$ is closed, convex and contains the origin. From Problem 5, Section 32.1, we have $A \subset (A^\circ)^\circ$. It remains to show that $(A^\circ)^\circ \subset A$. Suppose, there is a point $u \in (A^\circ)^\circ$, such that $u \notin A$. Since A is closed and convex, by Theorem 24.4, there is a closed hyperplane H which strictly separates A and x. In other words, there is a continuous linear function $\phi : E \longrightarrow \mathbb{R}$ and a number $\alpha \in \mathbb{R}$, such that $\phi(u) > \alpha$ and $\phi(e) < \alpha$ for all $e \in A$. Since $0 \in A$, we conclude, that $\alpha > 0$. By Theorem 31.4, $\phi(e) = \langle e, f \rangle$ for some $f \in F$. Let $v = \alpha^{-1} f$. Then $\langle e, v \rangle < 1$ for all $e \in A$ and, therefore, $v \in A^\circ$. On the other hand, $\langle u, v \rangle > 1$, that is $u \notin (A^\circ)^\circ$, which is a contradiction. Hence $(A^\circ)^\circ \subset A$, and the proof follows. □

PROBLEMS.

1°. Prove, that $((A^\circ)^\circ)^\circ = A^\circ$ for every non-empty set A.

2. Prove, that for any non-empty set $(A^\circ)^\circ = \operatorname{cl}\big(\operatorname{conv}(A, 0)\big)$.

3. Let $A = \big\{e \in E : \langle e, f_i \rangle \leq 1 : i = 1, \ldots, m\big\}$. Prove, that $A = \operatorname{conv}(0, f_1, \ldots, f_m)$.

4. Let $\langle\rangle : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $d = n(n+1)/2$ be the duality of Example 31.2.2. Let $K = \{X : X \succeq 0\}$ be the cone of positive semidefinite matrices. Prove, that $K^\circ = \{-X : X \succeq 0\}$ is the cone of negative semidefinite matrices.

5. Let $\langle\rangle : L^1[0,1] \times L^\infty[0,1] \longrightarrow \mathbb{R}$ be the duality of Example 31.2.3. Let $K \subset L^1[0,1]$, $K = \{f \in L^1[0,1] : f(\tau) \geq 0 \text{ for almost all } \tau \in [0,1]\}$. Prove, that $K^\circ = \{g \in L^\infty[0,1] : g(\tau) \leq 0 \text{ for almost all } \tau \in [0,1]\}$.

6. Let $K_1, K_2 \subset E$ be cones and let $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ be their Minkowski's sum. Prove, that $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$.

7. Suppose, that $K_1, K_2 \subset E$ are closed convex cones. Prove, that $(K_1 \cap K_2)^\circ = \operatorname{cl}(K_1^\circ + K_2^\circ)$.

8. Construct an example of closed convex cones $K_1, K_2 \subset \mathbb{R}^3$, such that $K_1 + K_2$ is not closed.

(32.3) Polarity in the Euclidean Space

If $A \subset \mathbb{R}^d$, then $A^\circ \subset \mathbb{R}^d$ is defined as $A^\circ = \{x \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for each } y \in A\}$, cf. Example 31.2.1 and Definition 32.1.

PROBLEMS.

1°. Let $B = \{x : \|x\| \leq 1\}$ be the unit ball. Prove, that $B^\circ = B$.

2. Suppose that $A^\circ = A$ for $A \subset \mathbb{R}^d$. Prove, that A is the unit ball.

3. Let $A = \{(\xi_1, \ldots, \xi_d) : -1 \leq \xi_i \leq 1\colon i = 1, \ldots, d\}$ and let $B = \{(\xi_1, \ldots, \xi_d) : |\xi_1| + \ldots + |\xi_d| \leq 1\}$. Prove, that $A^\circ = B$ and that $B^\circ = A$.

4. Let $p > 1, q > 1$ be such that $1/p + 1/q = 1$. Let $A_p = \{(\xi_1, \ldots, \xi_d) : |\xi_1|^p + \ldots + |\xi_d|^p \leq 1\}$. Prove, that $(A_p)^\circ = A_q$.

5. Let $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a non-degenerate linear transformation and let $A \subset \mathbb{R}^d$ be a set. Prove, that $(T(A))^\circ = (T^*)^{-1}(A^\circ)$.

6*. For a subset $A \subset \mathbb{R}^d$, let $\chi_A : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function, such that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be closed convex cones and let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_1\chi_{K_1} + \ldots + \alpha_n\chi_{K_n} = 0$. Prove, that $\alpha_1\chi_{K_1^\circ} + \ldots + \alpha_n\chi_{K_n^\circ} = 0$.

7*. Let $K \subset \mathbb{R}^d$ be a convex compact set, which is centrally symmetric with respect to the origin (that is, $-K = K$). Prove, that K° is a centrally symmetric convex compact set, and that

$$\operatorname{vol}(K) \cdot \operatorname{vol}(K^\circ) \leq \operatorname{vol}(B) \cdot \operatorname{vol}(B^\circ),$$

where "vol" is the volume and $B \subset \mathbb{R}^d$ is the unit ball (Santalo's inequality).

8**. Let $K \subset \mathbb{R}^d$ be a centrally symmetric convex compact set, and let $I \subset \mathbb{R}^d$ be the cube: $I = \{(\xi_1, \ldots, \xi_d) :\ -1 \leq \xi_i \leq 1,\ i = 1, \ldots, d\}$. Is it true that

$$\operatorname{vol}(I) \cdot \operatorname{vol}(I^\circ) \leq \operatorname{vol}(K) \cdot \operatorname{vol}(K^\circ)?$$

(Mahler's Conjecture).

9*. Let $A, B \subset \mathbb{R}^d$ be convex compact set. For $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, let $\alpha A + \beta B = \{\alpha x + \beta y : x \in A, y \in B\}$. Prove, that

$$\ln \operatorname{vol}(\alpha A + \beta B) \geq \alpha \ln \operatorname{vol}(A) + \beta \ln \operatorname{vol}(B)$$

(Brunn-Minkowski's inequality) and that

$$\ln \operatorname{vol}\big((\alpha A+\beta B)^\circ\big) \le \alpha \ln \operatorname{vol}(A^\circ) + \beta \ln \operatorname{vol}(B^\circ)$$

(Firey's inequality). For Problems 7–9, see R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge Univ. Press, 1993.

10°. Let $L \subset \mathbb{R}^d$ be a subspace. Prove, that L° is the orthogonal complement to L.

11. Let us define the *projective plane* $\mathbb{RP}^2$ as follows: the points of $\mathbb{RP}^2$ are the straight lines in $\mathbb{R}^3$, passing through the origin. Straight lines in $\mathbb{RP}^2$ are the planes in $\mathbb{R}^3$ passing through the origin. As usual, a line in $\mathbb{RP}^2$ (a plane in $\mathbb{R}^3$) consists of points of $\mathbb{RP}^2$ (lines in $\mathbb{R}^3$).

Prove, that for every two distinct points A and B of $\mathbb{RP}^2$, there is a unique straight line $c \subset \mathbb{RP}^2$ that contains A and B.

Prove, that every two distinct lines $a, b \in \mathbb{RP}^2$ intersect in a unique point C.

There is the *polarity correspondence* between points and lines of $\mathbb{RP}^2$: a point $A \in \mathbb{RP}^2$ corresponds to a line $a \subset \mathbb{RP}^2$ if and only if A, considered as a line in $\mathbb{R}^3$, is orthogonal to a, considered as a plane in $\mathbb{R}^3$. Prove, that a point C is the intersection of two distinct lines a and b if and only if the distinct points A and B lie on the straight line C.

The Euclidean plane $\mathbb{R}^2$ can be embedded in $\mathbb{RP}^2$ in the following way. Let us identify $\mathbb{R}^2$ with a two dimensional affine plane in $\mathbb{R}^3$, not passing through the origin. A point $x \in \mathbb{R}^2$ is identified with the straight line $X \subset \mathbb{R}^3$, passing through x and the origin. Hence, x is identified with a point of $\mathbb{RP}^2$. Prove, that the straight lines in $\mathbb{R}^2$ are identified with straight lines in $\mathbb{RP}^2$. Describe the points of $\mathbb{RP}^2$ that are not identified with any point of $\mathbb{R}^2$ and the straight lines in $\mathbb{RP}^2$ that are not identified with any straight line of $\mathbb{R}^2$.

Wednesday, March 18

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33. Polarity and Polyhedra

Recall that a polyhedron in $\mathbb{R}^d$ is the set of solutions to a finite system of linear inequalities (see Definition 12.1) and that a polytope in $\mathbb{R}^d$ is the convex hull of finitely many points (see Definition 4.3).

(33.1) Weyl-Minkowski Theorem. *A bounded polyhedron is a polytope. A polytope is a polyhedron.*

Proof. We have already proved that a bounded polyhedron is a polytope (see Corollary 12.3). Let us prove that a polytope is a polyhedron. Let $P = \operatorname{conv}(v_1, \dots, v_m)$, $P \subset \mathbb{R}^d$ be a polytope. Without loss of generality, we assume that $0 \in \operatorname{int}(P)$ (otherwise, we can choose the smallest affine subspace containing P and choose the origin in the interior of P, see Theorem 10.4). Suppose that $B_r \subset P \subset B_R$, where $B_l = \big\{x \in \mathbb{R}^d : \|x\| \le l\big\}$ is the ball of the radius l centered at the origin. Then $(B_R)^\circ \subset P^\circ \subset (B_r)^\circ$ (see Problem 3, Section 32.1). Since $(B_l)^\circ = B_{1/l}$ (cf. Problem 4, Section 32.1 and Problem 1, Section 32.3), we have $B_{1/R} \subset P^\circ \subset B_{1/r}$. Hence the polar P° is bounded. Furthermore,

$$P^\circ = \big\{x \in \mathbb{R}^d : \langle v_i, x\rangle \le 1, \ i = 1, \dots, m\big\}$$

is a polyhedron. Therefore, P° is a bounded polyhedron, and hence is a polytope by the first part of the theorem:

$$P^\circ = \operatorname{conv}(u_1, \dots , u_n)$$

for some $u_1, \dots , u_n \in \mathbb{R}^d$. Therefore, $(P^\circ)^\circ$ is a polyhedron:

$$(P^\circ)^\circ = \big\{x \in \mathbb{R}^d : \langle x, u_i\rangle \le 1, \ i = 1, \dots , n\big\}.$$

By Theorem 32.2, $(P^\circ)^\circ = P$, and the proof follows. $\square$

PROBLEMS.

1°. Check, that $(B_r)^\circ = B_{1/r}$, where $B_l \subset \mathbb{R}^d$ is the ball of the radius l centered at the origin.

2. Prove, that if $P \subset \mathbb{R}^d$ is a polyhedron, then P° is a polyhedron in $\mathbb{R}^d$.

3. Let $P = \operatorname{conv}(v_1, \dots , v_m)$, $v_i \in \mathbb{R}^d$ be a polytope and let $F \subset P$ be a face of P. Prove, that $F = \operatorname{conv}(v_i : v_i \in F)$.

In what follows, we agree that $\emptyset$ and P are faces of a polytope $P \subset \mathbb{R}^d$. We also agree that $\dim\ \emptyset = -1$.

(33.2) Theorem. *Let $P \subset \mathbb{R}^d$ be a polytope, containing the origin in its interior and let $Q = P^\circ$. Then Q is a polytope. For a face F of P, let*

$$\widehat{F} = \Big\{x \in Q : \langle x, y\rangle = 1 \quad \textit{for each} \quad y \in F\Big\}$$

(we agree that $\widehat{\emptyset} = Q$ and that $\widehat{P} = \emptyset$). Then

(1) *The set $\widehat{F}$ is a face of Q and*

$$\dim\ F + \dim\ \widehat{F} = d - 1;$$

(2) *If $F \subset G$ are faces of P then $\widehat{G} \subset \widehat{F}$;*

(3) *We have $\widehat{\widehat{F}} = F$, or, in other words,*

$$F = \Big\{y \in P : \langle x, y\rangle = 1 \quad \textit{for each} \quad x \in \widehat{F}\Big\}.$$

Proof. The proof that Q is a polytope is incorporated in the proof of Theorem 33.1. Suppose that $P = \operatorname{conv}(v_1, \dots , v_m)$. Let $I \subset [1 : m]$ be the set of indices of the points $v_i \in F$. We have $F = \operatorname{conv}(v_i : i \in I)$, see Problem 3, Section 33.1. Let $v = \dfrac{1}{|I|}\sum_{i\in I} v_i$, $v \in F$. We claim that

$$\widehat{F} = \big\{x \in Q : \langle x, v\rangle = 1\big\}. \tag{33.2.1}$$

Indeed, $v \in F$, so $\langle x, v\rangle = 1$ for any $x \in \widehat{F}$. On the other hand, for any $x \in Q$ and any v_i, we have $\langle x, v_i\rangle \le 1$. Therefore, if $x \in Q$ and $\langle x, v\rangle = 1$, then $\langle x, v_i\rangle = 1$ for $i \in I$ and hence $x \in \widehat{F}$. By (33.2.1), we conclude that $\widehat{F}$ is a face of Q. Let us prove that $\dim\ F + \dim\ \widehat{F} = d-1$. Since F is a face, there is a vector $c \in \mathbb{R}^d$ and a number α, such that

$$\langle c, v_i \rangle = \alpha \quad \text{for} \quad i \in I \qquad \text{and} \qquad \langle c, v_i \rangle < \alpha \quad \text{for} \quad i \notin I.$$

Since $0 \in \operatorname{int}(P)$, $\alpha > 0$. Scaling, if necessary, we can assume that $\alpha = 1$. Then $c \in \widehat{F}$. Suppose that $\dim\ F = k$. For any $x \in \widehat{F}$, we must have $\langle x, v_i \rangle = 1$ for $i \in I$, or, in other words, $\langle x - c, v_i \rangle = 0$ for $i \in I$. Since $\dim \operatorname{Span}(F) = k+1$, we get that $\dim\ \widehat{F} \leq d - k - 1$. On the other hand, if $y \in \mathbb{R}^d$ is a vector such that $\langle y, v_i \rangle = 0$ for $i \in I$, then $x = c + \epsilon y \in \widehat{F}$ for a sufficiently small $\epsilon > 0$. Hence $\dim\ \widehat{F} = d - k - 1$. Part 1 is proved.

Let us prove Part 2 of the Theorem. For any $x \in \widehat{G}$, we have $\langle x, y \rangle = 1$ for any $y \in G$. Since $F \subset G$, we have $\langle x, y \rangle = 1$ for any $y \in F$. Therefore, $x \in \widehat{F}$ and the proof of Part 2 follows.

By Theorem 32.2, $Q^\circ = P$. Then $\widehat{\widehat{F}}$ is a face of P and it is clear that $F \subset \widehat{\widehat{F}}$. Let $c \in \mathbb{R}^d$ be a vector such that $\langle c, x \rangle = 1$ for $x \in F$ and $\langle c, x \rangle < 1$ for $x \notin F$ (see the proof of Part 2 above). Then $c \in \widehat{F}$. Therefore, for $x \in \widehat{\widehat{F}}$ we have $\langle x, c \rangle = 1$, so $x \in F$. Therefore, $\widehat{\widehat{F}} \subset F$. Hence $F = \widehat{\widehat{F}}$ and the proof of Part 3 follows. $\square$

PROBLEMS.

