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Combinatorial Complexity of Orbits in Representations of the Symmetric Group

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ABSTRACT. A special class of convex polytopes is considered, whose elements are convex hulls of orbits of a vector in some real representations of a finite group. It is shown that quite a few problems of combinatorial optimization may be posed as linear programming problems on such polytopes in the case of the symmetric group. Generally, these polytopes correspond to NP-hard problems, so a system of approximations is constructed for them, thus providing an approximation algorithm with estimates of errors and complexity. All the orbits and special functions on them that correspond to polynomial-time problems known at present to the author are listed. They turn out to be selected by certain algebraic conditions: the statistical sum constructed in a special way using the orbit is a relative invariant of the general linear group and may therefore be computed in polynomial time.

§1. Introduction

Let $\varkappa: G \to GL(V)$ be a representation of a finite group G in a real vector space V endowed with a G-invariant scalar product $\langle\ ,\ \rangle$. The following two polytopes are the main object considered in this paper:

The convex hull of the orbit of a fixed vector $v \in V$,

$$P_\varkappa(v) = \operatorname{conv}\{\varkappa(g)\,v : g \in G\} \subset V. \tag{1.1}$$

The convex hull of the operators of the representation $\varkappa$ considered as points of the space $V \otimes V$ (one may identify V^* and V, $\operatorname{End}(V) = V^* \otimes V$ and $V \otimes V$ via the scalar product),

$$P_\varkappa = \operatorname{conv}\{\varkappa(g) : g \in G\} \subset V \otimes V. \tag{1.2}$$

It is easy to see that $P_\varkappa = P_\xi(E)$, where $\xi = \varkappa \otimes \mathrm{id}_V$ (here id_V denotes the trivial representation of G in V) and E is the identity operator in $\operatorname{End}(V)$. The polytope $P_\varkappa(v)$ is the image of the polytope $P_\varkappa$ under the projection

$$\mathrm{pr}: \operatorname{End}(V) \to V, \qquad \mathrm{pr}(A) = Av, \quad \forall A \in \operatorname{End}(V).$$

We are interested in the "complexity" of the combinatorial structure of the polytopes (1.1), (1.2). One of the possible approaches to define the complexity of a polytope P is as follows (see [1], [2]). Let us assign to each P the family of optimization problems

$$\text{given } c \in V \text{, find } \max\{\langle c, x\rangle : x \in P\}. \tag{1.3}$$

The complexity of P is understood as the complexity of the problem (1.3) for a generic $c \in V$. By the complexity of an algorithm we mean the number of operations from a given list that it performs. In §§1–5, this list will include arithmetic operations over real numbers (addition, subtraction, multiplication, and division) as well as the comparison of real numbers. In §6, where the statistical sums on a polytope are computed, the list is naturally expanded by including the operation of taking the exponential function for real and complex numbers. The corresponding complexity model is widely used in computational geometry (see [3], [4]).

Note that

$$\max\{\langle c, x\rangle : x \in P_\varkappa(v)\} = \max\{\langle c \otimes v, x\rangle : x \in P_\varkappa\},$$

so that the structure of the polytope (1.1) is not more "complex" than that of the polytope (1.2).

In fact, we shall never deal with individual polytopes P. Instead, we consider a certain natural series of polytopes $\{P_n\}$, $n \in \mathbb{N}$, of the form (1.1) or (1.2) corresponding to a series $\{\varkappa_n\}$ of representations of groups $\{G_n\}$ in the spaces $\{V_n\}$. In any case we shall have the inequality $\dim V_n \leqslant t(n)$, where t is some polynomial, and the functional $\langle c_n, \cdot\rangle$ will be determined by its values on the elements of some natural basis of the space V_n. Of particular interest for us are series of "simple" polytopes $\{P_n\}$ for which an algorithm solving the problem (1.3) with polynomially bounded complexity in n exists. Such (nontrivial) series being rare, we consider approximate solutions of the problem (1.3) as well. This approach yields a finer partition of the set of all polytopes (1.1), (1.2) into complexity classes.

The main example is the case $G_n = S_n$, where S_n is the symmetric group (i.e., the group of all permutations of the set $\{1, 2, \dots, n\}$). The representation $\varkappa_n$ is either an irreducible representation or a sum of a fixed number of irreducible representations corresponding to Young diagrams whose first row increases with n. The necessary notions concerning representation theory may be found in [5], [6], [7].

The question of solving the problem (1.3) for polytopes of the form (1.1), (1.2) has been considered previously by the author and A. M. Vershik in the context of combinatorial optimization problems [8], [9]. In particular, it is shown in [9], [10] that almost all combinatorial optimization problems may be put in the following form: find the maximum of a given linear form on the orbit of a vector in a representation of the symmetric group. It was A. M. Vershik who gave the impetus to begin the study of algebraic methods in optimization, and this study is continued in the present paper. In particular, he has posed the optimization problem for a linear form on an orbit in a representation of a finite group.

§2. Examples

In the examples below we omit the index $n \in \mathbb{N}$ unless this might lead to misunderstandings.

(2.1) EXAMPLE. Let $V_n = \mathbb{R}^n$, and let the group S_n act in the space V_n by coordinate permutations (this representation is denoted by ρ below),

$$(\rho(\sigma)x)_i = x_{\sigma^{-1}(i)}, \qquad x = (x_1, \dots, x_n) \in V_n, \qquad \sigma \in S_n.$$

(2.1.1) Let $v = (v_1, \dots, v_n)$. The combinatorial structure of the polytope

$$P_\rho(v) = \operatorname{conv}\{(v_{\sigma(1)}, \dots, v_{\sigma(n)}) : \sigma \in S_n\}$$

has been intensively studied (see [11, Russian pp. 181–186]). One may assume without loss of generality that

$$v_1 \geqslant v_2 \geqslant \dots \geqslant v_n.$$

In this case the problem (1.3) for the vector $c = (c_1, \dots, c_n)$ is none other than the ordering problem for the components of the vector c, since the maximum in (1.3) is equal to $\sum_{i=1}^n v_i c_{\sigma(i)}$, where $\sigma \in S_n$ is such that $c_{\sigma(1)} \geqslant c_{\sigma(2)} \geqslant \dots \geqslant c_{\sigma(n)}$. The complexity of this problem is $O(n \ln n)$. If there are at least two different numbers among $v_1, \dots, v_n$, we have $\dim P_\rho(v) = (n-1)$.

(2.1.2) Consider the polytope (1.2)

$$P_\rho \subset \mathbb{R}^{n^2}, \qquad \dim P_\rho = (n-1)^2.$$

By the Birkhoff-von Neumann theorem (see [12]) the polytope P_ρ is described by the system of equations

$$\forall i, \quad \sum_{i=1}^n x_{ij} = 1, \qquad \forall j, \quad \sum_{i=1}^n x_{ij} = 1$$

and inequalities

$$\forall i, j, \quad x_{ij} \geqslant 0.$$

The problem (1.3) is called the *assignment problem* and admits an algorithm whose complexity is $O(n^3)$ (see [12]).

