Exponential Sums and Integrals over Convex Polytopes

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1. Notation. Let $\langle \cdot , \cdot \rangle$ be the standard scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$, and let $\mathbb{Z}^n$ be the standard integral lattice in $\mathbb{R}^n$. By $\mathrm{ex}(P)$ we denote the set of vertices of a convex n-dimensional polytope $P \subset \mathbb{R}^n$. For $v \in \mathrm{ex}(P)$, define K_v as the smallest convex cone containing P and having its vertex at v. Finally, let dx be the Lebesgue measure on $\mathbb{R}^n$.

2. Main results.

Theorem 1. *Let $P \subset \mathbb{R}^n$ be a convex polytope, $\dim P = n$ and let $\rho\colon \mathbb{R}^n \to \mathbb{C}$ be a polynomial. Then there exist meromorphic functions $s_v(c) = \exp\{\langle c, v\rangle\} q_v(c)$, where $q_v(c)$ are rational functions, $v \in \mathrm{ex}(P)$, $c \in \mathbb{C}^n$, such that*

$$\int_P \exp\{\langle c, x\rangle\}\rho(x)\,dx = \sum_{v\in \mathrm{ex}(P)} s_v(c) \tag{1}$$

for all $c \in \mathbb{C}^n$ that are regular points for all functions $s_v(c)$. Here

$$s_v(c) = \int_{K_v} \exp\{\langle c, x\rangle\}\rho(x)\,dx \tag{2}$$

if the function $\langle c, \cdot\rangle$ decreases along the extreme rays of K_v (and therefore the integral in (2) exists).

Theorem 2. *Suppose that under the conditions of Theorem 1 $\rho(x) = 0$ for all $x \in \partial P$. Then there exist analytic functions $\sigma_v(c)$, $v \in \mathrm{ex}(P)$ defined in the region*

$$U = \mathbb{C}^n \setminus \bigcup_{j\in J,\, k\in\mathbb{Z}^n} H_j + 2\pi i k, \tag{3}$$

where $\{H_j : j \in J\}$ is a finite set of hyperplanes in $\mathbb{C}^n$, such that

$$\sum_{x\in P\cap\mathbb{Z}^n} \exp\{\langle c, x\rangle\}\rho(x) = \sum_{v\in\mathrm{ex}(P)} \sigma_v(c)$$

for all $c \in U$. Here

$$\sigma_v(c) = \sum_{x\in K_v\cap\mathbb{Z}^n} \exp\{\langle c, x\rangle\}\rho(x), \tag{4}$$

if the function $\langle c, \cdot\rangle$ decreases along the extreme rays of K_v (and therefore the series in (4) converges).

Theorem 1 was proved independently by M. Brion [1] (only for rational polytopes P) by means of toric varieties, by A. G. Khovanskii and A. V. Pukhlikov (an elementary proof was presented by A. G. Khovanskii at V. I. Arnol'd's seminar (1989), his lectures at Harvard University and at the Moscow State University students' conference (1990)), and later by the author [2]. These proofs are essentially different. Below we present the author's proof, based on a simple geometric idea. Theorem 2 is apparently a new one. We sketch its proof.

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3. Proofs.

Lemma 1. *Let $\{\Gamma_i\}$, $i = 1, \ldots, m$, be the set of all facets of a polytope P and let μ_i be the Lebesgue measure on the affine hull of Γ_i induced from dx in $\mathbb{R}^n$. Denote by $\mathbf{n}_i$ the unit outer normal to Γ_i. Then*

$$\int_P \exp\{\langle c, x\rangle\}\, dx = \frac{1}{\langle \lambda, c\rangle} \sum_{i=1}^{m} \langle \lambda, \mathbf{n}_i\rangle \int_{\Gamma_i} \exp\{\langle c, x\rangle\}\, d\mu_i$$

for all $c \in \mathbb{C}^n$ and $\lambda \in \mathbb{R}^n$ such that $\langle \lambda, c\rangle \neq 0$.

Proof. Without loss of generality, we may assume that $\langle \lambda, \lambda\rangle = 1$. Put $\omega(x) = \exp\{\langle c, x\rangle\} u_1 \wedge \cdots \wedge u_{n-1}$, where $\langle \lambda, u_i\rangle = 0$ for $i = 1, \ldots, n-1$, and $\lambda \wedge u_1 \wedge \cdots \wedge u_{n-1} = dx_1 \wedge \cdots \wedge dx_n$ is the standard exterior form of the oriented volume on $\mathbb{R}^n$ (here we identify the cotangent space $\mathbb{R}^{n*}$ with $\mathbb{R}^n$). Applying Stokes formula to P and ω, we obtain the desired result.