1°. From Section 15.3, we know that the cyclic polytope $C(d, n)$ has the largest number of facets (faces of dimension $d-1$) among all polytopes with n vertices (faces of dimension 0) in $\mathbb{R}^d$.

Construct a polytope in $\mathbb{R}^d$ that has the largest number of vertices among all polytopes in $\mathbb{R}^d$ with n facets.

2. A polytope $P \subset \mathbb{R}^d$ is called *self-dual*, if it contains the origin as its interior point and P° can be obtained from P by a non-degenerate linear transformation. Suppose that $P = \operatorname{conv}(v_1, \ldots, v_{d+1})$, where $v_1, \ldots, v_{d+1}$ are affinely independent and $0 = v_1 + \ldots + v_{d+1}$. Prove, that P is self-dual.

3. Prove, that for $d > 2$, the cube $I_d = \big\{(\xi_1, \ldots, \xi_d) : -1 \leq \xi_i \leq 1, i = 1, \ldots, d\big\}$ is not self-dual. Prove, that I_1 and I_2 are self-dual.

4. Prove, that a regular polygon in $\mathbb{R}^2$, containing the origin as its center, is self-dual.

5. Let e_1, e_2, e_3, e_4 be the standard orthonormal basis of $\mathbb{R}^4$. The polytope

$$P = \operatorname{conv}\big(e_i + e_j, -e_i - e_j, e_i - e_j : \quad 1 \leq i \neq j \leq 4\big)$$

is called the *24-cell*. Prove, that P is self-dual.

Polytope P is a regular 4-dimensional polytope having 24 vertices and 24 facets.

6. Let $K \subset \mathbb{R}^d$ be a convex compact set containing the origin in its interior. Let $Q_0 = K^\circ$ and let Q_p be the polar of K with the origin moved to p: $Q_p = \big\{x \in \mathbb{R}^d : \langle x, y - p \rangle \leq 1 \text{ for any } y \in K\big\}$. Prove, that Q_p is the image of Q_0 under the *projective transformation*:

$$x \longmapsto \frac{x}{1 - \langle x, p \rangle}.$$

7*. Let $K \subset \mathbb{R}^d$ be a convex set with non-empty interior. Let us define a function $\phi : \operatorname{int}(K) \longrightarrow \mathbb{R}$ by $\phi(p) = \ln \operatorname{vol}(Q_p)$, where "vol" is the volume and Q_p is defined in Problem 6. Prove, that ϕ is a convex function, that is

$$\phi(\alpha x + \beta y) \leq \alpha \phi(x) + \beta \phi(y),$$

for any $x, y \in \text{int}(K)$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

See the "universal barrier" in Yu. Nesterov, A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, PA, 1994.

Friday, March 20

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34. Application: The Characterization of the Moments of a Measure

(34.1) Definition. Let $\langle \rangle : E \times F \longrightarrow \mathbb{R}$ be a duality and let $K \subset E$ be a convex cone. The set

$$K^* = \Big\{ f \in F : \langle e, f \rangle \geq 0 \quad \text{for} \quad \text{any} \quad e \in K \Big\}$$

is called the *dual cone*. In other words, $K^* = -K^\circ$. If $K \subset F$ is a convex cone, the dual cone $K^* = -K^\circ \subset E$ is defined similarly.

PROBLEMS.

1°. Check, that K^* is a convex closed cone.

2°. Let $\mathbb{R}^d_+ = \Big\{ (\xi_1, \ldots, \xi_d) : \xi_i \geq 0 \text{ for } i = 1, \ldots, d \Big\}$ be the non-negative orthant. Prove, that $(\mathbb{R}^d_+)^* = \mathbb{R}^d_+$.

3°. Let K be a closed convex cone. Prove, that $(K^*)^* = K$.

Let us fix the interval $[0, 1] \subset \mathbb{R}^1$. We recall (see Definition 16.1) that the *moment cone* $M_{d+1} \subset \mathbb{R}^{d+1}$ is the conic hull of the curve $\widehat{g}(\tau) = (1, \tau, \ldots, \tau^d) : 0 \leq \tau \leq 1$:

$$M_{d+1} = \text{cone}\Big((1, \tau, \ldots, \tau^d) : \quad 0 \leq \tau \leq 1 \Big) \subset \mathbb{R}^{d+1}.$$

We recall that $K_+ \subset \mathbb{R}^{d+1}$ is the cone of all polynomials $p(\tau) = \gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d$ that are non-negative on $[0, 1]$:

$$K_+ = \Big\{ (\gamma_0, \ldots, \gamma_d) : \ \gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d \geq 0 \quad \text{for any} \quad 0 \leq \tau \leq 1 \Big\} \subset \mathbb{R}^{d+1}$$

(see Section 18).

(34.2) Proposition. *We have $K_+ = (M_{d+1})^*$ and $M_{d+1} = (K_+)^*$. Let $a = (\alpha_0, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$ be a point. Then $a \in M_{d+1}$ if and only if the following quadratic forms in $\xi_0, \ldots, \xi_m$*

$$\sum_{i,j=0}^{m} \alpha_{i+j} \xi_i \xi_j, \ 2m \leq d; \qquad \sum_{i,j=0}^{m} \alpha_{i+j+1} \xi_i \xi_j, \ 2m + 1 \leq d;$$

$$\sum_{i,j=0}^{m} (\alpha_{i+j} - \alpha_{i+j+1}) \xi_i \xi_j, \ 2m + 1 \leq d; \qquad \sum_{i,j=0}^{m} (\alpha_{i+j+1} - \alpha_{i+j+2}) \xi_i \xi_j, \ 2m + 2 \leq d$$

are positive semidefinite.

Proof. Let $c = (\gamma_0, \ldots, \gamma_d) \in \mathbb{R}^d$. Then $\langle c, \widehat{g}(\tau) \rangle = \gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d$. Therefore, $c \in (M_{d+1})^*$ if and only the polynomial $\gamma_0 + \gamma_1 \tau + \ldots + \gamma_d \tau^d$ is non-negative on the interval $[0, 1]$. Therefore, $K_+ = (M_{d+1})^*$. Since M_{d+1} is convex and closed (see

Problem 1 of Section 16.1), by Theorem 32.2 $(K_+)^* = M_{d+1}$ (see also Problem 3 of Section 34.1). Let $q(\tau) = \xi_0 + \xi_1\tau + \ldots + \xi_m\tau^m : 2m \leq d$ be any polynomial. Then $p_1(\tau) = q^2(\tau) \geq 0$ for any $\tau \in [0,1]$. We may write $p_1(\tau) = \gamma_0 + \gamma_1\tau + \ldots + \gamma_d\tau^d$, where

$$\gamma_k = \sum_{i+j=k} \xi_i\xi_j, \quad k = 0, \ldots, d.$$

Therefore, $c = (\gamma_0, \ldots, \gamma_d) \in K_+$. Since $K_+ = (M_{d+1})^*$, for any point $a \in M_{d+1}$ we must have $\langle a, c\rangle \geq 0$ and so the quadratic form

$$\sum_{i,j=0}^{m} \alpha_{i+j}\xi_i\xi_j$$

must be positive semidefinite.

Similarly, since the polynomial $p_2(\tau) = \tau q^2(\tau)$ is non-negative on $[0,1]$, the form

$$\sum_{i,j=0}^{m} \alpha_{i+j+1}\xi_i\xi_j, \quad 2m+1 \leq d$$

must be positive semidefinite for any $a \in M_{d+1}$; since the polynomial $p_3(\tau) = (1-\tau)q^2(\tau)$ is non-negative on $[0,1]$, the form

$$\sum_{i,j=0}^{m} (\alpha_{i+j} - \alpha_{i+j+1})\xi_i\xi_j, \quad 2m+1 \leq d$$

is positive semidefinite for any $a \in M_{d+1}$; and since $p_4(\tau) = \tau(1-\tau)q^2(\tau)$ is non-negative on $[0,1]$, the form

$$\sum_{i,j=0}^{m} (\alpha_{i+j+1} - \alpha_{i+j+2})\xi_i\xi_j, \ 2m+2 \leq d$$

is positive semidefinite for any $a \in M_{d+1}$.

We proved the "only if" part. Let us proved the "if" part. Suppose, that $a \in \mathbb{R}^{d+1}$ is such, that all the listed forms are positive semidefinite. Therefore, $\langle a, c\rangle \geq 0$ for any $c = (\gamma_0, \ldots, \gamma_d)$, such that the polynomial $p(\tau) = \gamma_0 + \ldots + \gamma_d\tau^d$ has the form $p_1(\tau) = q^2(\tau)$, $p_2(\tau) = \tau q^2(\tau)$, $p_3(\tau) = (1-\tau)q^2(\tau)$ or $p_4(\tau) = \tau(1-\tau)q^2(\tau)$, where $q(\tau)$ is any polynomial. On the other hand, by Problem 4, Section 18.6, every non-negative polynomial $p(\tau)$ is a conic combination of polynomials $p_1(\tau)$, $p_2(\tau)$, $p_3(\tau)$, and $p_4(\tau)$. Therefore, $\langle a, c\rangle \geq 0$ for any $c = (\gamma_0, \ldots, \gamma_d)$ such that the polynomial $\gamma_0 + \ldots + \gamma_d\tau^d$ is non-negative on the interval $[0,1]$. In other words, $a \in (K_+)^* = M_{d+1}$ and the proof follows. □

PROBLEM.

1*. Prove a stronger criterion: Let $a = (\alpha_0, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$ be a point. Suppose that d is even and that the quadratic forms

$$\sum_{i,j=0}^{d/2} \alpha_{i+j}\xi_i\xi_j \quad \text{and} \quad \sum_{i,j=0}^{d/2-1} (\alpha_{i+j+1} - \alpha_{i+j+2})\xi_i\xi_j$$

are positive semidefinite. Then $a \in M_{d+1}$. Suppose that d is odd and that the quadratic forms

$$\sum_{i,j=0}^{(d-1)/2} \alpha_{i+j+1}\xi_i\xi_j \quad \text{and} \quad \sum_{i,j=0}^{(d-1)/2} (\alpha_{i+j} - \alpha_{i+j+1})\xi_i\xi_j$$

are positive semidefinite. Then $a \in M_{d+1}$ (cf. Problem 4, Section 18.6).

2. Let $a = (\alpha_0, \ldots, \alpha_d)$. Let

$$\Delta_{1,m}(a) = \det \begin{pmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_m \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{m+1} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_m & \alpha_{m+1} & \ldots & \alpha_{2m} \end{pmatrix}, 2m \leq d;$$

$$\Delta_{2,m}(a) = \det \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{m+1} \\ \alpha_2 & \alpha_3 & \ldots & \alpha_{m+2} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m+1} & \alpha_{m+2} & \ldots & \alpha_{2m+1} \end{pmatrix}, 2m+1 \leq d;$$

$$\Delta_{3,m}(a) = \det \begin{pmatrix} \alpha_0 - \alpha_1 & \alpha_1 - \alpha_2 & \ldots & \alpha_m - \alpha_{m+1} \\ \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \ldots & \alpha_{m+1} - \alpha_{m+2} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_m - \alpha_{m+1} & \alpha_{m+1} - \alpha_{m+2} & \ldots & \alpha_{2m} - \alpha_{2m+1} \end{pmatrix}, 2m+1 \leq d;$$

$$\Delta_{4,m}(a) = \det \begin{pmatrix} \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \ldots & \alpha_{m+1} - \alpha_{m+2} \\ \alpha_2 - \alpha_3 & \alpha_3 - \alpha_4 & \ldots & \alpha_{m+2} - \alpha_{m+3} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m+1} - \alpha_{m+2} & \alpha_{m+2} - \alpha_{m+3} & \ldots & \alpha_{2m+1} - \alpha_{2m+2} \end{pmatrix},$$

$2m + 2 \leq d$.

Prove, that $a \in M_{d+1}$ if and only if all the determinants $\Delta_{1,m}(a)$, $\Delta_{2,m}(a)$, $\Delta_{3,m}(a)$, and $\Delta_{4,m}(a)$ are non-negative.

Hint: Use Sylvester's criterion.

(34.3) Moments of a measure

One can show (cf. Section 30.2) that $a = (\alpha_0, \ldots, \alpha_d) \in M_{d+1}$ if and only if there exists a Borel measure μ on the interval $[0, 1]$, such that

$$\alpha_i = \int_0^1 \tau^i \, d\mu \quad \text{for} \quad i = 0, \ldots, d.$$

Proposition 34.2 and Problem 2 provide a computationally efficient method to check whether $(\alpha_0, \ldots, \alpha_d)$ are the moments of some Borel measure on $[0, 1]$.

PROBLEM.

1. Prove, that $a \in M_{d+1}$ if and only if there exist a measure μ which is a non-negative linear combination of $(d+1)/2$ (for d odd) or $(d+2)/2$ (for d even) delta-measures δ_{τ_i}, $\tau_i \in [0, 1]$, such that $\displaystyle\int_0^1 \tau^i \, d\mu = \alpha_i$ for $i = 0, \ldots, d$.

Hint: Use Corollary 16.4.

Monday, March 23

35. Linear Inequalities and Linear Programming

(35.1) Cones and Orders

Let E be a (topological) vector space and let $K \subset E$ be a convex cone. The cone K defines an *order* on E as follows: we say that $x \leq y$ (sometimes we write $x \leq_K y$) provided $y - x \in K$. Similarly, $x \geq y$ provided $x - y \in K$. Thus, $K = \Big\{x \in E : x \geq 0\Big\}$.

PROBLEMS.

Let E be a vector space, let $K \subset E$ be a convex cone, and let $\leq$ be the corresponding order on E.

1°. Prove, that $x \geq x$ for any $x \in E$.

2°. Prove, that if $x \leq y$ and $y \leq z$, then $x \leq z$.

3°. Prove, that if $x \leq y$ and $\alpha \geq 0$, then $\alpha x \leq \alpha y$.

4°. Prove, that if $x_1 \leq y_1$ and $x_2 \leq y_2$ then $x_1 + x_2 \leq y_1 + y_2$.

5°. Suppose, that K does not contain a straight line. If $x \leq y$ and $y \leq x$ then $x = y$.

6°. Suppose, that E is a topological vector space and $K \subset E$ is a closed cone. Then the order $\leq$ is *continuous*: suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences and that $x = \lim_{n \longrightarrow \infty} x_n$ and $y = \lim_{n \longrightarrow \infty} y_n$. Suppose, that $x_n \leq y_n$ for all n. Then $x \leq y$.

(35.2) Examples.

(35.2.1) Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and let ${\mathbb{R}^d}_+ = \Big\{(\xi_1, \ldots, \xi_d) : \xi_i \geq 0 \text{ for } i = 1, \ldots, d\Big\}$.

(35.2.2) Let $\mathbb{R}^d$, $d = n(n+1)/2$ be the space of all $n \times n$ symmetric matrices and let $S_+ = \Big\{X \in \mathbb{R}^d : X \text{ is positive semidefinite } \Big\}$.

(35.2.3) Let $L^1[0,1]$ be the space of all integrable functions on the interval $[0,1]$ and let $L^1_+ = \Big\{f \in L^1[0,1] : f(\tau) \geq 0 \text{ almost everywhere } \Big\}$.

(35.2.4) Let $L^\infty[0,1]$ be the space of all bounded measurable functions on the interval $[0,1]$ and let $L^\infty_+ = \Big\{f \in L^\infty[0,1] : f(\tau) \geq 0 \text{ almost everywhere } \Big\}$.