(2.2) EXAMPLE. Let $V_n = \mathbb{R}^{n^2}$. The space V_n will be interpreted as the space of $n \times n$ matrices. The group S_n acts in V_n by simultaneous permutations of rows and columns of the matrices. We denote this representation by τ, $\tau = \rho^{\otimes 2}$,

$$(\tau(\sigma)x)_{ij} = x_{\sigma^{-1}(i)\sigma^{-1}(j)}, \qquad \sigma \in S_n, \quad x = (x_{ij}) \in \mathbb{R}^{n^2}, \quad 1 \leqslant i, j \leqslant n.$$

(2.2.1) Let n be even, $v \in V_n$,

$$v_{ij} = \begin{cases} 1 & \text{if } i = 2k-1,\ j = 2k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

The problem (1.3) with the functional $c = (c_{ij})$ on the polytope (1.1) may be interpreted as the *weighted matching problem*: find the maximal total weight of the partition of the set $\{1, 2, \dots, n\}$ into $n/2$ disjoint ordered pairs $\{(i_k, j_k),\ k = 1, \dots, n/2\}$ provided that the weight of an ordered pair (i, j) is considered to be equal to c_{ij}. There is an $O(n^3)$ algorithm to solve this problem (see [12]). The combinatorial structure of the polytope $P_\tau(w)$, where $w = v + v^t$, corresponding to the weighted matching problem for unordered pairs has been thoroughly studied in the literature. Note that

$$\dim P_\tau(w) = (n^2 - 3n)/2,$$

since the vector w lies in a component of the sum of the irreducible representation of the group S_n with Young diagram $(n-2, 2)$ and the trivial representation. See [6, p. 77]; [7, pp. 56–57] for the formula giving the dimension of an irreducible representation of the symmetric group.

(2.2.2) Let $v \in V_n$,

$$v_{ij} = \begin{cases} 1 & \text{if } j \equiv (i+1) \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

The polytope $P_\tau(v)$ is called the polytope of the *nonsymmetric travelling salesman problem*, and the problem (1.3) itself is known as the nonsymmetric travelling salesman problem. With the functional c given by its matrix (c_{ij}), (1.3) is formulated as the problem of finding the Hamiltonian path of maximal weight in the complete digraph with n vertices and given edge weight matrix (c_{ij}). This is an NP-hard problem, an algorithm with complexity $O(n^2 2^n)$ being nevertheless known for it. The element $v \in V$ lies in the component of the sum of irreducible representations with Young diagrams $(n-2, 2)$, $(n-2, 1, 1)$, (n). Therefore, $\dim P_\tau(v) = n^2 - 3n + 1$.

Set $w = v + v^t$. The polytope $P_\tau(w)$ is called the polytope of the *symmetric travelling salesman problem*. Note that $\dim P_\tau(w) = (n^2 - 3n)/2$, since the element w lies in the component of the sum of the irreducible representation with Young diagram $(n-2, 2)$ and the trivial representation. The combinatorial structure of the polytopes $P_\tau(v)$, $P_\tau(w)$ has been studied in quite a few papers (see, e.g., [13], [14]).

(2.2.3) The problem (1.3) for the polytope P_τ (see (1.2)) is one of the most complicated problems of combinatorial optimization known as the *quadratic assignment problem*. The author knows no algorithms for this problem more effective than exhaustive search in the set of the vertices of the polytope P_τ.

Using the decomposition of the representation τ into a sum of irreducible representations, we obtain (the irreducible representations are denoted by the corresponding Young diagrams):

$$\tau = (n-2, 1, 1) \oplus (n-2, 2) \oplus 3(n-1, 1) \oplus 2(n),$$

$$\dim P_\tau = \left(\frac{n^2 - 3n}{2} + 1\right)^2 + \left(\frac{n^2 - 3n}{2}\right)^2 + (n-1)^2$$

(by the Frobenius and Schur theorems, see [5, Chapter 4, §27]).

(2.3) Example. Let us fix $l \in \mathbb{N}$, and let $V_n = (\mathbb{R}^n)^{\otimes l}$. Consider the representation $\nu = \rho^{\otimes l}$ of the group S_n in V_n,

$$(\nu(\sigma)x)_{i_1, \dots, i_l} = x_{\sigma^{-1}(i_1), \dots, \sigma^{-1}(i_l)}, \quad x = (x_{i_1, i_2, \dots, i_l}) \in (\mathbb{R}^n)^{\otimes l}, \quad \sigma \in S_n.$$

In particular, $\nu = \rho$ for $l = 1$ (Example 2.1), and $\nu = \tau$ for $l = 2$ (Example 2.2). Suppose that $n = lm$, $m \in \mathbb{N}$. Let us fix a tensor $v \in V_n$,

$$v_{i_1, \dots, i_l} = \begin{cases} 1 & \text{if } \exists k \in \mathbb{N},\ 0 \leqslant k \leqslant m-1,\ \forall j,\ i_j = lk + j, \\ 0 & \text{otherwise.} \end{cases}$$

The problem (1.3) for the polytope $P_\nu(v)$ and the functional $c = (c_{i_1}, \dots, c_{i_l})$, $1 \leqslant i_j \leqslant n$, is known as the *weighted packing problem*: find the maximal weight of a partition of the set $\{1, 2, \dots, n\}$ into m ordered disjoint l-tuples

$$\{(i_{1j}, i_{2j}, \dots, i_{lj});\ j = 1, \dots, m\},$$

assuming the weight of the tuple $(i_1, \dots, i_l)$ to be equal to $c_{i_1, \dots, i_l}$.

For $l = 2$ one gets the weighted matching problem (Example 2.2.1). Problem (1.3) is NP-hard for $l > 2$, an algorithm of complexity $\exp\{n + O(\log n)\}$ being known for it. The problem (1.3) for the polytope P_ν is called the *assignment problem of degree* l (see [15]).

(2.4) Example. Let $G_n = W_n$ be the Weyl group of the irreducible root system of one of the types A_n, B_n, C_n, D_n, and let ρ be the natural action of W_n in the Cartan subalgebra V_n (see [17]). Polytopes of the form (1.1) have been considered in the literature. In particular, algorithms for solving problem (1.3) were studied in [16].

(2.4.1) The A_n series: $W_n = S_{n-1}$ (see Example 2.1).

(2.4.2) The B_n, C_n series. The group W_n acts in $\mathbb{R}^n = V_n$ in the following way:

$$(\rho(\sigma, \varepsilon)x)_i = \varepsilon_i x_{\sigma^{-1}(i)},$$

where $x = (x_1, \dots, x_n)$, $\sigma \in S_n$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i = \pm 1$.

(2.4.3) The D_n series. The group W_n acts in $V_n = \mathbb{R}^n$ just as in the case (2.4.2), with the only condition $\varepsilon_1 \cdot \varepsilon_2 \cdot \dots \cdot \varepsilon_n = 1$.

We shall consider the polytopes P_ρ, $P_\rho(v)$, where $v \in V_n^*$ is a weight.

§3. Constructions in the group algebra

It is shown in [9], [10] that the problem (1.3) on the polytope of the form (1.2) is NP-hard for most of the irreducible representations of the group

S_n, and consequently the polytope (1.2) is rather complicated. The precise formulation of this result is as follows.

(3.1) THEOREM [9, 10]. *Let a partition* $\lambda_2 + \cdots + \lambda_s = k$ *of a number* $k \in N$ *be fixed. Then the problem* (1.3) *for the polytope* (1.2) *of the irreducible representation* $\varkappa$ *of the group* S_n *with Young diagram* $(n-k, \lambda_2, \ldots, \lambda_s)$ *is NP-hard for* $k > 1$.