Proof of Theorem 1. Consider the case $\rho \equiv 1$, since for an arbitrary polynomial density it suffices to apply the differential operator $D_\rho = \rho(\partial/\partial c_1, \ldots, \partial/\partial c_n)$ to (1) and (2). Let us choose $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$, so that λ is not orthogonal to any edge of P. Consecutively applying Lemma 1 to P, then to its facets and so on, we finally obtain the decomposition (1), where $q_v(tc) = t^{-n} q_v(c)$ for all $v \in \operatorname{ex}(P)$, $c \in \mathbb{C}^n$, $t \in \mathbb{R}$. To prove (2), let us choose for each $v \in \operatorname{ex}(P)$ an open set $U_v \subset \mathbb{R}^n$ such that $w \neq v \implies \langle c, w\rangle < \langle c, v\rangle$ for all $c \in U_v$, $w \in \operatorname{ex}(P)$, and U_v contains no singular points of $s_w(c)$. Then, for all $c \in U_v$, we have

$$\begin{aligned}\lim_{t\to+\infty} t^n \exp\{-t\langle c, v\rangle\} \int_P \exp\{t\langle c, x\rangle\}\, dx &= \lim_{t\to+\infty} t^n \exp\{-t\langle c, v\rangle\} \int_{K_v} \exp\{t\langle c, x\rangle\}\, dx \\ &= \int_{K_v - v} \exp\{\langle c, x\rangle\}\, dx.\end{aligned}$$

The first equation holds since the integral of $\exp\{t\langle c, x\rangle\}$ outside an arbitrary small neighborhood of the vertex v is $O(\exp\{t(\langle c, v\rangle - \varepsilon)\})$ for some $\varepsilon > 0$ as $t \to +\infty$. The second equation is obvious. On the other hand,

$$\lim_{t\to+\infty} t^n \exp\{-t\langle c, v\rangle\} \sum_{w \in \operatorname{ex}(P)} s_w(tc) = q_v(c)$$

for all $c \in U_v$. Therefore $q_v(c) = \int_{K_v - v} \exp\{\langle c, x\rangle\}\, dx$, and the proof follows.

Proof of Theorem 2. Let us sum (1) and (2) over the points $c + 2\pi i k$, $k \in \mathbb{Z}^n$. We will prove that the series

$$\sum_{k\in\mathbb{Z}^n} s_v(c + 2\pi i k), \qquad c = (c_1, \ldots, c_n) \in \mathbb{C}^n, \tag{5}$$

converges uniformly on any compact set in a region of the form (3). First, suppose that $K_v = \{x \in \mathbb{R}^n : \langle a_i, x - v\rangle \leq 0,\ i = 1, \ldots, n\}$ is a simplicial cone. Put $\varphi(x) = \prod_{i=1}^n \langle a_i, x - v\rangle$. If $\rho = \varphi$, then $s_v(c) = |a_1 \wedge \cdots \wedge a_n|^{-1} \exp\{\langle c, v\rangle\} \prod_{i=1}^n \langle a_i^*, c\rangle^{-2}$, where $\langle a_i^*, a_j\rangle = \delta_{ij}$, and the desired convergence of (5) follows easily. The case of the density $\rho(x) = \varphi(x)\psi(x)$ reduces to the application of the differential operator $D_\psi = \psi(\partial/\partial c_1, \ldots, \partial/\partial c_n)$ to each term of (5), which preserves uniform convergence on compact sets. Finally, if K_v is not simplicial, then it can be represented as a linear combination of simplicial cones bounded by supporting hyperplanes of facets of K_v (see [3, Theorem 17]). Denote the sum of (5) by $\sigma_v(c)$. Now Theorem 2 follows from the Poisson summation formula.

Remark. If $\operatorname{ex}(P) \subset \mathbb{Z}^n$, then the condition $\rho(x) = 0$ for all $x \in \partial P$ can be omitted. In this case the functions $\sigma_v(c)$, $c = (c_1, \ldots, c_n)$, are rational in $\exp\{c_1\}, \ldots, \exp\{c_n\}$. This result was proved by M. Brion by means of toric varieties. An elementary proof was given by A. G. Khovanskii and A. V. Pukhlikov.