(35.2.5) Let $C[0,1]$ be the space of all continuous functions on the interval $[0,1]$ and let $C_+ = \Big\{f \in C[0,1] : f(\tau) \geq 0 \text{ for every } \tau \in [0,1]\Big\}$.

(35.2.6) Let $V[0,1]$ be the space of all signed Borel measures μ on the interval $[0,1]$ (the space of all continuous linear functionals $C[0,1] \longrightarrow \mathbb{R}$) and let $V_+ = \Big\{\mu \in V[0,1] : \int_0^1 f \, d\mu \geq 0 \text{ for every } f \in C_+\Big\}$.

PROBLEMS.

1°. Prove, that the cones of examples (35.2.1)–(35.2.6) are closed convex cones without straight lines.

For a cone $K \subset E$, let $K - K = \{x - y : x \in K, y \in K\}$.

2°. Prove, that $K - K = E$ in examples (35.2.1)–(35.2.5).

3. Prove, that $K - K = E$ in Example 35.2.6.

Let E be a vector space, let $K \subset E$ be a cone and let $\leq$ be the corresponding order.

4°. Suppose, that $K = \{0\}$. Prove, that $x \leq y$ if and only if $x = y$.

5°. Suppose, that $K = E$. Prove, that $x \leq y$ for any two $x \in y$.

6. Let us consider the following *lexicographic order* $\succ$ on $\mathbb{R}^d$: we say that $x \succ y$, where $x = (\xi_1, \ldots, \xi_d)$ and $y = (\eta_1, \ldots, \eta_d)$, if there is a $1 \leq k \leq d$ such that $\xi_k > \eta_k$ and $\xi_i = \eta_i$ for all $i < k$. Prove, that there is a convex cone $K \subset \mathbb{R}^d$ such that $x \succ y$ if and only if $x \geq_K y$. Prove, that the cone K is not closed.

(35.3) A general problem of linear programming

A general problem of linear programming is formulated as follows. Let E_1 and E_2 be vector spaces and let $A : E_1 \longrightarrow E_2$ be a linear transformation. Let $K_1 \subset E_1$ and $K_2 \subset E_2$ be convex cones. Let $b \in E_2$ be a vector and let $\phi : E_1 \longrightarrow \mathbb{R}$ be a linear function.

Find $\inf \phi(x)$,

where $x \in E_1$ is subject to the constraints:

$x \geq_{K_1} 0$ and $A(x) \geq_{K_2} b$

If $K_2 = \{0\}$, then we get a problem of linear programming in the *canonical form* (cf. Problem 4, Section 35.2).

Find $\inf \phi(x)$,

Subject to: $x \geq_{K_1} 0$ and $A(x) = b$.

If $K_1 = E_1$, we get a problem of linear programming in the *standard form* (cf. Problem 5, Section 35.2).

Find $\inf \phi(x)$,

Subject to: $A(x) \geq_{K_2} b$.

An important particular case is when $E_2 = \mathbb{R}^m$ and $A(x) = \big(\phi_1(x), \ldots, \phi_m(x)\big)$, where ϕ_i are linear functions.

(35.4) Problems and dual problems

Let $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$ be dualities. Let us fix closed convex cones $K_1 \subset E_1$ and $K_2 \subset E_2$ and let $K_1^* = \{f \in F_1 : \langle e, f\rangle \geq 0 \text{ for any } e \in K_1\}$ and $K_2^* = \{f \in F_1 : \langle e, f\rangle \geq 0 \text{ for any } e \in K_2\}$ be the dual cones. Hence $K_1^* \subset F_1$ and $K_2^* \subset F_2$. Let $A : E_1 \longrightarrow E_2$ be a continuous linear transformation and let $A^* : F_2 \longrightarrow F_1$ be the dual transformation (see Section 31.5). As we know (see Theorem 31.4), any continuous linear function $\phi : E_1 \longrightarrow \mathbb{R}$ can be written as $\phi(x) = \langle x, f\rangle_1$ for some $f \in F_1$. Let us fix $b \in E_2$ and $c \in F_1$.

A general problem of linear programming is written in the form:

Find $\inf\langle x, c\rangle_1$,

Subject to: $x \geq_{K_1} 0$ and $A(x) \geq_{K_2} b$,

where $x \in E_1$ is a variable. The *dual* problem of linear programming is

Find $\sup\langle b, p\rangle_2$,

Subject to: $p \geq_{K_2^*} 0$ and $A^*(p) \leq_{K_1^*} c$,

where $p \in F_2$ is a variable.

An important particular case, $K_2 = \{0\}$, so $K_2^* = F_2$ and we have a standard linear programming problem

Find $\inf\langle x, c\rangle_1$,

Subject to: $x \geq_{K_1} 0$ and $A(x) = b$

and the dual canonical linear programming problem

Find $\sup\langle b, p\rangle_2$,

Subject to: $A^*(p) \leq_{K_1^*} c$.

In particular, if $E_2 = F_2 = \mathbb{R}^m$ and $A(x) = \big(\langle x, a_1\rangle_1, \ldots, \langle x, a_m\rangle_1\big)$, we have the following pair of problems (cf. Problem 3, Section 31.5):

Find $\inf\langle x, c\rangle_1$,

Subject to: $x \geq_{K_1} 0$ and $\langle x, a_i\rangle = \beta_i$ for $i = 1, \ldots, m$

and

Find $\sup \displaystyle\sum_{i=1}^m \beta_i \pi_i$,

Subject to: $\displaystyle\sum_{i=1}^m \pi_i a_i \leq_{K_1^*} c$.

Wednesday, March 25

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36. Duality Theorems for Linear Programming

(36.1) The problems

Let $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$ be dualities. Let us choose a closed convex cone $K \subset E_1$ and let $K^* \subset F_1$:

$$K^* = \big\{f \in F_1 : \langle e, f\rangle \geq 0 \quad \text{for any} \quad e \in K_1\big\}$$

be the dual cone. Let $A : E_1 \longrightarrow E_2$ be a continuous linear transformation and let $A^* : F_2 \longrightarrow F_1$ be the dual transformation. We write simply Ax and A^*p instead of $A(x)$ and $A^*(p)$ for $x \in E_1$ and $p \in F_2$. Let us choose a $c \in F_1$ and a $b \in E_2$. We consider a pair of linear programming problems:

Problem I.

Find $\gamma = \inf\langle x, c\rangle_1$

Subject to: $Ax = b$ and $x \geq_K 0$

and

Problem II.

Find $\beta = \sup\langle b, p\rangle_2$

Subject to: $A^*p \leq_{K^*} c$

A point $x \in E_1$, which satisfies the conditions $Ax = b$ and $x \geq_K 0$, is called a *feasible* point in Problem I. A point $p \in F_2$, which satisfies the conditions $A^*p \leq_{K^*} c$, is called a *feasible* point in Problem II. If Problem I does not have a feasible solution, we agree that $\gamma = +\infty$. If Problem II does not have a feasible solution, we agree that $\beta = -\infty$.

(36.2) Lemma ("Weak Duality"). *For any feasible point x of Problem I and any feasible point p of Problem II one has*

$$\langle x, c\rangle_1 \geq \langle b, p\rangle_2.$$

In particular, $\gamma \geq \beta$.

Proof. We have

$$\langle b, p\rangle_2 = \langle Ax, p\rangle_2 = \langle x, A^*p\rangle_1$$

(see Section 31.5). Now, since $A^*p \leq_{K^*} c$, we have $c - A^*p \in K^*$ and hence $\langle x, c - A^*p\rangle \geq 0$. Therefore,

$$\langle b, p\rangle_2 = \langle x, A^*p\rangle_1 = \langle x, c\rangle_1 - \langle x, c - A^*p\rangle_1 \geq \langle x, c\rangle_1$$

and the proof follows. □

PROBLEMS.

1. Let $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$ be dualities. Let us fix cones $K_1 \subset E_1$ and $K_2 \subset E_2$ and let $K_1^* \subset F_1$ and $K_2^* \subset F_2$ be dual cones. Let $A : E_1 \longrightarrow E_2$ be a linear transformation and let $c \in F_1$ and $b \in E_2$. Consider the problems:

Problem 1: Find $\inf\langle x, c\rangle_1$ subject to the constraints $Ax \geq_{K_2} b$ and $x \geq_{K_1} 0$

and

Problem 2: Find $\sup\langle b, p\rangle_2$ subject to the constraints $A^*p \leq_{K_1^*} c$ and $p \geq_{K_2^*} 0$.

Prove, that for every feasible point x of the first problem and every feasible point of the second problem, one has $\langle x, c\rangle_1 \geq \langle b, p\rangle_2$.

2°. Suppose, that x_0 is a feasible point of Problem I and that p_0 is a feasible point of Problem 2, Section 36.1, such that $\langle x_0, c\rangle_1 = \langle b, p_0\rangle$. Prove, that x_0 is an optimal solution of Problem 1 and p_0 is an optimal solution of Problem 2.

3°. Suppose that $\gamma = -\infty$ in Problem I, Section 36.1. Prove, that Problem II does not have a feasible point. Suppose, that $\beta = +\infty$ in Problem II. Prove, that Problem I does not have a feasible point.

(36.3) Corollary ("Optimality Criterion"). *Suppose that x_0 is a feasible solution in Problem I and that p_0 is a feasible solution in Problem II such that*

$$\langle x_0, c - A^*p_0\rangle_1 = 0.$$

Then x_0 is an optimal solution in Problem I and p_0 is an optimal solution in Problem II. Suppose that $\gamma = \beta$ and that x_0 is an optimal solution in Problem I and p_0 is an optimal solution in Problem II. Then

$$\langle x_0, c - A^* p_0 \rangle_1 = 0.$$

Proof. From the proof of Lemma 36.2, it follows that $\langle x, c - A^* p\rangle \geq 0$ for any feasible x and p and that $\langle x, c - A^* p\rangle = 0$ if and only if $\langle x, c\rangle_1 = \langle b, p\rangle_2$, in which case x and p must be optimal solutions (cf. Problem 2, Section 36.2). $\square$

Corollary 36.3 is also known as the "Complementary slackness" condition.

(36.4) Definition. Let E_1 and E_2 be topological vector spaces. The *direct sum* $E \oplus E_2$ consists of all pairs (x_1, x_2), where $x_1 \in E_1$ and $x_2 \in E_2$ with the addition $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and the scalar multiplication $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$. The open sets in $E_1 \oplus E_2$ are the unions of the elementary open sets: $U_1 \times U_2 = \big\{(x_1, x_2) : x_1 \in U_1, x_2 \in U_2\big\}$, where U_1 is an open set in E_1 and U_2 is an open set in E_2.

PROBLEMS.

1. Prove, that if E_1 and E_2 are locally convex topological vector spaces then $E_1 \oplus E_2$ is a locally convex topological vector space.
2. Let E_1 and E_2 be topological vector spaces. Prove, that every continuous linear function $\phi : E_1 \oplus E_2 \longrightarrow \mathbb{R}$ has the form $\phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$ for some continuous linear functions $\phi_1 : E_1 \longrightarrow \mathbb{R}$ and $\phi_2 : E_2 \longrightarrow \mathbb{R}$.

(36.5) Theorem ("Strong Duality"). *Let $\widehat{A} : E_1 \longrightarrow E_2 \oplus \mathbb{R}$ be the linear transformation*

$$\widehat{A}x = \big(Ax, \langle x, c\rangle_1\big)$$

for $x \in E_1$. Suppose that the image $\widehat{A}(K) = \big\{\widehat{A}x : x \in K\big\}$ is a closed set in $E_2 \oplus \mathbb{R}$ and that there is a feasible point x in Problem I. Then $\gamma = \beta$. If $\gamma > -\infty$, then there is an optimal solution x is Problem I.

Proof. If $\gamma = -\infty$ then there are no feasible points p in Problem II (see Problem 3, Section 36.2), and hence $\beta = -\infty$. Therefore, we can assume that $\gamma > -\infty$.

In the space $E_2 \oplus \mathbb{R}$, consider the straight line $L = (b, \tau)$, where $-\infty < \tau < +\infty$. Then the intersection $L \cap \widehat{A}(K)$ is a closed set. The intersection $L \cap \widehat{A}(K)$ is the set of points $\big(b, \langle x, c\rangle_1\big)$, where x is a feasible point in Problem I. Since there are feasible points and the objective function is bounded from below, this set is a closed bounded interval or a closed ray bounded from below. Therefore, there is a feasible point x such that $\langle c, x\rangle_1 = \gamma$.

From Lemma 36.1, $\gamma \geq \beta$. Let us prove, that for any $\epsilon > 0$ there is a feasible point p in Problem II, such that $\langle b, p\rangle_2 \geq \gamma - \epsilon$. This would imply that $\beta \geq \gamma - \epsilon$, and, therefore, $\beta = \gamma$. We have

$$(b, \gamma - \epsilon) \notin L \cap \widehat{A}(K).$$

Since the cone $\widehat{A}(K_1)$ is closed, the point $(b, \gamma - \epsilon)$ can be strictly separated from $\widehat{A}(K)$ by a hyperplane (Theorem 24.4). In other words, there exists a pair (p, σ), $p \in F_2$, $\sigma \in \mathbb{R}$ such that

$$\langle b, p\rangle_2 + \sigma(\gamma - \epsilon) > 0$$

and

$$\langle Ax, p\rangle_2 + \sigma\langle x, c\rangle_1 \leq 0$$

for all $x \in K$. In particular, if x_0 is an optimal solution in Problem I, then $\langle x_0, c\rangle_1 = \gamma$ and $Ax_0 = b$, so

$$\langle b, p\rangle_2 + \sigma\gamma \leq 0.$$

Therefore, $\sigma < 0$ and, by scaling (p, σ), if necessary, we can assume that $\sigma = -1$. Thus we have

$$\langle b, p\rangle_2 - (\gamma - \epsilon) > 0$$

and

$$\langle Ax, p\rangle_2 - \langle x, c\rangle_1 = \langle x, A^*p\rangle_1 - \langle x, c\rangle_1 = \langle x, A^*p - c\rangle_1 \leq 0$$

for all $x \in K$. Therefore, $c - A^*p \in K^*$, that is $A^*p \leq_{K^*} c$. Hence we conclude that p is a feasible point in Problem II and that $\langle b, p\rangle_2 > \gamma - \epsilon$. □

PROBLEMS.

1. Consider the problems:

Problem 1: Find $\gamma = \inf\langle x, c\rangle_1$ subject to the constraints $Ax \geq_{K_2} b$ and $x \geq_{K_1} 0$ and

Problem 2: Find $\beta = \sup\langle b, p\rangle_2$ subject to the constraints $A^*p \leq_{K_1^*} c$ and $p \geq_{K_2^*} 0$.

(see Problem 1, Section 36.2). Let $\mathcal{K} \subset E_2 \oplus \mathbb{R}$ be the cone:

$$\mathcal{K} = \Big\{\big(Ax - y, \langle x, c\rangle_1 + \tau\big) : x \in K_1, y \in K_2, \tau \geq 0\Big\}.$$

Suppose, that $\mathcal{K}$ is closed and that there is a feasible solution in Problem 1. Prove, that $\gamma = \beta$.

2. Consider a linear programming problem in $C[0, 1]$:

Find $\gamma = \inf \displaystyle\int_0^1 \tau x(\tau)\, d\tau$

Subject to: $\displaystyle\int_0^1 x(\tau)\, d\tau = 1$ and $x(\tau) \geq 0$ for all $0 \leq \tau \leq 1$.