It is assumed here that the vector c in (1.3) is defined by the rational coefficients of its expansion with respect to the standard basis of the space $V_\varkappa \otimes V_\varkappa$ (see [6], [7]). Note that for $k = 1$ a polynomial algorithm for this problem does exist.

In this section we discuss the possibility of solving the problem (1.3) approximately, replacing the polytopes (1.1), (1.2) by simpler ones.

We shall assume that the field $\mathbb{R}$ of real numbers is a splitting field for the group G. The following notation will be used:

$\mathbb{R}G$ is the group algebra of a finite group G, i.e., the linear space consisting of formal linear combinations $r = \sum_{g\in G} r(g)\, g$ with the multiplication operation (convolution) $(r_1 r_2)(g) = \sum r_1(h_1)\, r_2(h_2)\colon h_1 h_2 = g$;

$(\mathbb{R}G)_+ = \{r \in \mathbb{R}G : \forall g,\ r(g) \geqslant 0\}$ is the nonnegative orthant in the space $\mathbb{R}G$;

$\Delta = \{r \in (\mathbb{R}G)_+ : \sum_{g\in G} r(g) = 1\}$ is the unit simplex in $\mathbb{R}G$;

$e = |G|^{-1} \sum_{g\in G} g \in \Delta$ is the barycenter of the simplex Δ.

Let $\varkappa\colon G \to GL(V)$ be a representation of the group G, and let $v \in V$. Set

$$L_\varkappa(v) = \operatorname{lin}\{\varkappa(g)v : g \in G\} \subset V$$
$$L_\varkappa = \operatorname{lin}\{\varkappa(g) : g \in G\} \subset V \otimes V.$$

Evidently, $P_\varkappa(v) \subset L_\varkappa(v)$, $P_\varkappa \subset L_\varkappa$. It is convenient to think of the conjugate spaces $L^*_\varkappa(v)$, $L^*_\varkappa$ as of subsets of the group algebra $\mathbb{R}G$. Let us define the mappings $\varphi : L^*_\varkappa(v),\ L^*_\varkappa \to \mathbb{R}G$ by

$$\varphi(c) = |G|^{-1} \sum_{g\in G} \langle c, \varkappa(g)v\rangle\, g,$$
$$\varphi(c) = |G|^{-1} \sum_{g\in G} \langle c, \varkappa(g)\rangle\, g \tag{3.2}$$

for a linear functional $\langle c, \cdot\rangle \in L^*_\varkappa(v)$, $L^*_\varkappa$, respectively. Note that the space $L^*_\varkappa(v)$ is a left G-module isomorphic to $L_\varkappa(v)$, while $L^*_\varkappa$ is a bimodule.

(3.3) LEMMA. *The mapping* φ *defined by formulas* (3.2) *is an isomorphism of the left module* $L^*_\varkappa(v)$ *(respectively, the bimodule* $L^*_\varkappa$*) onto a left ideal (respectively, a two-sided ideal) of the group algebra* $\mathbb{R}G$. *If* $\varkappa$ *is an irreducible representation and* $v \neq 0$, *the ideal* $\varphi(L^*_\varkappa(v))$ *is generated by the primitive idempotent* $\langle v, v\rangle^{-1} \dim \varkappa \cdot \varphi(v^*)$, *where* v^* *is the linear functional* $\langle v, \cdot\rangle$ *on the space* $L_\varkappa(v)$.

PROOF. The first statement is evident. Let us verify the second one. Set

$$u = \langle v, v\rangle^{-1} \dim \varkappa \cdot \varphi(v^*).$$

It is clear that $u \in \varphi(L^*_\varkappa(v))$. Since $\varphi(L^*_\varkappa(v)) \subset \mathbb{R}G$ is a simple ideal, it suffices to verify that $u^2 = u$. Indeed,

$$u^2(g) = \sum_{h\in G} u(gh)\, u(h^{-1})$$
$$= \frac{\dim^2 \varkappa}{|G|^2} \sum_{h\in G} \langle v, v\rangle^{-1} \langle v, \varkappa(gh)v\rangle \cdot \langle v, v\rangle^{-1} \langle v, \varkappa(h^{-1})v\rangle.$$

Since $\varkappa$ is an irreducible representation, the orthogonality relations for matrix elements (see [5], Chapter 5, §31) imply that the latter sum is equal to

$$|G| \cdot (\dim \varkappa)^{-1} \langle v, v\rangle^{-1} \langle v, \pi(g)v\rangle.$$

Hence $u^2 = u$, and the lemma is proved.

Thus, we shall identify the spaces $L^*_\varkappa(v)$, $L^*_\varkappa$ with their images in $\mathbb{R}G$ under the inclusion φ.

Now let us describe the objects dual to the polytopes (1.1), (1.2).

Without loss of generality we may assume that $\sum_{g\in G} \varkappa(g)\, v = 0$ for the polytope (1.1), and $\sum_{g\in G} \varkappa(g) = 0$ for (2.2) (one may always achieve this by shifting the polytopes $P_\varkappa$, $P_\varkappa(v)$). For a polytope P, $0 \in P \subset L$, set

$$P^* = \{c \in L^* : \forall x \in P,\ \langle c, x\rangle \geqslant -1\}.$$

Then

$$P^*_\varkappa(v) = \{r \in L^*_\varkappa(v) : r + e \in (\mathbb{R}G)_+\}, \tag{3.4}$$
$$P^*_\varkappa = \{r \in L^*_\varkappa : r + e \in (\mathbb{R}G)_+\}. \tag{3.5}$$

Our immediate aim is to approximate the polytopes (3.4), (3.5) and the dual polytopes (1.1), (1.2) by polytopes whose structure is less complicated.

(3.6) DEFINITION. A convex closed cone $K \subset (\mathbb{R}G)_+$ with the vertex at the origin is said to be *invariant* if $\forall g \in G$, $gK = Kg = K$.

In particular, $(\mathbb{R}G)_+$ itself is an invariant cone.

Let us consider the following general situation. Let $L^* \subset \mathbb{R}G$ be a linear space satisfying

$$\forall r \in L^*, \qquad \sum_{g\in G} r(g) = 0.$$

Denote

$$P^* = \{r \in L^* : r + e \in (\mathbb{R}G)_+\},$$

and set

$$P^*(K) = \{r \in L^* : r + e \in K\}$$

for any invariant cone K. It is clear that $P^*(K) \subset P^*$. In order to minimize the complexity of the polytope $P^*(K)$, we shall choose the cones K

possessing as few extremal rays as possible. The following lemma allows us to estimate the "gap" between the polytopes P^* and $P^*(K)$.

(3.7) LEMMA. *Let $u \in \mathbb{R}G$ be an element of the group algebra such that*

$$\forall r \in L^*, \quad ru = r \quad and \quad eu = e, \tag{3.7.1}$$

$$u + ke \in K \quad for\ some\ k > 0. \tag{3.7.2}$$

Then

$$P^*(K) \subset P^* \subset (k+1)P^*(K).$$

PROOF. Let $r \in P^*$. Since $r+e \in (\mathbb{R}G)_+$, $u+ke \in K$, and the cone K is invariant, we have $(r+e)(u+ke) \in K$, so that $(k+1)^{-1}(r+e)(u+ke) \in K$. Since $ru = r$, $eu = e$ (by (3.7.1)), $re = 0$ (in view of $\sum r(g) = 0$), and $ee = e$, we have

$$(k+1)^{-1}(r+e)(u+ke) = (k+1)^{-1}r + e.$$

Thus, $(k+1)^{-1}r + e \in K$ and $r \in (k+1)P^*(K)$. The inclusion $P^*(K) \subset P^*$ is evident.