Another elementary proof can be obtained on the same lines as the above proof of Theorem 1. Namely, let $\forall m \in \mathbb{N}$, $mP = \{mx : x \in P\}$. The classical results on the integer points in polytopes (see, for example, [4, Chap. 4]) imply that

$$\sum_{x\in mP\cap\mathbb{Z}^n} \exp\{\langle c, x\rangle\} = \sum_{x\in\operatorname{ex}(P)} \exp\{m\langle c, v\rangle\} Q_v(m; c),$$

for all $m \in \mathbb{N}$, where $Q_v(m; c)$ is a polynomial in m whose coefficients are rational functions in $\exp\{c_1\}, \dots, \exp\{c_n\}$. To complete the proof it remains to consider the asymptotic behavior of the above sums as $m \to +\infty$.

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On the Uniqueness of the Solution of the Inverse Exact Interpolation Problem

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In the present paper we study the following problem [1]: *is a Banach couple uniquely determined by the collection of all interpolation spaces generated by it*? The authors are familiar with only two results concerning this question. There are a rather special result due to Aronszajn and Gagliardo, cited in the survey [1], and a theorem by V. G. Zobina [4] asserting that the couple of finite-dimensional spaces (l_1^n, l_∞^n) is uniquely determined by the collection of its exact interpolation spaces.

We say that two pairs (X_1, X_2) and (Y_1, Y_2) of normed spaces are *homothetic* if, possibly after interchanging Y_1 with Y_2, X_i coincides with Y_i as a linear space and the norm in X_i is a multiple of the norm in Y_i, $i = 1, 2$. The same notation will be used for the norm in a normed space and the corresponding norm for linear operators in this space.

Theorem 1. *Let $\|\cdot\|_i$, $i = 1, 2, 3, 4$, be norms in a finite-dimensional space E, and let $E_i = (E, \|\cdot\|_i)$. Suppose that*

$$\max\{\|T\|_1, \|T\|_2\} = \max\{\|T\|_3, \|T\|_4\}$$

for any linear operator T in E. Then the couples (E_1, E_2) and (E_3, E_4) are homothetic.

Outline of proof. For each of the norms $\|\cdot\|_i$ on E, denote by $\|\cdot\|_i^*$ the dual norm on the dual space E' of linear functionals on E. By considering operators of rank 1, we find that for any $x \in E$ and $\varphi \in E'$ we have

$$\max\{\|x\|_1\|\varphi\|_1^*, \|x\|_2\|\varphi\|_2^*\} = \max\{\|x\|_3\|\varphi\|_3^*, \|x\|_4\|\varphi\|_4^*\}.$$

Denote the left and the right sides of the last equation by $M(x, \varphi)$ and $N(x, \varphi)$, respectively. Set

$$\alpha_{ik} = \max\{\|x\|_i/\|x\|_k \mid x \in E \setminus \{0\}\}, \quad \beta_{ik} = \max\{\|\varphi\|_i^*/\|\varphi\|_k^* \mid \varphi \in E' \setminus \{0\}\}, \qquad i, k = 1, 2, 3, 4.$$

Let x_{12} be an element of E realizing the maximum in the definition of α_{12}. Then we have $M(x_{12}, \varphi) = \|x_{12}\|_1\|\varphi\|_1^*$ for any $\varphi \in E'$. Next we prove that $\|x_{12}\|_1/\|x_{12}\|_3 = \alpha_{13}$ and $\|x_{12}\|_1/\|x_{12}\|_4 = \alpha_{14}$. From now on, denote x_{12} by x_1. Interchanging E with E', we find an element $\varphi_1 \in E'$ realizing the maxima in the definitions of β_{12}, β_{13}, and β_{14}. Let $N(x_1, \varphi_1) = \|x_1\|_3\|\varphi_1\|_3^*$; then $\|x_1\|_1/\|x_1\|_3 = \|\varphi_1\|_3^*/\|\varphi_1\|_1^* = 1/\beta_{13}$, so that $\beta_{31}\beta_{13} = 1$ and there exists a $c > 0$ such that $\|\cdot\|_1^* = c\|\cdot\|_3^*$. It is now not hard to conclude that the couples (E_1, E_2) and (E_3, E_4) are homothetic.

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