Prove, that $\gamma = 0$ and that there is no optimal solution.
The dual problem is

Find $\beta = \sup p$,

Subject to: $p \in \mathbb{R}$ and $\tau\, d\tau - p\, d\tau$ is a non-negative measure on the interval $0 \leq \tau \leq 1$.

Prove, that $\beta = 0$ and that $p = 0$ is the optimal solution.

37. The "Standard" Linear Programming

(37.1) Problems

Let $E_1 = F_1 = \mathbb{R}^n$ and $E_2 = F_2 = \mathbb{R}^m$. We consider the standard dualities $\langle\rangle_1$ and $\langle\rangle_2$ (see Example 31.2.1). We will denote the scalar product in $\mathbb{R}^n$ and $\mathbb{R}^m$ simply by $\langle\rangle$. Let $K = \mathbb{R}^n_+$ (see Example 35.2.1). Then $K^* = \mathbb{R}^n_+$ and instead of writing $x \geq_K 0$ and $x \geq_{K^*} 0$ we simply write $x \geq 0$. Let us choose $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We think of a vector from $\mathbb{R}^n$ (resp. $\mathbb{R}^m$) as an n-column (resp. m-column) of real numbers. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. We can think of A as an $n \times m$ matrix. Then $A^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is represented by the transposed $m \times n$ matrix. We consider a pair of linear programming problems:

Problem I.

Find $\gamma = \langle c, x\rangle$
Subject to: $Ax = b$, $x \geq 0$

and

Problem II.

Find $\beta = \inf\langle b, p\rangle$
Subject to: $A^*p \leq c$.

(37.2) Theorem. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let $P \subset \mathbb{R}^n$ be a polyhedron. Then the image $T(P) \subset \mathbb{R}^m$ is a polyhedron.*

Proof. First, we prove that if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$, $T(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_{n-1})$, then $T(P)$ is a polyhedron. Suppose that

$$P = \Big\{\langle a_i, x\rangle \leq \beta_i : i = 1, \ldots, k\Big\},$$

where $a_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$. Let us choose $x = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Then $x \in T(P)$ if and only if there exists a $\tau \in \mathbb{R}$ such that $(x, \tau) \in P$, that is, if (x, τ) satisfies the system of linear inequalities:

$$\sum_{j=1}^{n-1} \alpha_{ij}\xi_j + \alpha_{in}\tau \leq \beta_i \quad \text{for} \quad i = 1, \ldots, k.$$

Let $I_+ = \{i : \alpha_{in} > 0\}$, $I_- = \{i : \alpha_{in} < 0\}$, and $I_0 = \{i : \alpha_{in} = 0\}$. Then $x \in T(P)$ if and only if

$$\sum_{j=1}^{n-1} \alpha_{ij}\xi_j \leq \beta_i \quad \text{for} \quad i \in I_0$$

and there exists a $\tau \in \mathbb{R}$ such that

$$\tau \leq \frac{\beta_i}{\alpha_{in}} - \sum_{j=1}^{n-1} \frac{\alpha_{ij}}{\alpha_{in}}\xi_j \quad \text{for} \quad i \in I_+$$

and

$$\tau \geq \frac{\beta_i}{\alpha_{in}} - \sum_{j=1}^{n-1} \frac{\alpha_{ij}}{\alpha_{in}} \xi_j \quad \text{for} \quad i \in I_-.$$

Therefore, $x \in T(P)$ if and only if

$$\sum_{j=1}^{n-1} \alpha_{ij} \xi_j \leq \beta_i \quad \text{for} \quad i \in I_0$$

and

$$\frac{\beta_{i_1}}{\alpha_{i_1 n}} - \sum_{j=1}^{n-1} \frac{\alpha_{i_1 j}}{\alpha_{i_1 n}} \xi_j \leq \frac{\beta_{i_2}}{\alpha_{i_2 n}} - \sum_{j=1}^{n-1} \frac{\alpha_{i_2 j}}{\alpha_{i_2 n}} \xi_j \quad \text{for all pairs} \quad i_1 \in I_- \quad \text{and} \quad i_2 \in I_+.$$

We got a system of linear inequalities describing $T(P)$, so $T(P)$ is a polyhedron.

Second, we prove that if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is a projection $T(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_d)$ and $P \subset \mathbb{R}^n$ is a polyhedron, then $T(P)$ is a polyhedron. The proof follows since T can be written as a composition of projections $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-1} \longrightarrow \ldots \longrightarrow \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$.

Finally, since every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be written as a projection $T : \mathbb{R}^n \longrightarrow \mathbb{R}^d = \text{image}(T)$ in an appropriate (not necessary orthogonal) coordinate system, the proof follows. □

(37.3) Theorem. *Suppose that there exists a feasible point x in Problem I, Section 37.1. Then $\gamma = \beta$. If $-\infty < \gamma = \beta < +\infty$, then there exist optimal solutions in Problems I or II.*

Proof. The cone $K = \mathbb{R}^n_+$ is a polyhedron. Therefore, by Theorem 37.2 the image $\widehat{A}(K) = \big\{(Ax, \langle c, x \rangle) : x \in K\big\}$ is a polyhedron, and hence is closed. Theorem 36.5 implies that $\gamma = \beta$ and that if $\gamma > -\infty$, then there is an optimal solution x in Problem I.

Let us introduce "slack" vectors $y \in \mathbb{R}^n$ and $q_1, q_2 \in \mathbb{R}^m$. Since every vector $p \in \mathbb{R}^m$ can be written as a difference $q_1 - q_2$ for some $q_1, q_2 \in \mathbb{R}^m_+$ and the inequality $A^*p \leq c$ is equivalent to $A^*p + y = c, y \geq 0$, we can construct a linear programming problem in the canonical form, which is equivalent to Problem II:

Find $\beta = \sup\langle b, q_1 - q_2 \rangle$

Subject to: $A^*(q_1 - q_2) + y = c, \quad q_1, q_2, y \geq 0.$

As we proved above, for linear programs in the canonical form, there is an optimal solution, provided $-\infty < \beta < +\infty$. Thus there exists an optimal solution (q_1, q_2, y) to the problem. Therefore, $p = q_1 - q_2$ is an optimal solution to Problem II. □

PROBLEMS.

1. Prove, that if there is a feasible point in Problem II, Section 37.1, then $\beta = \gamma$.
2. Construct an example of Problems I and II, such that neither has a feasible point.
3. Consider a pair of of dual problems in the general form:

Find $\gamma = \inf\langle c, x \rangle$

Subject to: $Ax \geq b$ and $x \geq 0$

and

Find $\beta = \langle b, p\rangle$

Subject to: $A^*p \leq c$ and $p \geq 0$.

Prove, that if there is a feasible point in one of the problems, then $\gamma = \beta$ and if $-\infty < \beta = \gamma < +\infty$ then there exist optimal solutions in the problems.

4. Prove, that the Minkowski sum $P_1 + P_2 = \{x + y : x \in P_1, y \in P_2\}$ of polyhedra $P_1, P_2 \subset \mathbb{R}^d$ is a polyhedron.

5*. Prove, that every polyhedron $P \subset \mathbb{R}^d$ can be represented as a sum $P = L + K + Q$, where $L \subset \mathbb{R}^d$ is an affine subspace, $K \subset \mathbb{R}^d$ is a convex cone without straight lines and $Q \subset \mathbb{R}^d$ is a polytope.

6*. Prove, that a closed convex set $P \subset \mathbb{R}^d$ is a polyhedron, if and only if it has finitely many faces.

7*. Prove, that the polar of a polyhedron is a polyhedron.

Let us consider Problems I and II of Section 37.1. Let $c = (\gamma_1, \ldots, \gamma_n)$, let $x = (\xi_1, \ldots, \xi_n)$, and let $a_1, \ldots, a_n \in \mathbb{R}^m$ be the columns of the matrix A. So the constraint $x \geq 0$ can be rewritten as $\xi_i \geq 0$ for $i = 1, \ldots, n$ and the constraint $A^*p \leq c$ can be rewritten as $\langle a_i, p\rangle \leq \gamma_i$ for $i = 1, \ldots, n$.

(37.4) Corollary ("Optimality Criterion"). *Let $x = (\xi_1, \ldots, \xi_n)$ be a feasible point in Problem I and let p be a feasible point in Problem II, Section 37.1. Then x is an optimal solution in Problem I and p is an optimal solution in Problem II if and only if*

$$\xi_i > 0 \quad \textit{implies} \quad \langle a_i, p\rangle = \gamma_i \qquad \textit{and} \qquad \langle a_i, p\rangle < \gamma_i \quad \textit{implies} \quad \xi_i = 0.$$

Proof. Follows by Corollary 36.3 and Theorem 37.3. □

PROBLEM.

Consider a pair of of dual problems in the general form:

Find $\gamma = \inf\langle c, x\rangle$

Subject to: $Ax \geq b$ and $x \geq 0$

and

Find $\beta = \langle b, p\rangle$

Subject to: $A^*p \leq c$ and $p \geq 0$.

Let $c = (\gamma_1, \ldots, \gamma_n)$, let $b = (\beta_1, \ldots, \beta_m)$, let $a_1, \ldots, a_n \in \mathbb{R}^m$ be the columns of A and let $a'_1, \ldots, a'_m \in \mathbb{R}^n$ be the rows of A. Suppose that $x = (\xi_1, \ldots, \xi_n)$ is a feasible point in the first problem and that $p = (\pi_1, \ldots, \pi_m)$ is a feasible point in the second problem. Prove, that x is an optimal solution in the first problem and p is an optimal solution in the second problem, if and only if:

$$\xi_i > 0 \quad \text{implies} \quad \langle a_i, p\rangle = \gamma_i \qquad \text{and} \qquad \langle a_i, p\rangle < \gamma_i \quad \text{implies} \quad \xi_i = 0;$$

and

$$\pi_j > 0 \quad \text{implies} \quad \langle a'_j, x\rangle = \beta_j \qquad \text{and} \qquad \langle a'_j, x\rangle > \beta_j \quad \text{implies} \quad \pi_j = 0.$$

38. Application: the Transportation Problem

Recall from Section 14 the Transportation Problem: we are given a directed graph $G(V, E)$ with n vertices ($V = \{1, \ldots, n\}$ is the set of vertices= cities) and m edges ($E = \{i \to j\}$ is the set of edges). We suppose that G does not contain loops. To every vertex i, a number a_i ("demand", if $a_i > 0$ and "supply", if $a_i < 0$) is attached. To every edge $i \to j$, a non-negative number c_{ij} (the cost per unit of transportation) is attached. The objective is to find a feasible flow, minimizing the total cost of transportation.

(38.1) The Transportation Problem.

Find $\gamma = \min \sum_{i \to j \in E} c_{ij} x_{ij}$

Subject to:

$$\sum_{j : j \to i \in E} x_{ji} - \sum_{k : i \to k \in E} x_{ik} = a_i \quad \text{for every vertex} \quad i \in V$$

and

$$x_{ij} \geq 0 \quad \text{for all} \quad i \to j \in E.$$

Problem 38.1 is a problem of linear programming in the canonical form:

Find $\min \langle \mathbf{c}, \mathbf{x} \rangle$

Subject to: $A\mathbf{x} = \mathbf{a}$ and $\mathbf{x} \geq 0$, where $\mathbf{c} = (c_{ij})$, $\mathbf{x} = (x_{ij})$, $\mathbf{a} = (a_1, \ldots, a_n)$ and $A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$:

$$A\mathbf{x} = \Big(\sum_{j : j \to 1 \in E} x_{j1} - \sum_{k : 1 \to k \in E} x_{1k}, \ldots, \sum_{j : j \to n \in E} x_{jn} - \sum_{k : n \to k \in E} x_{nk} \Big).$$

Hence we can write the dual program:

(38.2) The Dual Program.

Find $\alpha = \sup \sum_{i=1}^{n} a_i p_i$

Subject to: $p_j - p_i \leq c_{ij}$ for every edge $i \to j \in E$.

Problem 38.2 has the following interpretation: the variable p_i is the price of the commodity in the i-th city. A vector $\mathbf{p} = (p_1, \ldots, p_n)$ of prices is feasible, provided one can not gain by buying the commodity in one city and selling it in another city (taking into account the transportation costs).

PROBLEM.

1°. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a feasible point in Problem 38.2 and let c be a constant. Prove, that $(p_1 + c, \ldots, p_n + c)$ is a feasible point in Problem 38.2.

(38.3) Proposition. *Suppose that there is a feasible point in Problem 38.1. Then there is an optimal solution in Problem 38.1, there is an optimal solution in Problem 38.2, and $\gamma = \alpha$. Let $\mathbf{x} = \big(x_{ij} : i \rightarrow j \in E\big)$ be a feasible solution in Problem 38.1 and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a feasible solution in Problem 38.2. Then $\mathbf{x}$ is an optimal solution in Problem 38.1 and $\mathbf{p}$ is an optimal solution in Problem 38.2 if and only if*

$$x_{ij} > 0 \quad \textit{implies} \quad p_j - p_i = c_{ij} \qquad \textit{and} \qquad p_i - p_j < c_{ij} \quad \textit{implies} \quad x_{ij} = 0.$$

Proof. Since the costs c_{ij} are non-negative, $\gamma > -\infty$ and the result follows by Theorem 37.3 and Corollary 37.4. □

Suppose that G does not contain directed cycles (cf. Problem 4, Section 14.2). Proposition 38.3 suggests an algorithm for solving Problem 38.1. Proposition 14.3 implies that if $\mathbf{x}$ is an extreme point of the transportation polytope, then the edges $i \rightarrow j$, where $x_{ij} > 0$ form a forest in G. Suppose that $\mathbf{x}$ is a feasible point in Problem 38.2 such that the set of edges $i \rightarrow j$, where $x_{ij} > 0$ form a *spanning tree* T in G, that is a tree such that every vertex i of V is a vertex of the tree. Let us assign the price p_1 arbitrarily. Then, if $i \rightarrow j$ is an edge of T and p_i is computed, we compute $p_j = p_i + c_{ij}$. Thus we compute the prices p_i for $i = 1, \ldots, n$. If the vector $\mathbf{p} = (p_1, \ldots, p_n)$ is feasible in Problem 38.2, then, by Proposition 38.3, $\mathbf{x}$ is an optimal flow. Otherwise, there is an edge $i \rightarrow j \in E$, $i \rightarrow j \notin T$, such that $p_j - p_i > c_{ij}$. Adding this edge to the tree generates a cycle. We change the flow on the edges of the cycle in such a way that the overall balance is preserved: we increase the flow on $i \rightarrow j$ until the flow on some other edge of the cycle drops to 0. We delete the edge and get a new tree T' and a new feasible flow with a smaller total cost. To construct the initial solution $\mathbf{x}$, we add a new vertex 0 with $a_0 = 0$ and introduce new edges $i \rightarrow 0$ if $a_i < 0$ and $0 \rightarrow i$ if $a_i > 0$. We let c_{0i} and c_{j0} to be very big numbers, greater, say, than $\sum_{1 \leq i \neq j \leq n} c_{ij}$. The initial solution is $x_{i0} = -a_i$, if $a_i < 0$, $x_{0i} = a_i$, if $a_i > 0$, and $x_{ij} = 0$ otherwise. It is easy to see, that if the original problem has a feasible solution, then for any optimal solution in the modified problem we have $x_{0i} = x_{i0} = 0$.