Let us describe the system of invariant cones to be used in the sequel.

(3.8) DEFINITION. Let $H \leqslant G$ be a subgroup of the group G. Define the invariant cone $K^*(H)$ by the formula

$$K^*(H) = \mathrm{co}\left\{ g_1 \left(\sum_{h \in H} h \right) g_2 ;\ g_1, g_2 \in G \right\}.$$

Note that the number of extremal rays of the cone $K^*(H)$ does not exceed $|G:H|^2$.

If $\varkappa$ is an irreducible representation, P^* is a polytope of the form (3.5), and $K = K^*(H)$, the estimate in Lemma 3.7 may be put in more explicit form.

Denote by $\pi(H)$ the representation of the group G induced by the trivial representation of the subgroup H; we denote by $(\alpha : \beta)$ the multiplicity of the representation β in the representation α. Next, we write simply $P^*(H)$ instead of $P^*(K)$, $K = K^*(H)$.

(3.9) LEMMA. *Let $\varkappa$ be an irreducible representation of the group G, $H \leqslant G$, and $(\pi(H) : \varkappa) \neq 0$. Let $k = \dim \varkappa / (\pi(H) : \varkappa) - 1$ and $P^* = P^*_\varkappa$. Then $P^*(H) \subset P^* \subset (k+1)P^*(H)$.*

PROOF. Consider the element

$$u = \frac{(k+1)}{|G| \cdot |H|} \sum_{g \in G} \sum_{h \in H} g^{-1}hg - ke$$

of the group algebra $\mathbb{R}G$. Evidently, u and k satisfy condition (3.7.2). Let us verify condition (3.7.1). Denote by

$$e_\alpha = \frac{\dim \alpha}{|G|} \sum_{g \in G} \chi_\alpha(g)\, g$$

the central idempotent of a representation α with the character χ_α (see [5], Chapter 5, §33). The element $u \in \mathbb{R}G$ lies in the center of the group algebra. Consequently, it may be expanded into a linear combination of central idempotents of irreducible representations,

$$u = \frac{k+1}{|G|} \sum_{g \in G} \chi_{\pi(H)}(g) - ke = e_\varkappa + e_\varepsilon + \sum_{\beta \neq \varkappa, \varepsilon} r_\beta e_\beta, \qquad r_\beta \in \mathbb{R}$$

(here ε denotes the trivial representation, $e_\varepsilon = e$). Since for irreducible representations α, β we have $e_\alpha \cdot e_\beta = \delta_{\alpha,\beta} e_\alpha$, condition (3.7.1) is also valid.

(3.10) REMARK. The estimate given in Lemma 3.9 is not sharp; e.g., for the polytope $P^* = P^*_p$ (see (2.1.2)) and $H = S_{n-1} \leqslant S_n$ Lemma 3.9 gives

$$P^*(H) \subset P^*_p \subset (n-1)P^*(H),$$

while in fact the equality $P^*(H) = P^*_p$ holds (this is one of the reformulations of the Birkhoff-von Neumann theorem mentioned above). Nevertheless, the estimate of Lemma 3.9 is "asymptotically sharp", since $\dim \varkappa = (\pi(H) : \varkappa)$ and $k = 0$ if H is a trivial subgroup. In this case we, of course, have $P^*(H) = P^*_\varkappa$.

Set

$$\bar{P}^*(H) = \left\{ r \in \mathbb{R}G : \sum_{g \in G} r(g) = 0,\ r + e \in K^*(H) \right\}.$$

Thus,

$$P^*_\varkappa(H) = \bar{P}^*(H) \cap L^*_\varkappa.$$

Hence the polytope $P_\varkappa(H)$ is a projection of some polytope $\bar{P}(H)$ dual to $\bar{P}^*(H)$ onto the space $L_\varkappa$. Since the polytope $\bar{P}^*(H)$ has at most $|G:H|^2$ vertices, $\bar{P}(H)$ has at most $|G:H|^2$ facets. Let us describe the polytope $\bar{P}(H)$ and the projection pr: $\bar{P}(H) \to P_\varkappa(H)$ explicitly under the assumption that $(\pi(H) : \varkappa) > 0$.

Let $V_{\pi(H)}$ be the space of the representation $\pi(H)$ of the group G. We assume that the orthonormal basis is chosen in $V_{\pi(H)}$ indexed by left cosets in G/H with the natural action of G : $\gamma(gH) = (\gamma g)H$. Further, let $W = V_{\pi(H)} \otimes V_{\pi(H)}$ be the space of representation operators for $\pi(H)$ with the basis $\{g_1 H \otimes g_2 H\}$; the nonnegative orthant in this basis will be denoted by W_+. Just as above, we set

$$L_{\pi(H)} = \mathrm{lin}\{\pi(H)(g) : g \in G\}$$

and identify $L^*_{\pi(H)}$ with an ideal of the group algebra (see (3.2)). Let us prove now that

$$K^*(H) = (L_{\pi(H)} \cap W_+)^*.$$

(3.11) LEMMA. *We have*

$$K^*(H) = \{c \in L^*_{\pi(H)} : \forall x \in L_{\pi(H)} \cap W_+ ,\ \langle c, x\rangle \geqslant 0\}.$$

PROOF. Consider the linear functional $\langle \cdot, g_1H \otimes g_2H\rangle$ on the space $L_{\pi(H)}$. We have

$$L_{\pi(H)} \cap W_+ = \{x \in L_{\pi(H)} : \forall g_1, g_2, \langle x, g_1H \otimes g_2H\rangle \geqslant 0\}.$$

The element

$$|G|^{-1} \sum_{\sigma\in G} \langle \pi(H)(\sigma),\ g_1H \otimes g_2H\rangle \sigma = |G|^{-1} \sum_{h\in H} g_2 h g_1^{-1}$$

is the image of the linear functional $\langle g_1H \otimes g_2H, \cdot\rangle$ under the identification (3.2). Therefore,

$$(L_{\pi(H)} \cap W_+)^* = \operatorname{co}\left\{\sum_{h\in H} g_2 h g_1^{-1} : g_1, g_2 \in G\right\} = K^*(H).$$

The lemma is proved.

Thus the notation $K(H)$ for the cone $L_{\pi(H)} \cap W_+$ is validated.

To within a shift to the origin, the polytope $\bar{P}(H) \subset V_{\pi(H)} \otimes V_{\pi(H)}$ is the intersection of the affine hull of the representation operators of $\pi(H)$ with W_+. If $(\pi(H) : \varkappa) > 0$, the polytope $P_\varkappa(H)$ is a projection of the polytope $\bar{P}(H)$ onto the space $V_\varkappa \otimes V_\varkappa$.

Let us state the main results of this section in the form of a theorem.

(3.12) THEOREM. *Let $\varkappa$ be a nontrivial irreducible representation of the group G in a real vector space $V_\varkappa$. Let*

$$P_\varkappa = \operatorname{conv}\{\varkappa(g) : g \in G\} \subset V_\varkappa \otimes V_\varkappa$$

be the convex hull of the representation operators. For any subgroup $H \subset G$ there exists a polytope $\bar{P}(H) \subset V_{\pi(H)} \otimes V_{\pi(H)}$, where $\pi(H)$ is a representation induced by the trivial representation of the subgroup H, such that

(3.12.1) *$\bar{P}(H)$ has at most $|G : H|^2$ facets;*

(3.12.2) *if $(\pi(H) : \varkappa) \neq 0$, then the image $P_\varkappa(H)$ of the polytope $\bar{P}(H)$ under the projection $\operatorname{pr}: V_{\pi(H)} \otimes V_{\pi(H)} \to V_\varkappa \otimes V_\varkappa$ satisfies the conditions*

$$\frac{(\pi(H) : \varkappa)}{\dim \varkappa} P_\varkappa(H) \subset P_\varkappa \subset P_\varkappa(H).$$

(3.13) COROLLARY. *Replacing the polytope $P_\varkappa$ of the form (1.2) by $\bar{P}(H)$ and the functional c by $\operatorname{pr}^*(c)$ in the problem (1.3), we come to a linear programming problem whose size does not exceed $|G : H|^2 \times |G : H|^2$, and the upper estimate $\dim \varkappa / (\pi(H) : \varkappa)$ is valid for the relative error due to replacement.*

In the next section we consider a particular case of the above construction. Namely, we apply it to the symmetric group.

§4. The case of the symmetric group

The irreducible representation of the symmetric group S_n corresponding to the Young diagram

$$\Lambda = (\lambda_1, \dots, \lambda_s), \qquad \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_s > 0, \quad \sum_{i=1}^n \lambda_i = n,$$

will be also denoted by Λ. Let

$$S_\Lambda = S_{\{1,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \dots \times S_{\{n-\lambda_s+1,\dots,n\}} \subset S_n$$

be the Young subgroup corresponding to Λ and consisting of permutations preserving the row number for each element of the standard tableau of the form Λ. We have

$$|S_\Lambda| = \lambda_1!\lambda_2!\cdots\lambda_s!.$$

By $\pi(\Lambda)$ we shall denote the representation of the group S_n induced by the trivial representation of the subgroup S_Λ. Hence,

$$\dim \pi(\Lambda) = n! \Big/ \prod_{i=1}^s \lambda_i!.$$

If $\Gamma = (\gamma_1, \dots, \gamma_t)$ is another Young diagram with n nodes, the multiplicity $(\pi(\Lambda) : \Gamma)$ has an evident combinatorial meaning: $(\pi(\Lambda) : \Gamma)$ is the number of ways to arrange λ_1 ones, λ_2 twos, ..., λ_s numbers s in the Young diagram Γ in such a way that the numbers be nondecreasing from left to right in the rows and increasing downwards in the columns. In particular,

$$(\pi(\Lambda) : \Gamma) > 0 \Leftrightarrow \Gamma \geqslant \Lambda, \quad \text{i.e.,} \quad \forall j, \quad \sum_{i=1}^j \gamma_i \geqslant \sum_{i=1}^j \lambda_i.$$

(All these statements may be found in [6], [7].) If $H = S_\Lambda$, the cone $K^*(H)$ (see Definition 3.8) will be denoted by $K^*(\Lambda)$, and the polytopes $P(H)$, $P^*(H)$, $\bar{P}^*(H)$, and $\bar{P}(H)$ will be denoted by $P(\Lambda)$, $P^*(\Lambda)$, $\bar{P}^*(\Lambda)$, and $\bar{P}(\Lambda)$, respectively.

There exists a more explicit description of the polytope $P(\Lambda)$. Namely, it is easy to write down the equations defining the linear hull

$$L_{\pi(\Lambda)} = \operatorname{lin}\{\pi(\Lambda)(\sigma) : \sigma \in S_n\}$$

of the representation operators of $\pi(\Lambda)$ in the space $V_{\pi(\Lambda)} \otimes V_{\pi(\Lambda)}$. Let $\theta_i : V_{\mu_i} \to V_{\pi(\Lambda)}$ be all possible semistandard homomorphisms (see [6, §13]) mapping the space V_{μ_i} of the irreducible representation μ_i of the group S_n onto the corresponding component of the representation $\pi(\Lambda)$ so that

$$V_{\pi(\Lambda)} = \bigoplus_{i\in I} \theta_i(V_{\mu_i}).$$

Then, by the Frobenius and Schur theorems, the subspace

$$L_{\pi(\Lambda)} \subset V_{\pi(\Lambda)} \otimes V_{\pi(\Lambda)}$$

is defined by the simultaneous equations

$$\begin{aligned} (\theta_i^* \otimes \theta_j^*)x &= 0 \quad \text{if} \quad i \neq j, \\ (\theta_i^* \otimes \theta_i^* - \theta_j^* \otimes \theta_j^*)x &= 0 \quad \text{if the representations } \mu_i \\ &\qquad \text{and } \mu_j \text{ are equivalent.} \end{aligned} \tag{4.1}$$

The formulas for semistandard homomorphisms imply that the coefficients of this system are all equal to $0, \pm 1$. Hence the problem

(4.2) Find

$$\max\{\langle \mathrm{pr}^*(c), x\rangle : x \in \bar{P}(\Lambda)\}$$

(see Corollary 3.13) is a linear programming problem of size

$$\dim^2 \pi(\Lambda) \times \dim^2 \pi(\Lambda)$$

whose matrix consists only of the elements $0, \pm 1$. Consequently, the problem (4.2) may be solved (e.g., by the ellipsoid method [2]) in time, polynomial in

$$\dim \pi(\Lambda) = n!/\lambda_1! \cdots \lambda_s!.$$

We have proved the following statement:

(4.3) Theorem. *Let $\varkappa$ be an irreducible nontrivial representation of the symmetric group S_n with Young diagram K. For any Young diagram $\Lambda = (\lambda_1, \dots, \lambda_s)$ with n nodes, $\Lambda \leqslant K$, there exists an approximate algorithm for the problem (1.3) with the polytope (1.2) whose complexity does not exceed $t(\dim \pi(\Lambda))$, where t is some polynomial independent of n, Λ, K. The value of the objective function $c(\Lambda)$ given by this algorithm is related to its optimal value c_{opt} by the inequalities*

$$\frac{\dim \varkappa}{(\pi(\Lambda) : K)} \geqslant \frac{c(\Lambda)}{c_{\text{opt}}} \geqslant 1.$$

If $c_{\text{opt}} = 0$, then $c(\Lambda) = 0$, and vice versa.

(4.4) Example. Let $\varkappa$ be the irreducible representation of the group S_n with the Young diagram $(n-2, 2)$. Thus, $\varkappa$ is the first "nonpolynomial" representation of the group S_n; it corresponds to the symmetric quadratic assignment problem. Let us take the diagram Λ_j of the form

$$\Lambda_j = (n-j, 1^j), \qquad 1 < j < n,$$

for Λ. We have

$$\dim \varkappa = (n^2 - 3n)/2, \qquad (\pi(\Lambda_j) : \varkappa) = (j^2 - j)/2.$$

Thus, for $\varkappa = (n-2, 2)$, the following statements concerning the problem (1.3) for the polytope (1.2) are valid:

(4.4.1) for any polynomial $f(n) = \alpha n^2$, $\alpha > 0$, there exists a polynomial algorithm whose relative error does not exceed $f(n)$;

(4.4.2) for any function $f(n) = n^\alpha$, $\alpha > 0$, there exists an algorithm of subexponential complexity $O(\exp\{n^\beta\})$, $0 < \beta < 1$, whose relative error does not exceed $f(n)$.