Wednesday, April 1

Solutions to Some Problems from the Second Set

Friday, April 3

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39. Semidefinite Programming

Let $\mathbb{R}^d$, $d = n(n+1)/2$ be the space of $n \times n$ symmetric matrices. We recall that the scalar product in $\mathbb{R}^d$ is given by the formula:

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{tr}(AB),$$

where $A = (a_{ij})$ and $B = (b_{ij})$ (see Section 19.1). We write $X \succeq 0$ provided X is a positive semidefinite matrix. Let $\mathcal{S}_+ = \big\{X : X \succeq 0\big\}$ be the cone of positive semidefinite matrices (see Section 19.2). We consider the corresponding order $\succeq$ in $\mathbb{R}^d$ (see Example 35.2.2).

(39.1) Proposition. *The cone $\mathcal{S}_+ \subset \mathbb{R}^d$ of positive semidefinite matrices is self-dual:*

$$(\mathcal{S}_+)^* = \mathcal{S}_+.$$

Proof. Let us prove that $\mathcal{S}_+ \subset (\mathcal{S}_+)^*$. In other words, we should prove that $\langle A, B \rangle \geq 0$ for any $A, B \in \mathcal{S}_+$. Let us find an orthogonal matrix U, such that $UAU^{-1} = D$ is a diagonal matrix, $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are (non-negative) eigenvalues of A. Let $UBU^{-1} = C$. Then C is a positive semidefinite matrix and

$$\langle A, B \rangle = \langle D, C \rangle = \sum_{i=1}^{n} \lambda_i c_{ii} \geq 0,$$

since the diagonal of a positive semidefinite matrix is non-negative (see Sections 19.1 and 19.2).

Let us prove that $(\mathcal{S}_+)^* \subset \mathcal{S}_+$. Let $A \in (\mathcal{S}_+)^*$. Then $\langle A, B \rangle \geq 0$ for all positive semidefinite matrices B. Let $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ be a vector, and let $X = (x_{ij})$, where $x_{ij} = \xi_i \xi_j$ (in other words, $X = x \otimes x$, see Section 19.1). Then X is positive semidefinite, and, therefore,

$$\langle A, X \rangle = \sum_{ij=1}^{n} a_{ij} \xi_i \xi_j \geq 0.$$

Therefore, A is a positive semidefinite matrix (cf. Section 19.2). □

PROBLEM.

1°. Let B be a positive definite $n \times n$ matrix. Prove, that if X is an $n \times n$ positive semidefinite matrix, such that $\langle B, X \rangle = 0$, then $X = 0$.

Let E be a vector space and let $K \subset E$ be a convex cone. Let $C \subset E$ be a convex set and suppose that $K = \operatorname{cone}(C)$. We recall (Section 18.1) that if $0 \notin C$ and every non-zero vector $x \in K$ admits a unique representation $x = \lambda y$ for $\lambda > 0$ and $y \in C$, then C is called a base of K. We recall that in $\mathbb{R}^d$, a cone with a compact base is closed (cf. Problem 1, Section 18.1).

(39.2) Lemma. *Let $K \subset \mathbb{R}^d$ be a convex cone with a compact base and let $T : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a linear transformation such that $K \cap \ker T = 0$. Then the image $T(K) \subset \mathbb{R}^m$ is a cone with a compact base, and, therefore, is closed.*

Proof. Let $C \subset \mathbb{R}^d$ be a compact base of K. Then $T(C) \subset \mathbb{R}^m$ is a compact convex set, not containing the origin. Then there exists a hyperplane $H \subset \mathbb{R}^m$, strictly separating 0 from $T(C)$ (see Theorem 22.3). Suppose that $H = \{x : f(x) = \alpha\}$, where $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a linear function and $\alpha > 0$ is a number. Let $B \subset H$ be the "radial projection" of $T(C)$ onto H: $x \longmapsto \alpha x / f(x)$. Then B is a compact convex base of $T(K)$. □

PROBLEMS.

1. Construct an example of a convex closed cone $K \subset \mathbb{R}^3$ and a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, such that $T(K)$ is not closed.

2. Let $C \subset \mathbb{R}^d$ be a non-empty compact convex set, such that $0 \notin C$. Prove, that $K = \operatorname{cone}(C)$ is a cone with a compact base.

3. Let E be a locally convex topological vector space and let $C \subset E$ be a nonempty compact convex set, such that $0 \notin C$. Prove, that $K = \operatorname{cone}(C)$ is a cone with a compact base.

4. Let E be a locally convex topological vector space and let $K \subset E$ be a cone with a compact base. Prove, that K is closed.

5. Prove the following generalization of Lemma 39.2: let E_1 and E_2 be locally convex topological vector spaces, let $K \subset E_1$ be a cone with a compact base and let $T : E_1 \longrightarrow E_2$ be a continuous linear transformation, such that $K \cap \ker\ T = 0$. Then $T(K) \subset E_2$ is a closed cone.

(39.3) Semidefinite programs

Let us fix $n \times n$ symmetric matrices $A_1, \ldots, A_m, C$ and real numbers $\beta_1, \ldots, \beta_m$. Consider the problem in the canonical form:

Problem I

Find $\gamma = \inf\langle C, X\rangle$

Subject to: $\langle A_i, X\rangle = \beta_i$ for $i = 1, \ldots, m$ and $X \succeq 0$,

and the dual program in the standard form:

Problem II

Find $\beta = \sup \sum_{i=1}^{m} \beta_i p_i$

Subject to: $\sum_{i=1}^{m} p_i A_i \preceq C$

(39.4) Proposition. *Suppose that some linear combination*

$$B = \alpha_1 A_1 + \ldots + \alpha_m A_m + \rho C$$

is a positive definite matrix and that there is a feasible point in Problem I, Section 39.3. Then $\gamma = \beta$, and, if $\gamma > -\infty$, there is an optimal solution to Problem I.

Proof. We use Theorem 36.5. Consider the linear transformation $T : \mathbb{R}^d \longrightarrow \mathbb{R}^m$, where $d = n(n+1)/2$ and

$$T(X) = \big(\langle A_1, X\rangle, \ldots, \langle A_m, X\rangle, \langle C, X\rangle\big).$$

We claim that $\ker\ T \cap \mathcal{S}_+ = 0$. Indeed, if $X \in \ker\ T$, then

$$\langle C, X\rangle = \langle A_1, X\rangle = \ldots = \langle A_m, X\rangle = 0$$

and hence $\langle B, X\rangle = 0$. Since B is positive definite and X is positive semidefinite, by Problem 1, Section 39.1, it follows that $X = 0$. By Lemma 39.2, $T(\mathcal{S}_+)$ is a closed cone. We use Theorem 36.5. $\square$

PROBLEMS.

1. Let $n = 2, m = 1$, $\beta_1 = 1$, and let

$$C = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in Problems I, II of Section 39.3. Prove, that $\gamma = \beta = 1/2$, that there is no optimal solution in Problem I and that there is an optimal solution in Problem II.

2. Let $n = 2, m = 2$, $\beta_1 = 1$, $\beta_2 = 1/2$, and let

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 0 \end{pmatrix}$$

in Problems I, II of Section 39.3. Prove, that $\gamma = +\infty$ (there are no feasible points in Problem I) and that $\beta = 0$.

Some unrelated problems:

3. Let $c_0, \ldots, c_m \in \mathbb{R}^d$ be vectors and let $\langle c_i, x\rangle \leq 0; i = 0, \ldots, m$ be a system of linear inequalities in $\mathbb{R}^d$. The inequality $\langle c_0, x\rangle \leq 0$ is called *active*, provided there is an $x \in \mathbb{R}^d$, such that $\langle c_i, x\rangle \leq 0$ for $i = 1, \ldots, m$ and $\langle c_0, x\rangle > 0$. Prove the Farkas Lemma: the inequality $\langle c_0, x\rangle \leq 0$ is not active if and only if $c_0 = \lambda_1 c_1 + \ldots + \lambda_m c_m$ for some non-negative $\lambda_1, \ldots, \lambda_m$.

4. Write the problem dual to the Diet Problem (Example 12.4). Find an interpretation of the dual problem.

Monday, April 6

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40. Application: Cliques and the Chromatic Number of a Graph

(40.1) Definition. Let $G = (V, E)$ be a graph with the set of vertices $V = \{1, \ldots, n\}$ and the set of edges E. We assume that the edges are undirected and that there are no loops or multiple edges. A *clique* of G is a set of vertices $W \subset V$, such that every two distinct vertices $i, j \in W$ are connected by an edge of the graph: $(ij) \in E$. The largest number of vertices in a clique of G is called the *clique number* and denoted by $\omega(G)$. A *k-coloring* of G is a map $\phi : V \longrightarrow \{1, \ldots, k\}$, such that $\phi(i) \neq \phi(j)$ if $(ij) \in E$. The smallest k, such that a k-coloring exists, is a called the *chromatic number* of G and denoted $\chi(G)$.

PROBLEMS.

1°. Prove, that $\omega(G) \leq \chi(G)$.

2°. Construct an example of a graph G, such that $\omega(G) < \chi(G)$.

A graph $G = (V, E)$, $|V| = n$ is called *planar*, if one can choose n distinct points $v_i \in \mathbb{R}^2$, $i = 1, \ldots, n$ and, for every edge $(ij) \in E$, connect v_i and v_j by a smooth curve, in such a way that every two curves may intersect in their common vertex only.

3. Prove, that $\omega(G) \leq 4$ for any planar graph G.

4. Prove, that $\chi(G) \leq 5$ for any planar graph G.

5*. Prove, that $\chi(G) \leq 4$ for any planar graph G.

Remark. The problem of computing $\omega(G)$ and $\chi(G)$ for general graphs are computationally hard ("NP-hard"). Moreover, it turns out that even to approximate $\omega(G)$ and $\chi(G)$ within a factor of $n^{1-\epsilon}$, where $\epsilon > 0$ and n is the number of vertices, is hard.

(40.2) Semidefinite relaxation for the clique number

Let $d = n(n+1)/2$ and let $\mathbb{R}^d$ be the space of all $n \times n$ symmetric matrices. For $1 \leq i \neq j \leq n$, let A_{ij} be the $n \times n$ matrix such that the entries on the intersection of the i-th row and j-th column and the j-th row and i-th column are 1's and all other entries are 0. Let J be the $n \times n$ matrix filled by 1's and let I be the $n \times n$ identity matrix. Given a graph $G = (V, E)$, consider the following problem of semidefinite programming.

Find $\vartheta(G) = \sup\langle J, X\rangle$

Such that: $\langle A_{ij}, X\rangle = 0$ for every pair $(ij) \notin E$, $\langle I, X\rangle = 1$ and $X \succeq 0$.

In other words, we are seeking to maximize the sum of the entries of a positive semidefinite matrix $X = (x_{ij})$ of the trace 1, provided $x_{ij} = 0$ if (ij) is not an edge of G.

The value $\vartheta(G)$ was introduced by L. Lovász and is called *Lovász's theta-function*, see L. Lovász, On the Shannon capacity of a graph, *IEEE Transactions on Information Theory*, **25** (1979), 1–7. The function $\vartheta(G)$ is easy to compute.

(40.3) Proposition. *We have $\vartheta(G) \geq \omega(G)$.*

Proof. Let $W \subset V$ be a clique and let $|W| = k$. Let $x = (\xi_1, \ldots, \xi_n)$, where

$$\xi_i = \begin{cases} 1 & \text{if } i \in W \\ 0 & \text{if } \ i \notin W. \end{cases}$$

Let $X = (x_{ij})$, where $x_{ij} = (\xi_i \xi_j)/k$. Then X is a feasible point of the problem and $\langle J, X\rangle = k$. Therefore, $\vartheta(G) \geq \langle J, X\rangle = k = |W|$. □

Problem.

1. Let G be the pentagon, that is a graph with 5 vertices $1, 2, 3, 4, 5$ and 5 edges $(12), (23), (34), (45)$, and (15). Prove, that $\vartheta(G) = \sqrt{5}$.

(40.4) The dual program

The program of Section 40.2 can be written in the canonical form by replacing $\sup\langle J, X\rangle$ by $-\inf\langle -J, X\rangle$. Then the dual program is equivalent to the following:

Find $\inf \tau$

Subject to: $\tau I - J - \sum\limits_{(ij)\notin E} x_{ij} A_{ij} \succeq 0,$

where τ and x_{ij} for $(ij) \notin E$ are arbitrary real variables. We note that the matrix $\tau I - \Big(J + \sum\limits_{(ij)\notin E} x_{ij} A_{ij}\Big)$ is positive semidefinite, if and only if the largest eigenvalue of the matrix $\Big(J + \sum\limits_{(ij)\notin E} x_{ij} A_{ij}\Big)$ does not exceed τ. Hence the dual problem is equivalent to finding the infimum τ_{inf} of the maximum eigenvalue $\tau(Y)$ of a symmetric matrix $Y = (y_{ij})$ such that

$$y_{ij} = \begin{cases} 1 & \text{if } (ij) \in E \quad \text{or} \quad i = j \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

(40.5) Proposition. *We have* $\tau_{\text{inf}} = \vartheta(G)$.

Proof. Follows by Proposition 39.4. $\square$

Finally, we claim

(40.6) Proposition. *We have* $\vartheta(G) \leq \chi(G)$.

Sketch of Proof. Let us fix a k-coloring of G. The coloring induces a partition $V = V_1 \cup \ldots \cup V_k$ of the set of vertices, where V_s is the set of vertices of the s-th color. Using Proposition 40.5 and the definition of τ_{inf} (see Section 40.4), it suffices to construct an $n \times n$ symmetric matrix $Y = (y_{ij})$ whose eigenvalues do not exceed k and such that $y_{ij} = 1$ if $i = j$ or if i and j belong to different sets V_s of the partition. One should choose

$$y_{ij} = \begin{cases} -k+1 & \text{if } \ i \neq j \quad \text{and} \quad i, j \in V_s \quad \text{for some} \quad s \\ 1 & \text{otherwise.} \end{cases}$$

$\square$

Remark. We conclude, that although the clique number $\omega(G)$ and the chromatic number $\chi(G)$ is hard to compute, it is easy to compute a number $\vartheta(G)$ such that $\omega(G) \leq \vartheta(G) \leq \chi(G)$.

See M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, Berlin, 1988.

Wednesday, April 8

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41. Application: Linear Optimal Control

(41.1) Linear programs in general form

Recall from Section 35.4 the general form of a linear programming problem and its dual. We have two dualities $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ and $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$. Let us fix convex cones $K_1 \subset E_1$ and $K_2 \subset E_2$ and let $K_1^* \subset F_1$ and $K_2^* \subset F_2$ be the dual cones: $K_1^* = \big\{f \in F_1 : \langle e, f\rangle \geq 0 \text{ for all } e \in K_1\big\}$ and $K_2^* = \big\{f \in F_2 : \langle e, f\rangle \geq 0$ for all $e \in K_2\big\}$.

With a linear transformation $A : E_1 \longrightarrow E_2$, a vector $b \in E_2$, and a vector $c \in F_1$, we associate a pair of linear programs:

The primal program

Find $\gamma = \inf\langle x, c\rangle_1$

Subject to: $Ax \geq_{K_2} b, \quad x \geq_{K_1} b$

and

The dual program

Find $\beta = \sup\langle b, p\rangle_2$

Subject to: $A^*p \leq_{K_1^*} c, \quad p \geq_{K_2^*} 0.$

(41.2) Theorem. *For any feasible point x of the primal problem and any feasible point p in the dual problem, $\langle x, c\rangle_1 \geq \langle b, p\rangle_2$. If x is feasible in the primal problem and p is feasible in the dual problem and $\langle x, c\rangle = \langle b, p\rangle$, then x is optimal in the primal problem and p is optimal in the dual problem and*

$$\langle p, Ax - b\rangle_2 = \langle c - A^*p, x\rangle_1 = 0$$

("complementary slackness").

Proof. The proof is very similar to those of Lemma 36.2 and Corollary 36.3. If $x \in E_1$ is a feasible point in the primal problem and $p \in F_2$ is a feasible point in the dual problem, then

$$\langle x, c\rangle_1 = \langle x, A^*p\rangle_1 + \langle x, c - A^*p\rangle_1 = \langle Ax, p\rangle_2 + \langle x, c - A^*p\rangle_1$$

$$= \langle b, p\rangle_2 + \langle Ax - b, p\rangle_2 + \langle x, c - A^*p\rangle_1.$$

Since $Ax - b \in K_2$ and $p \in K_2^*$, we have $\langle Ax - b, p\rangle_2 \geq 0$. Since $x \in K_1$ and $c - A^*p \in K_1^*$, we have $\langle x, c - A^*p\rangle_1 \geq 0$. Now, if $\langle x, c\rangle_1 = \langle b, p\rangle_2$, we must have $\langle Ax - b, p\rangle_2 = \langle x, c - A^*p\rangle_1 = 0$. $\square$

(41.3) A problem of optimal control

As we discussed in Section 27.2, a typical problem of linear control can be stated as

Minimize $\displaystyle\int_0^1 g(\tau)u(\tau)\, d\tau$

Subject to: $\displaystyle\int_0^1 f_i(\tau)u(\tau)\, d\tau = \alpha_i$ for $i = 1, \ldots, m$ and $0 \leq u(\tau) \leq 1$ for almost all $\tau \in [0,1]$,

where $u \in L^\infty[0,1]$ is a variable ("control"), $g, f, f_1, \ldots, f_m \in L^1[0,1]$ are given functions, and $\alpha_1, \ldots, \alpha_m$ are given numbers.

We can consider it as a problem of linear programming. Let $E_1 = L^\infty[0,1]$, $F_1 = L^1[0,1]$ and let

$$\langle x, y\rangle_1 = \int_0^1 x(\tau)y(\tau)\, d\tau$$

for $x \in E_1$ and $y \in F_1$.

Let $K_1 = \big\{x \in E_1 : x(\tau) \geq 0 \text{ for almost all } \tau \in [0,1]\big\}$.

Then $K_1^* = \big\{y \in F_1 : y(\tau) \geq 0 \text{ for almost all } \tau \in [0,1]\big\}$.

Let $E_2 = \mathbb{R}^m \oplus L^\infty[0,1]$, $F_2 = \mathbb{R}^m \oplus L^1[0,1]$ and let

$$\langle (x,h), (y,p)\rangle_2 = \sum_{i=1}^m \xi_i\eta_i + \int_0^1 h(\tau)p(\tau)\, d\tau,$$

where $x = (\xi_1, \ldots, \xi_m)$ and $y = (\eta_1, \ldots, \eta_m)$.

Let $K_2 = \big\{(x,h) : x \in \mathbb{R}^m \text{ is any and } h(\tau) \geq 0 \text{ for almost all } \tau \in [0,1]\big\}$.

Then $K_2^* = \big\{(0,p) : p(\tau) \geq 0 \text{ for almost all } \tau \in [0,1]\big\}$.

Let $A : E_1 \longrightarrow E_2$ be the transformation

$$Au = \Big(\int_0^1 f_1(\tau)u(\tau)\ d\tau, \dots, \int_0^1 f_m(\tau)u(\tau)\ d\tau, -u\Big).$$

Let $b = (\alpha_1, \dots, \alpha_m, -1)$, where "$-\mathbf{1}$" is the function that is identically -1. Then the problem can be stated as

Find: $\gamma = \inf\langle g, u\rangle_1$

Subject to: $Au \geq_{K_2} b$, $u \geq_{K_1} 0$.

(41.4) The dual problem

For $y = (\eta_1, \dots, \eta_m)$ and $p \in L^1[0,1]$ we have

$$\langle Au, (y,p)\rangle_2 = \sum_{i=1}^m \xi_i \int_0^1 f_i(\tau)u(\tau)\ d\tau - \int_0^1 u(\tau)p(\tau)\ d\tau$$

$$= \int_0^1 \Big(\sum_{i=1}^m \xi_i f_i(\tau) - p(\tau)\Big)u(\tau)\ d\tau = \langle u, A^*(y,p)\rangle_1.$$

Therefore,

$$A^*(y,p) = -p(\tau) + \sum_{i=1}^m \xi_i f_i(\tau).$$

Hence the dual problem is

Find: $\beta = \sup \sum_{i=1}^m \alpha_i\lambda_i - \int_0^1 p(\tau)\ d\tau$

Subject to: $-p(\tau) + \sum_{i=1}^m \lambda_i f_i(\tau) \leq g(\tau)$ and $p(\tau) \geq 0$ for almost all $\tau \in [0,1]$,

where $\lambda_1, \dots, \lambda_m$ are real variables and $p(\tau) \in L^1[0,1]$ is a variable function.

(41.5) Proposition. *Suppose that $(\lambda_1, \dots, \lambda; p)$ is an optimal solution to the dual problem. Then*

$$p(\tau) = \max\Big\{0, -g(\tau) + \sum_{i=1}^m \lambda_i f_i(\tau)\Big\}$$

for almost all $\tau \in [0,1]$.

Proof. Obviously,

$$p(\tau) = \max\Big\{0, -g(\tau) + \sum_{i=1}^m \lambda_i f_i(\tau)\Big\}$$

is a feasible point in Problem 41.4 and for every feasible point p_1, we have $p_1(\tau) \geq p(\tau)$ for almost all $\tau \in [0,1]$. Unless $p_1(\tau) = p(\tau)$ for almost all $\tau \in [0,1]$, $\int_0^1 p(\tau)\ d\tau < \int_0^1 p_1(\tau)\ d\tau$, and the value of the objective function is smaller. $\square$

(41.6) Proposition. *Suppose that there is a strong duality between the problems (41.3) and (41.4), that is $\gamma = \beta$. Suppose further, that $u(\tau)$ is an optimal solution in the primal problem and $(\lambda_1, \dots, \lambda_m; p)$ is an optimal solution in the dual problem. Suppose that the set of roots of the function*

$$g(\tau) - \sum_{i=1}^{m} \lambda_i f_i(\tau)$$

has measure zero. Then

$$u(\tau) = \begin{cases} 1 & \text{if } p(\tau) > 0 \\ 0 & \text{if } p(\tau) = 0 \end{cases}$$

for almost all $u \in [0, 1]$.

Proof. We use the "complementary slackness" conditions of Theorem 41.2. We have

$$\int_0^1 p(\tau)\big(1 - u(\tau)\big)\ d\tau = 0$$

Therefore, if $p(\tau) > 0$, then $u(\tau) = 1$ (for almost all τ). Moreover,

$$\int_0^1 \Big(g(\tau) - \sum_{i=1}^{m} \lambda_i f_i(\tau) + p(\tau)\Big) u(\tau) = 0.$$

From Proposition 41.5,

$$p(\tau) = 0 \quad \text{implies} \quad \Big(g(\tau) - \sum_{i=1}^{m} \lambda_i f_i(\tau)\Big) > 0$$

for almost all τ. The proof follows. $\square$

PROBLEM.

1. Prove, that there is a strong duality, that is $\gamma = \beta$.
Hint: Rewrite the primal problem in the canonical form:

Find $\displaystyle\int_0^1 g(\tau) u(\tau)\ d\tau$

Subject to: $\displaystyle\int_0^1 f_i(\tau) u(\tau)\ d\tau = \alpha_i$ for $i = 1, \dots, m$, $u + v = 1$, and $u(\tau), v(\tau) \geq_K 0$ for almost all τ,

where $K \subset L^\infty[0, 1] \oplus L^\infty[0, 1]$ is the cone, consisting of pairs of functions (u, v) : $u(\tau) \geq 0, v(\tau) \geq 0$ for almost all τ and $u + v =$ constant function. Prove, that K has a compact base (cf. Section 26.1) and use Problem 5, Section 39.2.

42. Application: Chebyshev Approximation

(42.1) Chebyshev approximation as a linear programming problem

Recall from Section 7 the problem of Chebyshev approximation: suppose we are given continuous functions $f_1, \ldots, f_m$ on the interval $[0,1]$ and yet another continuous function g on $[0,1]$. We want to find a linear combination $\alpha_1 f_1 + \ldots + \alpha_m f_m$, such that the maximum deviation

$$\max_{0 \leq \tau \leq 1} \bigl|\alpha_1 f_1(\tau) + \ldots + \alpha_m f_m(\tau) - g(\tau)\bigr|$$

is the smallest possible.

Introducing a new real variable λ, we can write the problem in the following form:

Find $\inf \lambda$

Subject to:

$$\lambda + \sum_{i=1}^m \alpha_i f_i(\tau) \geq g(\tau) \quad \text{for all} \quad \tau \in [0,1]$$

and

$$\lambda - \sum_{i=1}^m \alpha_i f_i(\tau) \geq -g(\tau) \quad \text{for all} \quad \tau \in [0,1].$$

(42.2) The dual problem

Let $E_1 = \mathbb{R}^{m+1}$, let $F_1 = \mathbb{R}^{m+1}$ and let $\langle\rangle_1 : E_1 \times F_1 \longrightarrow \mathbb{R}$ be the standard duality (see Example 31.2).

Let $E_2 = C[0,1] \oplus C[0,1]$, where $C[0,1]$ is the space of continuous functions on the interval $[0,1]$ and let $F_2 = V[0,1] \oplus V[0,1]$, where $V[0,1]$ is the space of signed Borel measures on $[0,1]$ (see Section 29 and Example 31.2.4). Let $\langle\rangle_2 : E_2 \times F_2 \longrightarrow \mathbb{R}$ be the duality:

$$\bigl\langle (h_1, h_2), (\mu_1, \mu_2) \bigr\rangle_2 = \int_0^1 h_1 \, d\mu_1 + \int_0^1 h_2 \, d\mu_2.$$

Let $K \subset F_2$ be the cone

$$K = \Bigl\{ (h_1, h_2) : \quad h_1(\tau) \geq 0 \text{ and } h_2(\tau) \geq 0 \quad \text{for all} \quad \tau \in [0,1] \Bigr\}.$$

Then the dual cone is

$$K^* = \Bigl\{ (\mu_1, \mu_2) : \mu_1 \quad \text{and} \quad \mu_2 \quad \text{are (non-negative) measures} \Bigr\}.$$

Let $A : E_1 \longrightarrow E_2$ be the transformation

$$A\bigl(\alpha_1, \ldots, \alpha_m; \lambda\bigr) = \Bigl(\lambda \mathbf{1} + \sum_{i=1}^m \alpha_i f_i, \quad \lambda \mathbf{1} - \sum_{i=1}^m \alpha_i f_i \Bigr),$$

where $\mathbf{1}$ is the function identically equal to 1.

Then

$$\big\langle A(\alpha_1, \ldots, \alpha_m; \lambda), \quad (\mu_1, \mu_2)\big\rangle_2$$

$$= \sum_{i=1}^m \alpha_i \int_0^1 f_i \ d\mu_1 - \sum_{i=1}^m \alpha_i \int_0^1 f_i \ d\mu_2 + \lambda \int_0^1 \ d\mu_1 + \lambda \int_0^1 \ d\mu_2.$$

Therefore, the dual transformation $A^* : F_2 \longrightarrow F_1$ is given by the formula:

$$A^*(\mu_1, \mu_2)$$

$$= \Big(\int_0^1 f_1 \ d\mu_1 - \int_0^1 f_1 \ d\mu_2, \ldots, \int_0^1 f_m \ d\mu_1 - \int_0^1 f_m \ d\mu_2, \int_0^1 \ d\mu_1 + \int_0^1 \ d\mu_2\Big).$$

Let $c = (0, \ldots, 0, 1) \in \mathbb{R}^{m+1}$ and let $b = (g, -g) \in C[0,1] \oplus C[0,1]$.

Problem 42.1 can be written as a problem of linear programming in the standard form:

we want to find $\inf \lambda = \inf\langle x, c\rangle_1$, where $x \in E_1$ is subject to the constraint $Ax \geq_K b$.

Then the dual problem is maximize $\langle b, p\rangle_2$, where $p \in F_2$ is subject to the constraint $A^*p = c$ and $p \geq_{K^*} 0$. In other words, the dual problem is:

Find $\sup \displaystyle\int_0^1 g \ d\mu_1 - \int_0^1 g \ d\mu_2$

Subject to:

$$\int_0^1 f_i \ d\mu_1 - \int_0^1 f_i \ d\mu_2 = 0 \quad \text{for} \quad i = 1, \ldots, m$$

and

$$\int_0^1 \ d\mu_1 + \int_0^1 \ d\mu_2 = 1,$$

where μ_1 and μ_2 are positive measures on $[0,1]$.

(42.3) A reformulation of the dual problem

Let us introduce a norm $\|\cdot\|$ on the space $V[0,1]$:

$$\|\phi\| = \max\Big\{\Big|\int f \ d\phi\Big|, \quad \text{where} \quad |f(\tau)| \leq 1 \quad \text{for all} \quad 0 \leq \tau \leq 1\Big\}.$$

Then the problem of Section 42.3 can be rewritten as:

Find $\gamma = \sup \displaystyle\int_0^1 g \ d\phi$

Subject to:

$$\int_0^1 f_i \ d\phi = 0 \quad \text{for} \quad i = 1, \ldots, m \qquad \text{and} \qquad \|\phi\| \leq 1.$$

Indeed, we let $\phi = \mu_1 - \mu_2$.

(42.4) Proposition.

(1) *There is always an optimal solution in the primal problem 42.1 and there is always an optimal solution in the dual problem 42.3.*

(2) *An optimal solution in problem 42.3 can be chosen as a linear combination of not more than $m+1$ delta-measures:*

$$\phi = s_1\delta_{t_1} + \ldots + s_{m+1}\delta_{t_{m+1}},$$

where $t_i \in [0,1] : i = 1, \ldots, m$ and $|s_1| + \ldots + |s_m| = 1$.

(3) *There is always strong duality, that is the optimal values of the objective functions in problems 42.1 and 42.3 coincide.*

(4) *Let $f = \alpha_1 f_1 + \ldots + \alpha_m f_m$ and let $\lambda = \max_{0 \le \tau \le 1} |g(\tau) - f(\tau)|$. Then f is the best Chebyshev approximation to g if and only if there are $m+1$ points $t_i \in [0,1]$ and $m+1$ numbers s_i, not all zero, such that:*

$$\sum_{i=1}^{m+1} s_i f_j(t_i) = 0 \quad \text{for all} \quad j = 1, \ldots, m$$

and if $s_i > 0$ then $g(t_i) - f(t_i) = \lambda$ and if $s_i < 0$ then $g(t_i) - f(t_i) = -\lambda$.

The proof of Proposition 42.4 is in the problems below.

PROBLEMS.