(4.5) Example. Here we consider the polytope $P_\tau(w)$ of the symmetric travelling salesman problem (see example 2.2.2) shifted as usual to the origin by the vector

$$\bar{w} = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(\sigma) w.$$

Thus, we consider the convex hull of the orbit of the element

$$w = \{w_{ij}\}, \qquad 1 \leqslant i, j \leqslant n,$$

$$w_{ij} = \begin{cases} 1 - 2/(n-1) & \text{if } |i-j| \equiv 1 \pmod{n}, \\ 0 & \text{if } i = j, \\ -2/(n-1) & \text{otherwise}, \end{cases}$$

in the representation τ of the symmetric group S_n acting in the space of square matrices by simultaneous permutations of rows and columns (see Example 2.2).

Let us estimate the error appearing when we replace the polytope $P = P_\tau(w)$ of the problem (1.3) by the polytope $P(n-2, 2)$. The vector $w \in \mathbb{R}^{n^2}$ lies in the component of the irreducible representation $(n-2, 2)$. Thus, we use Lemma 3.7 to estimate the error, choosing the element $u \in \mathbb{R}S_n$ as follows:

$$u = \frac{\dim(n-2, 2)}{\langle w, w\rangle} \varphi(w) + e$$

(see Lemma 3.3). Thus

$$\begin{aligned} u &= \frac{n-1}{4n!} \sum_{\sigma \in S_n} \langle \sigma w, w\rangle \sigma + \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \\ &= \frac{n-1}{4n!} \left(\sum_{i,j=1}^{n} \sum_{\sigma\{i, i+1\} = \{j, j+1\}} \sigma \right) - (n-1)e. \end{aligned}$$

Therefore $u + (n-1)e \in K^*(n-2, 2)$. Let

$$\bar{c} = \langle c, \bar{w}\rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \langle c, \sigma w\rangle$$

be the average value of the objective function in the symmetric travelling salesman problem. Applying Lemma 3.7, we come to the following result.

(4.5.1) Corollary. *A polynomial approximate algorithm exists for the symmetric travelling salesman problem computing the value c_0 of the objective function satisfying the inequalities*

$$n \geqslant (c_0 - \bar{c})/(c_{\text{opt}} - \bar{c}) \geqslant 1,$$

where c_{opt} and $\bar{c}$ are the optimum and the average values, respectively.

(4.6) Example. Below we give the equations and inequalities describing the polytope $\bar{P}(n-2, 2) \subset V_{\pi(n-2,2)} \otimes V_{\pi(n-2,2)}$ used to construct the approximate algorithms in the examples (4.4), (4.5). The basis in the space $V_{\pi(n-2,2)}$ is indexed by the unordered pairs $\{i, j\}$, $1 \leqslant i \neq j \leqslant n$. These equations are obtained from the formulas (4.1) (see also [10]).

$\forall k,\ 1 \leqslant k \leqslant n, \quad \forall i, j,\ n \geqslant j > i \geqslant 3,$

$$\sum_{m:\ m \neq k} \left(x^{\{i,j\}}_{\{k,m\}} - x^{\{1,j\}}_{\{k,m\}} - x^{\{2,j\}}_{\{k,m\}} + x^{\{1,2\}}_{\{k,m\}} \right) = 0;$$

$\forall k,\ 1 \leqslant k \leqslant n, \quad \forall j,\ n \geqslant j \geqslant 4,$

$$\sum_{m:\ m \neq k} \left(x^{\{2,j\}}_{\{k,m\}} - x^{\{1,j\}}_{\{k,m\}} - x^{\{2,3\}}_{\{k,m\}} + x^{\{1,3\}}_{\{k,m\}} \right) = 0;$$

$\forall i,\ 1 \leqslant i \leqslant n, \quad \forall m, k,\ n \geqslant m > k \geqslant 3,$

$$\sum_{j:\ j \neq i} \left(x^{\{i,j\}}_{\{k,m\}} - x^{\{i,j\}}_{\{1,m\}} - x^{\{i,j\}}_{\{2,k\}} + x^{\{i,j\}}_{\{1,2\}} \right) = 0;$$

$\forall i,\ 1 \leqslant i \leqslant n, \quad \forall m,\ n \geqslant m \geqslant 4,$

$$\sum_{j:\ j \neq i} \left(x^{\{i,j\}}_{\{2,m\}} - x^{\{i,j\}}_{\{1,m\}} - x^{\{i,j\}}_{\{2,3\}} + x^{\{i,j\}}_{\{1,3\}} \right) = 0;$$

$\forall m,\ 1 \leqslant m \leqslant n, \quad \forall j,\ 1 \leqslant j \leqslant n,$

$$\sum_{\substack{k:\ k \neq m \\ \{i,j\}}} x^{\{i,j\}}_{\{k,m\}} = n - 1, \qquad \sum_{\substack{i:\ i \neq j \\ \{k,m\}}} x^{\{i,j\}}_{\{k,m\}} = (n-1);$$

$$\forall i, j,\ 1 \leqslant i \neq j \leqslant n, \quad \forall k, m,\ 1 \leqslant k, m \leqslant n, \qquad x^{\{i,j\}}_{\{k,m\}} \geqslant 0.$$

The polytope $\bar{P}(K)$, where $K = K^*(n-2, 2) + K^*(n-1, 1)$, is described by the above system together with the additional inequalities

$$\forall i,\ 1 \leqslant i \leqslant n, \quad \forall k,\ 1 \leqslant k \leqslant n, \qquad \sum_{j,m} x^{\{i,j\}}_{\{k,m\}} \geqslant 1.$$

§5. The combinatorial structure of the polytope of the representation operators

(5.1) Let us return to the examples (2.1.2), (2.2.3). As we have already mentioned, the polytope of the natural representation operators is completely described by the Birkhoff-von Neumann theorem. The facets of the polytope P_p are all the possible tuples of vertices

$$\Gamma_i^j = \{\sigma \in S_n : \sigma(i) \neq j\}, \qquad 1 \leqslant i, j \leqslant n.$$

The polytope P_τ (Example 2.2.3) is much more complicated. It is easy to verify that the sets

$$\Gamma^{j_1 j_2}_{i_1 i_2} = \{\sigma \in S_n : (\sigma(i_1) \neq j_1) \vee (\sigma(i_2) \neq j_2)\}, \qquad i_1 \neq i_2,\ j_1 \neq j_2, \tag{5.1}$$

are facets of the polytope P_τ, but they do not constitute a complete list of facets.

(5.2) Example. Let us fix a partition

$$I \cup J = \{1, 2, \dots, n-2\}, \qquad I \cap J = \varnothing,\ I \neq \varnothing,\ J \neq \varnothing.$$

Set $\Gamma_{I,J} = \{\sigma \in S_n : r(\sigma) = 0\}$, where

$$r = \sum_{\substack{\sigma(n)=n \\ \sigma(1) \in I}} \sigma + \sum_{\substack{\sigma(n-1)=n-1 \\ \sigma(1) \in J}} \sigma - \sum_{\substack{\sigma(n)=n \\ \sigma(n-1)=n-1}} \sigma.$$

It is easy to verify that for different I, J, the sets $\Gamma_{I,J}$ are different facets of the polytope P_τ (see [10]). Let us show that the family (5.1) includes the most degenerate facets of P_τ.