1. Prove, that there is always an optimal solution in the primal problem.

Hint: Without loss of generality, we may assume that $f_1, \ldots, f_m$ are linearly independent. For $a = (\alpha_1, \ldots, \alpha_m)$, let $l(a) = \max_{0 \le \tau \le 1} |\alpha_1 f_1 + \ldots + \alpha_m f_m|$. Prove, that $l(a)$ is a norm on $\mathbb{R}^m$. Let $G = \max_{0 \le \tau \le 1} |g(\tau)|$. Prove, that the primal problem reduces to finding the minimal value of a continuous function $d(a) = \max_{0 \le \tau \le 1} |\alpha_1 f_1(\tau) + \ldots + \alpha_m f_m(\tau) - g(\tau)|$ on the compact set $\{a : l(a) \le 2G\}$.

2. Prove, that there is always an optimal solution in the dual problem.

Hint: Show, that the dual problem is the problem of finding the maximum value of a continuous linear function on a non-empty compact set in the weak* topology in $V[0,1]$, cf. Section 29.3.

3. Prove Part 2 of Proposition 42.4.

Hint: Prove, that the extreme points of the set $\{\phi \in V[0,1] : \|\phi\| = 1\}$ are signed delta measures δ_t and $-\delta_t$ for some $t \in [0,1]$ (cf. Problem 1, Section 30.1). Use Theorem 28.2 (cf. Proposition 30.3).

4. Check, that there is a weak duality: if ϕ is a feasible point in problem 42.3 and $(\alpha_1, \ldots, \alpha_m; \lambda)$ is a feasible point in problem 42.1, then $\int_0^1 g \, d\phi \le \lambda$.

5. Prove Part 3 of Proposition 42.4.

Hint: Prove, that the cone $K^* = \Big\{(\mu_1, \mu_2) :$ μ_1 and μ_2 are non-negative Borel probability measures $\Big\}$ has a compact base (cf. Section 29.3). Use Theorem 36.5 and Problem 5, Lemma 39.2.

6. Prove Part 4 of Proposition 42.4.

Hint: Use the "complementary slackness", Theorem 41.2.

Suppose that $f_i = \tau^{i-1}$ for $i = 1, \ldots, m$. Then $f = \alpha_1 f_1 + \ldots + \alpha_m f_m$ is a polynomial.

See A.M. Vershik and V. Temel't, Certain questions on approximation of the optimal value of infinite-dimensional linear programming problems, (Russian) *Sibirsk. Mat. Z.*, **9**(1968), 790–803.

7. Prove, that for any continuous function g, the best approximating polynomial is unique, and is completely characterized by the property that there exist $m+1$ points $0 \leq t_1 < t_2 < \ldots < t_{m+1} \leq 1$ such that $f(t_i) - g(t_i) = \epsilon(-1)^i \lambda$, where $\lambda = \max\limits_{0 \leq \tau \leq 1} |f(\tau) - g(\tau)|$ and $\epsilon \in \{-1, 1\}$.

Monday, April 13

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43. The Schur-Horn Theorem

Let S_n be the set of all $n!$ permutations of the set $\{1, \ldots, n\}$, that is, the set of all one-to-one maps $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$. If $\sigma \in S_n$ is a permutation, let σ^{-1} be the inverse permutation, so $\sigma^{-1}(\sigma(i)) = i$ for $i = 1, \ldots, n$.

(43.1) Definition. Let us fix a vector $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. For a permutation $\sigma \in S_n$, let $\sigma(a) \in \mathbb{R}^n$ be the vector $\sigma(a) = \big(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}\big)$. The convex hull $P(a) = \operatorname{conv}\big(\sigma(a) : \sigma \in S_n\big)$ is called the *permutation polytope*. In other words, the permutation polytope $P(a)$ is the convex hull in $\mathbb{R}^n$ of the set of all vectors $\sigma(a)$, obtained from a by a permutation of the coordinates.

PROBLEMS.

1°. Prove, that $P(a)$ lies in the hyperplane $x = (\xi_1, \ldots, \xi_n) : \xi_1 + \ldots + \xi_n = \alpha_1 + \ldots + \alpha_n$.

2. Suppose, that not all the coordinates of a are equal. Prove, that $\dim\ P(a) = n - 1$.

3. Prove, that $P(a)$ has $n!$ vertices if and only if the coordinates $\alpha_1, \ldots, \alpha_n$ are distinct.

4. Let us choose $a = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} \geq \alpha_n$. Let $x = (\xi_1, \ldots, \xi_n)$ be a vector. Prove, that $y \in P(a)$, if and only if for any subset $\omega \subset \{1, \ldots, n\}$ one has $\displaystyle\sum_{i \in \omega} \xi_i \leq \sum_{i=1}^{|\omega|} \alpha_i$, where $|\omega|$ is the cardinality of the set ω. In particular, if $\xi_1 \geq \ldots \geq \xi_n$, then $x \in P(a)$ if and only if $\displaystyle\sum_{i=1}^{k} \xi_i \leq \sum_{i=1}^{k} \alpha_i$ for $k = 1, \ldots, n$ (Rado's Theorem).

See V.A. Emelichev, M.M. Kovalev, and M.K. Kravtsov, *Polytopes, Graphs, and Optimization*, Cambridge Univ. Press, 1984.

5. Let us choose $a = (0, 1 \ldots, n-1)$. The polytope $P_{n-1} = P(a)$ for this choice of a is called the *permutohedron*. Prove, that the permutohedron is a *zonotope*, that is the Minkowski sum of finitely many (namely, $\binom{n}{2}$) intervals.

Hint: If $e_1, \ldots, e_n$ is a standard basis of $\mathbb{R}^n$, for $i < j$ choose $I_{ij} = [(e_i - e_j)/2, (e_j - e_i)/2]$. Then

$$\frac{n-1}{2}(1, \ldots, 1) + \sum_{i<j} I_{ij} = P_{n-1},$$

see, for example, G.M. Ziegler, *Lectures on Polytopes*, Springer, 1995.

6. Draw a picture of the permutohedron P_2.

7. Draw a picture of the permutohedron P_3 (since P_3 lies in the hyperplane in $\mathbb{R}^4$, it is a 3-dimensional polytope).

8°. For a permutation $\sigma \in S_n$, let $X^\sigma = (x^\sigma_{ij})$ be the $n \times n$ permutation matrix:

$$x^\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Prove, that $\sigma(a) = X^\sigma a$, where vectors in $\mathbb{R}^n$ are interpreted as n-columns of real numbers.

(43.2) Schur-Horn Theorem. *Let us fix a positive integer n and n real numbers $\lambda_1, \ldots, \lambda_n$. Let $l = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be a vector.*

(1) *Let $A = (\alpha_{ij})$ be an $n \times n$ real symmetric (or complex Hermitian) matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the diagonal $a = (\alpha_{11}, \ldots, \alpha_{nn})$ lies in the permutation polytope $P(l)$: $a \in P(l)$ (Schur's Theorem).*

(2) *Let $a \in P(l)$ be a point from the permutation polytope. Then there exists an $n \times n$ real symmetric matrix $A = (\alpha_{ij})$ with the eigenvalues $\lambda_1, \ldots, \lambda_n$ and the diagonal $a = (\alpha_{11}, \ldots, \alpha_{nn})$ (Horn's Theorem).*

Proof. Let us prove Part 1. Let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Suppose that $A = (\alpha_{ij})$ is a real symmetric $n \times n$ matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ (the proof for complex Hermitian matrices is completely analogous). Then $A = UDU^t$ for some orthogonal matrix $U = (u_{ij})$. Hence the diagonal entries of A can be written as

$$\alpha_{kk} = \sum_{i=1}^n u^2_{ki}\lambda_i.$$

Let $B = (\beta_{ij})$ be the $n \times n$ matrix such that $\beta_{ij} = u^2_{ij}$. Hence we may write $a = Bl$, where a and l are interpreted as n-columns of real numbers. Since U is an orthogonal matrix, the matrix B is doubly stochastic (see Definition 13.1), that is, B is a non-negative matrix with all row and column sums equal to 1. By the Birkhoff - von Neumann Theorem (see Theorem 13.2), B can be written as a convex combination of permutation matrices X^σ, $\sigma \in S_n$. Using Problem 8, Section 43.1, we conclude that a is a convex combination of $\sigma(l)$, that is $a \in P(l)$.

For he proof of Part 2, see, A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York, 1979 and problems below. □

PROBLEMS.

1. Let $\gamma_1, \ldots, \gamma_{n-1}$ and $\lambda_1, \ldots, \lambda_n$ be real numbers such that $\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \ldots \geq \gamma_{n-1} \geq \lambda_n$. Prove, that the diagonal $n \times n$ matrix $D = \operatorname{diag}(\gamma_1, \ldots, \gamma_{n-1})$ can be appended to an $n \times n$ symmetric matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$.

2. Let $l = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Suppose that $a = (\alpha_1, \ldots, \alpha_n) \in P(l)$. Prove, that there exists a vector $c \in \mathbb{R}^{n-1}$, $c = (\gamma_1, \ldots, \gamma_{n-1})$, such that $\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \ldots \geq \gamma_{n-1} \geq \lambda_n$ and $\hat{a} = (\alpha_1, \ldots, \alpha_{n-1}) \in P(c)$.

3. Using Problems 1 and 2, prove Horn's Theorem (use induction on n).

(43.3) An application: Toda Lattices

Let us write $[A,B] = AB - BA$ for $n \times n$ matrices A and B. Let us fix $n-1$ real numbers $\beta_1, \dots, \beta_{n-1}$ and let $B = (b_{ij})$ be the skew-symmetric matrix

$$b_{ij} = \begin{cases} \beta_i & \text{if } i < n \text{ and } j = i+1 \\ -\beta_j & \text{if } \ j < n \ \text{ and } \ i = j+1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let us fix a tridiagonal symmetric matrix $L_0 = (l_{ij})$:

$$l_{ij} = \begin{cases} \alpha_i & \text{if } \ i = j \\ \beta_i & \text{if } \ i < n \text{ and } \ j = i+1 \\ \beta_j & \text{if } \ j < n \ \text{ and } \ i = j+1 \\ 0 & \text{elsewhere,} \end{cases}$$

where $\alpha_1, \dots, \alpha_n$ are some fix numbers. Consider a one-parametric family $L(t)$ of matrices that satisfy the differential equation

$$\frac{d}{dt} L(t) = [B, L(t)] \quad \text{and} \quad L(0) = L_0. \tag{43.3.1}$$

PROBLEMS.

1. Prove, that every $L(t)$ is a tridiagonal symmetric matrix.
2. Prove, that the eigenvalues of $L(t)$ do not change (equivalently, that the trace of L^k is preserved along the trajectory for any positive integer k).
3. Prove, that the solution $L(t)$ can be obtained as follows: we compute the matrix $\exp(tL_0)$. Then we factor $\exp(tL_0) = U(t)\Delta(t)$, where $U(t)$ is an orthogonal matrix and $\Delta(t)$ is an upper triangular matrix. Then $L(t) = U(t)^{-1} L_0 U(t)$

Hence it follows that the diagonal of $L(t)$ remains inside the permutation polytope $P(l)$, where l is the vector of the eigenvalues of L_0.

The equation describes the behavior of a mechanical system, called the *Toda lattice.* Consider n particles on the line of a unit mass each, with the coordinates $x_1(t), \dots, x_n(t)$ and impulses $p_1(t), \dots, p_n(t)$. The coordinates and the impulses change as time t goes. The evolution of the system is described by the Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n-1} \exp(x_k - x_{k-1}),$$

where $\exp(x_k - x_{k-1})$ stands for the energy of interaction between the k-th and the $k-1$st particles (the i-th and j-th particle do not interact for $|i-j| > 1$). Hence the behavior of the system is described by a system of equations

$$\frac{d}{dt} x_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad \frac{d}{dt} p_k = -\frac{\partial H}{\partial x_k}.$$

If we let

$$\beta_k = \frac{1}{2} \exp\{(x_k - x_{k-1})/2\} \quad \text{and} \quad \alpha_k = -\frac{1}{2} p_k$$

and construct the matrices B and L as described above, we get a solution to equation (43.3.1).

The Schur Theorem implies that the vector of velocities (=impulses) $p_1, \ldots, p_n$ changes inside a certain permutation polytope. The vertices of the polytope correspond to the configurations of equilibrium. There is only one (stable) equilibrium, attained asymptotically, where $x_1 > x_2 > \ldots > x_n$ and $p_1 \geq p_2 \geq \ldots \geq p_n$.

Wednesday, April 15

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44. The Toeplitz-Hausdorff Theorem

(44.1) The space of Hermitian matrices

We recall that a complex $n \times n$ matrix $A = (a_{ij})$ is called *Hermitian* provided $a_{ji} = \overline{a_{ij}}$ for all $i, j = 1, \ldots, n$. With a Hermitian matrix one can associate a *Hermitian form*

$$\sum_{i,j=1}^{n} a_{ij} \zeta_i \overline{\zeta_j},$$

where $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ are complex variables. We see that the value of a Hermitian form at any point $(\zeta_1, \ldots, \zeta_n)$ is a real number. The matrix A is called *positive semidefinite*, provided

$$\sum_{i,j=1}^{n} a_{ij} \zeta_i \overline{\zeta_j} \geq 0$$

for any set of numbers $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$. The space of all $n \times n$ Hermitian matrices can be identified with $\mathbb{R}^{n^2}$. Indeed, we have n degrees of freedom choosing the diagonal entries (each diagonal entry must be a real number) and then $n^2 - n$ degrees of freedom choosing the upper diagonal entries (each such an entry can be a complex number, that is, a pair of real numbers). In particular, the space of all Hermitian 2×2 matrices

$$\begin{pmatrix} \gamma & \alpha + i\beta \\ \alpha - i\beta & \delta \end{pmatrix}$$

is identified with the space $\mathbb{R}^4$ of the 4-tuples $(\alpha, \beta, \gamma, \delta)$.

(44.2) Lemma.

(1) *The set B of 2×2 positive semidefinite Hermitian matrices of the trace 1 is defined by the inequalities:*

$$B = \left\{ \begin{pmatrix} \gamma & \alpha + i\beta \\ \alpha - i\beta & \delta \end{pmatrix} : \gamma + \delta = 1 \quad \textit{and} \quad \alpha^2 + \beta^2 + (\gamma - 1/2)^2 \leq 1/4 \right\}.$$

Geometrically, B is the 3-dimensional ball in the affine hyperplane

$$H = \left\{ \begin{pmatrix} \gamma & \alpha + i\beta \\ \alpha - i\beta & \delta \end{pmatrix} : \gamma + \delta = 1 \right\}$$

centered at the matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

and of the radius 1/2.

(2) *The boundary $S = \partial B$ of the ball consists of the 2×2 positive semidefinite Hermitian matrices of the trace 1 and rank 1. In other words, any matrix from S is represented in the form*

$$\begin{pmatrix} \zeta_1\overline{\zeta_1} & \zeta_1\overline{\zeta_2} \\ \zeta_2\overline{\zeta_1} & \zeta_2\overline{\zeta_2} \end{pmatrix}$$

for some $\zeta_1, \zeta_2 \in \mathbb{C}$, such that $|\zeta_1|^2 + |\zeta_2|^2 = 1$.