(5.3) Theorem. *Let $P_\tau^* \subset \mathbb{R}S_n$ be the polytope dual to P_τ defined by the formula* (3.5). *Then*

$$\forall r \in P_\tau^*, \quad \forall \sigma \in S_n, \qquad 0 \leqslant r(\sigma) + \frac{1}{n!} \leqslant \frac{1}{(n-2)!}.$$

Proof. The inequality $r(\sigma) + 1/n! \geqslant 0$ follows from (3.5). Since the polytope P_τ^* is S_n-invariant, it is sufficient to prove that

$$r(\varepsilon) + \frac{1}{n!} \leqslant \frac{1}{(n-2)!},$$

where ε is the unit element of the group S_n. Set

$$\bar{r} = \frac{1}{n!} \sum_{\sigma \in S_n} h(r+e)h^{-1}.$$

The element $\bar{r}$ lies in the center of the group algebra $\mathbb{R}S_n$, $\bar{r} \in \Delta$, and $\bar{r}(\varepsilon) = r(\varepsilon) + 1/n!$. Therefore, it is sufficient to prove that $\bar{r}(\varepsilon) \leqslant 1/(n-2)!$.

One may choose the basis of the center of the group algebra $\mathbb{R}S_n$ consisting of elements of the form

$$f_\Lambda = \frac{1}{n!} \sum \chi_{\pi(\Lambda)}(\sigma)\, \sigma,$$

where Λ is a Young diagram with n nodes, and $\chi_{\pi(\Lambda)}$ is the character of the induced representation $\pi(\Lambda)$. Since

$$\tau = (n-2, 1, 1) \oplus (n-2, 2) \oplus 3(n-1, 1) \oplus 2(n),$$

we have $\bar{r} = \sum \alpha_\Lambda f_\Lambda$, where Λ ranges over the diagrams $(n-2, 1, 1)$, $(n-2, 2)$, $(n-1, 1)$, (n). Next, $\sum \alpha_\Lambda = 1$ since $\bar{r} \in \Delta$ and $f_\Lambda \in \Delta$. Note that $\forall \sigma \in S_n$, $\bar{r}(\sigma) \geqslant 0$.

Let us consecutively substitute certain elements of the group S_n for σ. Namely,

$$\sigma = (1\,2 \cdots n-3)(n-2 \quad n-1)(n),$$

$$\bar{r}(\sigma) = \frac{1}{n!}(\alpha_{(n)} + \alpha_{(n-1,1)} + \alpha_{(n-2,2)}) \geqslant 0,$$

i.e., $\alpha_{(n-2,1,1)} \leqslant 1$;

$$\sigma = (1\,2 \cdots n-1)(n), \qquad \bar{r}(\sigma) = \frac{1}{n!}(\alpha_{(n)} + \alpha_{(n-1,1)}) \geqslant 0,$$

i.e., $\alpha_{(n-2,2)} \leqslant 1 - \alpha_{(n-2,1,1)}$;

$$\sigma = (1\,2 \cdots n-2)(n-1\; n), \qquad \bar{r}(\sigma) = \frac{1}{n!}(\alpha_{(n)} + \alpha_{(n-2,2)}) \geqslant 0,$$

i.e., $\alpha_{(n-1,1)} \leqslant 1 - \alpha_{(n-2,1,1)}$.

Combining these inequalities with the equation

$$\alpha_{(n)} + \alpha_{(n-1,1)} + \alpha_{(n-2,2)} + \alpha_{(n-2,1,1)} = 1,$$

we obtain

$$\bar{r}(\varepsilon) = \frac{1}{n!}\left(\alpha_{(n)} + n\,\alpha_{(n-1,1)} + \binom{n}{2}\alpha_{(n-2,2)} + 2\binom{n}{2}\alpha_{(n-2,1,1)}\right)$$
$$\leqslant \frac{1}{n!}\cdot 2\cdot\binom{n}{2} = \frac{1}{(n-2)!}.$$

The theorem is proved.

(5.4) Corollary.

(5.4.1) *Every face of the polytope P_τ contains at most $n! - (n-2)!$ vertices. The faces defined by the formulas* (5.1) *contain $n! - (n-2)!$ vertices exactly.*

(5.4.2) *One may inscribe a ball into the polytope P_τ so that the supporting hyperplanes of the facets* (5.1) *are tangent to it.*

Proof. The statement (5.4.1) follows, by duality, from the fact that at most $n! - (n-2)!$ facets intersect at any point of the polytope P^*_τ, i.e.,

$$\forall r \in P^*_\tau, \qquad \operatorname{card}\left\{\sigma : r(\sigma) + \frac{1}{n!} = 0\right\} \leqslant n! - (n-2)!.$$

The statement (5.4.2) is valid since

$$\forall r \in P^*_\tau, \qquad \|r\|^2 = \sum_{\sigma\in S_n} r^2(\sigma) \leqslant \frac{1}{(n-2)!} - \frac{1}{n!},$$

and the equality is achieved for the elements

$$\frac{1}{(n-2)!}\sum_{\substack{\sigma(i_1)=j_1,\\ \sigma(i_2)=j_2}} \sigma - e$$

corresponding to the facets (5.1).

§6. Computation of the statistical sums

The problem (1.3) may be solved in polynomial time in n for the polytopes (2.1), (2.2.1), (2.4). While this phenomenon may be partially explained for the polytopes (2.1) from the viewpoint of approximations (cf. Remark 3.10), the polytope (2.2.1) corresponding to the matching problem is in no aspect distinguished from the viewpoint of the methods developped in §3. In this section we develop an alternative approach, requiring computation of a certain functional (statistical sum) on the group G.

(6.1) Definition. Let $P \subset V$ be a polytope with the set of vertices $\mathrm{Vert}(P)$. An arbitrary complex-valued function μ: $\mathrm{Vert}(P) \to \mathbb{C}$ will be referred to as a *charge*. Given an affine function $f(x) = \langle c, x\rangle + b$, $c \in V$, $b \in \mathbb{R}$, let us define the statistical sums

$$S_\mu(f; t) = \sum_{x\in\mathrm{Vert}(P)} \exp\{t\, f(x)\}\,\mu(x), \qquad t \in \mathbb{R}, \tag{6.1.1}$$

$$\sigma_\mu(f; m) = \sum_{x\in\mathrm{Vert}(P)} f^m(x)\,\mu(x), \qquad m \in \mathbb{N}. \tag{6.1.2}$$

The following evident result is valid.

(6.2) Proposition. *Let P be a polytope and μ be a charge such that $\forall x \in \mathrm{Vert}(P)\, \mu(x) \neq 0$ and $\forall x, y \in \mathrm{Vert}(P)\, x \neq y \Rightarrow f(x) \neq f(y)$. Then*

$$\lim_{t\to+\infty} t^{-1}\log|S_\mu(f; t)| = \max\{f(x) : x \in P\}. \tag{6.2.1}$$

(6.2.2) *Suppose additionally that $\forall x$, $f(x) > 0$. Then*

$$\lim_{m\to+\infty} |\sigma_\mu(f; m)|^{1/m} = \max\{f(x) : x \in P\}.$$

(6.3) Remarks. The injectivity condition is evidently the general position condition for the function f. It may be omitted if one supposes that $\forall x$, $\mu(x) > 0$. Due to the evident identities

$$S_\mu(f + a; t) = \exp\{ta\}\, S_\mu(f, t),$$

$$\sigma_\mu(f + a; m) = \sum_{k=0}^{m} \binom{m}{k} a^k \sigma_\mu(f; m-k), \qquad a \in \mathbb{R},$$

is suffices to be able to compute the sums (6.1) for a linear function

$$f(x) = \langle c, x\rangle;$$

they will be denoted by $S_\mu(c, t)$ and $\sigma_\mu(c, m)$, respectively.