(3) *The set K of all positive semidefinite 2×2 Hermitian matrices is the convex cone with the base B.*

(4) *The boundary ∂K consists of the 2×2 positive semidefinite Hermitian matrices whose rank does not exceed 1. In other words, any matrix from ∂K is represented in the form*

$$\begin{pmatrix} \zeta_1\overline{\zeta_1} & \zeta_1\overline{\zeta_2} \\ \zeta_2\overline{\zeta_1} & \zeta_2\overline{\zeta_2} \end{pmatrix}$$

for some $\zeta_1, \zeta_2 \in \mathbb{C}$.

Proof. To prove Part 1, we note that the matrix $A = \begin{pmatrix} \gamma & \alpha + i\beta \\ \alpha - i\beta & 1 - \gamma \end{pmatrix}$ is positive semidefinite if and only if $\det A = \gamma(1-\gamma)-\alpha^2-\beta^2 \geq 0$, that is, $\alpha^2+\beta^2+(\gamma-1/2)^2 \leq 1/4$. Next, $A \in \partial B$ if and only if $\det A = 0$, so rank $A = 1$ and the proof of Part 2 follows. Since every non-zero positive semidefinite matrix can be scaled to a matrix with the trace 1, the proof of Parts 3 and 4 follows. □

(44.3) Definition. Let $A = (a_{ij})$ be an $n \times n$ complex matrix. The set $R(A) \subset \mathbb{C}$ in the complex plane:

$$R(A) = \Big\{ \sum_{i,j=1}^{n} a_{ij}\zeta_i\overline{\zeta_j} : |\zeta_1|^2 + \ldots + |\zeta_n|^2 = 1 \Big\}$$

is called the *numerical range* of the matrix.

Let $\|z\|^2 = |\zeta_1|^2 + \ldots + |\zeta_n|^2$ for a $z = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$.

(44.4) The Toeplitz-Hausdorff Theorem. *The numerical range of a matrix is a convex set in the complex plane.*

Proof. First, we note that it suffices to prove the theorem for 2×2 matrices A. Indeed, let

$$f(z) = \sum_{i,j=1}^{n} a_{ij}\zeta_i\overline{\zeta_j},$$

where $z = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$. Let us choose $w_1, w_2 \in R(A)$. We want to prove that $[w_1, w_2] \subset R(A)$. Let us choose $z_1, z_2 \in \mathbb{C}^n$ such that $\|z_1\| = \|z_2\| = 1$

and $f(z_1) = w_1$ and $f(z_2) = w_2$. Let us consider a 2-dimensional complex (4-dimensional real) subspace H containing z_1 and z_2. Let us choose an orthonormal basis $\{v_1, v_2\}$ of H. Then every point $z \in H$ can be written as a linear combination $z = \xi_1 v_1 + \xi_2 v_2$, where $\xi_1, \xi_2 \in \mathbb{C}$ and

$$f(z) = \sum_{i,j=1}^{2} b_{ij} \xi_i \overline{\xi}_j$$

for some 2×2 matrix B. We observe that $w_1, w_2 \in R(B)$ and if we prove that $R(B)$ is convex, we will establish that $[w_1, w_2] \subset R(B) \subset R(A)$.

Hence we assume that $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$. Without loss of generality, we assume that not all α_{ij} are zero. Consider the space $\mathbb{R}^4$ of all 2×2 Hermitian matrices and a linear transformation $\mathcal{A} : \mathbb{R}^4 \longrightarrow \mathbb{R}^2 = \mathbb{C}$:

$$\mathcal{A} : \begin{pmatrix} \beta & \gamma \\ \overline{\gamma} & \delta \end{pmatrix} \longmapsto \alpha_{11}\beta + \alpha_{12}\gamma + \alpha_{21}\overline{\gamma} + \alpha_{22}\delta.$$

By Part 2 of Lemma 44.2, we conclude that the numerical range of A is the image of the sphere S under the transformation $\mathcal{A}$. Next, we claim that $\mathcal{A}(S) = \mathcal{A}(B)$, that is, that the images of the ball B and the sphere S coincide. Since $\mathcal{A}$ is a linear transformation and B is convex, the proof will follow. Consider the restriction of $\mathcal{A}$ onto the 3-dimensional hyperplane H containing B. The preimage $\mathcal{A}^{-1}(z) \subset H$ of every point $z \in \mathbb{C}$ is a straight line, say l_z. Hence $z \in \mathcal{A}(B)$ if and only if $l_z \cap B \neq \emptyset$ and $z \in \mathcal{A}(S)$ if and only if $l_z \cap S \neq \emptyset$. But every straight line which intersects a ball must also intersect its boundary. Hence $\mathcal{A}(S) = \mathcal{A}(B)$ and the proof follows. □

Some related convexity results.

PROBLEMS.

1. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be $n \times n$ complex Hermitian matrices. Consider the quadratic transformation $\phi : \mathbb{C}^n \longrightarrow \mathbb{R}^3$:

$$\phi(z) = \left(\sum_{i,j=1}^{n} a_{ij} \zeta_i \overline{\zeta_j}, \quad \sum_{i,j=1}^{n} b_{ij} \zeta_i \overline{\zeta_j}, \quad \sum_{i,j=1}^{n} c_{ij} \zeta_i \overline{\zeta_j} \right)$$

for $z = (\zeta_1, \ldots, \zeta_n)$. Prove, that the image $\phi(\mathbb{C}^n)$ is a convex cone in $\mathbb{R}^3$.

Hint: Reduce the problem to the case $n = 2$. Using Part 4 of Lemma 44.2, represent the image $\phi(\mathbb{C}^n)$ as the image of the boundary ∂K of the cone of positive semidefinite Hermitian matrices under a linear transformation. Prove, that the images of the boundary ∂K and the whole cone K coincide (using the fact that a straight line intersecting K must also intersect its boundary).

2. Let $q_1, q_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be real quadratic forms and let $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the unit ball. Consider the quadratic map $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^2$: $\phi(x) = \big(q_1(x), q_2(x)\big)$. Prove, that the image $\phi(B)$ is a convex set in $\mathbb{R}^2$ for $n \geq 2$.

Hint: Reduce to the case $n = 2$. Consider the set D of all 2×2 real positive semidefinite matrices X such that $\operatorname{trace}(X) \leq 1$. Prove, that D is a 3-dimensional "circular cone". Represent $\phi(B)$ as the image of the "side surface" of D under a linear transformation. Prove, that the image of the "side surface" and the image of D

under a linear transformation coincide (use the fact that a straight line intersecting D must also intersect its "side surface").

3^*. Let $q_1, q_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be real quadratic forms and let $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere. Consider the quadratic map $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^2$: $\phi(x) = \big(q_1(x), q_2(x)\big)$. Prove, that the image $\phi(S)$ is a convex set in $\mathbb{R}^2$ for $n \geq 3$.

Hint: Reduce to the case $n = 3$. It suffices to prove that the intersection of $\phi(S)$ with every straight line in $\mathbb{R}^2$ is connected. The preimage of a straight line is a quadric in $\mathbb{R}^3$. Its intersection with the unit sphere in $\mathbb{R}^3$ consists of at most two connected components that are symmetric about the origin.

4. Problem 6, Section 21.3.

5^*. An $n \times n$ matrix $A = (a_{ij})$ is called *3-diagonal* if $a_{ij} = 0$ provided $|i-j| \geq 2$. For a real symmetric matrix $A = (a_{ij})$, let $q_A : \mathbb{C}^n \longrightarrow \mathbb{R}$ be the quadratic form $q_A(z) = \sum\limits_{i,j=1}^{n} a_{ij} \zeta_i \overline{\zeta_j}$ for $z = (\zeta_1, \ldots, \zeta_n)$. Let us fix real symmetric $n \times n$ 3-diagonal matrices $A_1, \ldots, A_m$ and let us consider a map $\phi : \mathbb{C}^n \longrightarrow \mathbb{R}^m$:

$$\phi(z) = \big(q_{A_1}(z), \ldots, q_{A_m}(z)\big).$$

Prove, that the image $\phi(B)$ of the unit ball $B \subset \mathbb{C}^n$,

$$B = \big\{z = (\zeta_1, \ldots, \zeta_n) : |\zeta_1|^2 + \ldots + |\zeta_n|^2 \leq 1\big\}$$

is a convex set in $\mathbb{R}^m$.

For these and related results, see

P.R. Halmos, *A Hilbert Space Problem Book*, Graduate Texts in Mathematics, **19**; Encyclopedia of Mathematics and its Applications, **17**, Springer-Verlag, New York-Berlin, 1982.

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A.I. Barvinok, Problems of distance geometry and convex properties of quadratic maps, *Discrete Comput. Geom.* **13** (1995), no. 2, 189–202.

Friday, April 17

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45. The Lyapunov Theorem

(45.1) Theorem. *Let us fix m functions $f_1, \ldots, f_m \in L^1[0,1]$. For a Lebesgue measurable subset $A \subset [0,1]$, let $\phi(A) \in \mathbb{R}^m$ be the point*

$$\phi(A) = (\xi_1, \ldots, \xi_m), \quad \textit{where} \quad \xi_i = \int_A f_i(\tau)\, d\tau \quad \textit{for} \quad i = 1, \ldots, m.$$

Then the set

$$X = \Big\{\phi(A) : \quad A \subset [0,1] \quad \textit{is Lebesgue measurable}\Big\}$$

is a convex compact set in $\mathbb{R}^m$*.*

Proof. Let us extend ϕ to a map $\phi : L^\infty[0,1] \longrightarrow \mathbb{R}^m$,

$$\phi(u) = (\xi_1, \ldots, \xi_m), \quad \text{where} \quad \xi_i = \int_0^1 u(\tau) f_i(\tau)\, d\tau \quad \text{for} \quad i = 1, \ldots, m.$$

Let

$$B = \Big\{ u \in L^\infty[0,1] : \quad 0 \leq u(\tau) \leq 1 \quad \text{for almost all} \quad \tau \in [0,1] \Big\}.$$

Then B is convex and weak* compact (see Section 26.1). Since ϕ is linear and continuous, the image $\phi(B) \subset \mathbb{R}^m$ is a compact convex set.

Next, we claim that $\phi(B) = X$. Clearly, $X \subset \phi(B)$. Let us choose $a = (\alpha_1, \ldots, \alpha_m) \in \phi(B)$. Then the set

$$B_a = \Big\{ u \in B : \int_0^1 u(\tau) f_i(\tau)\, d\tau = \alpha_i, \quad \text{for} \quad i = 1, \ldots, m \Big\}$$

is non-empty and weak* compact. Therefore, by the Krein-Milman Theorem (Theorem 25.1), there is an extreme point $u_a \in B_a$. By Proposition 26.2, we have $u_a(\tau) \in \{0,1\}$ for almost all τ. Let $A = \big\{\tau : u_a(\tau) = 1\big\}$. Then $\phi(A) = a$. Therefore, $\phi(B) \subset X$ and the proof follows. □

(45.2) Corollary. *Let us fix a Lebesgue measurable set* $S \subset \mathbb{R}$ *and let* $F_i(x,\tau)$, $i = 1, \ldots, m$ *be Lebesgue measurable functions in 2 real variables* $x \in S$ *and* $\tau \in [0,1]$*. Suppose that the functions* $F_i(x,\tau)$ *are integrable on* $S \times [0,1]$*. For a Lebesgue measurable function* $x(\tau)$ *on the interval* $[0,1]$ *such that* $x(\tau) \in S$ *for almost all* $\tau \in [0,1]$*, let*

$$\psi(x) = (\xi_1, \ldots, \xi_m), \quad \textit{where} \quad \xi_i = \int_0^1 F_i\big(x(\tau), \tau\big)\, d\tau \quad \textit{for} \quad i = 1, \ldots, m.$$

Then the set $X \subset \mathbb{R}^m$*:*

$$X = \Bigg\{ \psi(x), \quad \textit{where} \quad x(\tau) \quad \textit{is a Lebesgue measurable function on} \quad [0,1]$$

$$\textit{and } x(\tau) \in S \textit{ for almost all } \tau \in [0,1] \Bigg\}$$

is convex.

Proof. Let us choose two points $a, b \in X$. Hence we have $a = \psi(x)$ and $b = \psi(y)$ for some Lebesgue measurable functions x and y on the interval $[0,1]$, such that $x(\tau), y(\tau) \in S$ for almost all τ. We will construct a convex set $Y \subset \mathbb{R}^m$, such that $Y \subset X$ and $a, b \in Y$. This will prove that X is convex.

To do that, for a Lebesgue measurable subset $A \subset [0,1]$, let us define a function $z_A(\tau)$:

$$z_A(\tau) = \begin{cases} x(\tau) & \text{if } \tau \in A \\ y(\tau) & \text{if } \tau \notin A. \end{cases}$$

Obviously, z_A is Lebesgue measurable and $z_A(\tau) \in S$ for almost all $\tau \in [0,1]$. Let

$$Y = \Big\{ \psi(z_A) \quad \text{for some Lebesgue measurable} \quad A \subset [0,1] \Big\}.$$

Since $z_{[0,1]}(\tau) = x(\tau)$ and $z_\emptyset(\tau) = y(\tau)$, we have $a, b \in Y$. Since $z_A(\tau) \in S$ for almost any τ, we conclude that $Y \subset X$. Let us show that Y is convex. Let us define functions $f_i \in L^1[0,1]$, $i = 1, \ldots, m$ by the formulas:

$$f_i(\tau) = F_i\big(x(\tau), \tau\big) - F_i\big(y(\tau), \tau\big) \quad \text{for} \quad i = 1, \ldots, m.$$

For a Lebesgue measurable set $A \subset [0,1]$, let

$$\phi(A) = (\xi_1, \ldots, \xi_m), \quad \text{where} \quad \xi_i = \int_A f_i(\tau)\, d\tau \quad \text{for} \quad i = 1, \ldots, m.$$

Let

$$c = (\gamma_1, \ldots, \gamma_m), \quad \text{where} \quad \gamma_i = \int_0^1 F_i\big(y(\tau), \tau\big)\, d\tau \quad \text{for} \quad i = 1, \ldots m.$$

Then $\psi(z_A) = \phi(A) + c$. Therefore, the set Y is a translation (by c) of the set

$$\Big\{ \phi(A) : \quad A \subset [0,1] \quad \text{is Lebesgue measurable} \Big\}.$$

The latter set is convex by Theorem 45.1. $\square$

PROBLEMS.

1°. Let us consider the set $X \subset \mathbb{R}^3$ consisting of the points $a = (\alpha_1, \alpha_2, \alpha_3)$, such that

$$\alpha_1 = \int_0^1 \cos\big(\tau x(\tau)\big)\, d\tau, \quad \alpha_2 = \int_0^1 \tau^3 x^2(\tau)\, d\tau, \quad \alpha_3 = \int_0^1 \tau^2 2^{x(\tau)}\, d\tau,$$

where $x(\tau)$ is a Lebesgue measurable function on the interval $[0,1]$, such that $x(\tau) \in (0, e) \cup (\pi, 4)$ for almost all τ. Prove, that X is a convex set.

2^*. Draw a picture of the set X from Problem 1.

Corollary 45.2 has some interesting applications to problems of optimal control, see A. Megretsky and S. Treil, Power distribution inequalities in optimization and robustness of uncertain systems, *J. Math. Systems Estim. Control*, **3** (1993), no. 3, 301–319.

Monday, April 20

Solutions to Some Problems from the Third Set

Some General References

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Infinite-dimensional convexity:

W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

Many of the ideas and approaches used in this course I learned from my teacher, A.M. Vershik. In 1983, I attended his lectures at the Leningrad (now St.Petersburg) University and worked with him since then on various projects in convex geometry.