We do not dwell here on the purely technical questions concerning the estimates of the convergence rate in Proposition 6.2. As for the development and applications of the statistical sum method, see [18], [19]. In particular, it is shown in these papers that the design of effective algorithms for the problem (1.3) requires only the ability to compute sums of the form (6.1.1), (6.1.2) for an appropriate charge μ. It will be shown below how to construct a charge $\mu : \forall x, \mu(x) \neq 0$ for the polytopes (2.1.2), (2.2.1), and (2.4) in such a way that the sum (6.1.1) can be evaluated effectively for any linear functional $f = \langle c, x \rangle$.

Case (2.1.2). Let us define the charge μ on the vertices of the polytope P_ρ by the formula $\mu(\rho(\sigma)) = \operatorname{sgn}\sigma$, where $\operatorname{sgn}\sigma = 1$ if σ is an even permutation, and $\operatorname{sgn}\sigma = -1$ if σ is an odd permutation. Let a linear functional be given by its matrix $c = (c_{ij})$, $1 \leqslant i, j \leqslant n$. Set

$$C_{ij} = \exp\{tc_{ij}\}, \qquad C = (C_{ij}).$$

Then $S_\mu(c, t) = \det C$. The determinant of a matrix of size $n \times n$ is well-known to be computable in $O(n^3)$ arithmetic operations. Consequently, the sum $S_\mu(c, t)$ may be computed in $O(n^3)$ arithmetic operations and applications of the exponential functions.

Case (2.2.1). Let is define a charge μ by the formula $\mu(\tau(\sigma)v) = \operatorname{sgn}\sigma$. The stabilizer of the element v being formed by even permutations only, this formula defines the charge on the polytope's vertices correctly. Set

$$C_{ij} = \exp\{tc_{ij}\}$$

for a linear functional with the matrix $c = (c_{ij})$, $1 \leqslant i, j \leqslant n$. We have $S_\mu(c, t) = \operatorname{Pf}(c)$. The Pfaffian of a matrix of size $n \times n$ may be computed in $O(n^3)$ arithmetic operations (see [20, pp. 318–329]).

Case (2.4). Let the charge μ be defined by the formula

$$\mu(\rho(w)) = \det\rho(w), \qquad w \in W_n.$$

We list below the expressions for the sums (6.1.1).

The A_n series (see (2.1)).

The B_n, C_n series. For a linear functional with matrix $c = (c_{ij})$, $1 \leqslant i, j \leqslant n$, we have

$$S_\mu(c; t) = \det(\exp\{tc_{ij}\} - \exp\{-tc_{ij}\}), \qquad 1 \leqslant i, j \leqslant n.$$

The D_n series. For a linear functional defined by the matrix $c = (c_{ij})$ we have

$$S_\mu(c; t) = \frac{1}{2}\det(\exp\{tc_{ij}\} - \exp\{-tc_{ij}\}) + \frac{1}{2}\det(\exp\{tc_{ij}\} + \exp\{-tc_{ij}\}), \qquad 1 \leqslant i, j \leqslant n.$$

Given a weight v, let us introduce a statistical sum with respect to some measure over the polytope $P_\rho(v)$. Let λ be a half-sum of positive roots, and denote by $(v : u)$ the multiplicity of the weight u in the representation corresponding to the weight v. By the Weyl formula [17], the identity

$$\sum_{u \in P_\rho(v)} \exp\{t\langle c, u\rangle\}(v : u) = (S_\mu(c; t)|_{P = P_\rho(\lambda)})^{-1} \cdot S_\mu(c; t)|_{P = P_\rho(v+\lambda)}$$

is valid for a linear functional $c = (c_1, \dots, c_n)$. Here the sum (6.1.1) for the polytope $P = P_\rho(\lambda)$ stands in the denominator, the charge μ being the one described above; the numerator contains the sum (6.1.1) for the polytope $P = P_\rho(v+\lambda)$ with the same charge μ (here we do not consider the degenerate case of vanishing denominator). Note that by the formulas of example (2.4) both sums may be computed by an algorithm with a polynomial in n number of arithmetic operations and applications of the exponential function. The latter example proves to be rather useful in combinatorial optimization (it is considered in detail in the author's paper [19] for $W_n = S_{n-1}$).

Thus, in the examples (2.1.2), (2.2.1), and (2.4) the sum (6.1.1) is computable, for an appropriate nontrivial charge μ, in a polynomial in n number of arithmetic operations and applications of the exponential function. This fact might serve as an algebraic "explanation" of the relative simplicity of the corresponding polytope's structure.

Except for trivial ones, the author does not know any examples of polytopes of the form $P_\varkappa(v)$, $P_\varkappa$ admitting a polynomial-time algorithm for the problem (1.3) and distinct from the above (or their evident modifications).

However, other polytope series may also admit such algorithms for linear functionals of a special form. Below we consider the problem of computing the sum (6.1.1) in the example (2.3) and the sum (6.1.2) in the example (2.2.3), the charge μ and the functional c of special form being chosen appropriately. The corresponding algorithmic results for the problems (1.3) have been proved in the author's papers [19] and [21].

Case (2.5). Suppose that l is even. In this case the stabilizer of the element v consists of even permutations only, and the formula $\mu(\nu(\sigma)) = \operatorname{sgn}\sigma$ correctly defines a charge. Set

$$C_{i_1, i_2, \dots, i_l} = \exp\{tc_{i_1, \dots, i_l}\}.$$

Then, evidently, $S_\mu(c; t) = P(C)$, where P is some polynomial in the coefficients of the tensor C. Consider the space $V_n = (\mathbb{R}^n)^{\otimes l}$ and define the action of the general linear group $GL(n, \mathbb{R})$ in V_n as the lth tensor power of its natural action in $\mathbb{R}^n$.

(6.4) PROPOSITION (see [7, p. 327]). *The polynomial P is a relative invariant of the group $GL(n, \mathbb{R})$, namely,*

$$\forall G \in GL(n; \mathbb{R}), \qquad P(G(C)) = \det G \cdot P(C).$$

The problem of computation of the invariant P is NP-hard when $l > 2$. The difference from example (2.2.1) (the case $l = 2$) is that for $l > 2$ there are no more "normal forms" for tensors $c \in V_n$ with respect to the action of $GL(n, \mathbb{R})$ described above

(6.5) Definition. The *rank* of the tensor C is the minimal number r, $r = \operatorname{rank} C$, $r \in \mathbb{N}$, such that C possesses a representation of the form

$$C = \sum_{i=1}^{r} u^{i1} \otimes \cdots \otimes u^{il}, \quad \text{where} \quad u^{ij} \in \mathbb{R}^n. \tag{6.5.1}$$

Similarly, the *2-rank* of a tensor C is a minimal number $r = \operatorname{rank}_2 C$ such that C possesses a representation of the form

$$C = \sum_{i=1}^{r} a^{i1} \otimes \cdots \otimes a^{il/2}, \tag{6.5.2}$$

where $a^{ij} \in (\mathbb{R}^n)^{\otimes 2}$ (here l is assumed to be even).

(6.6) Theorem. *Let a number $k \in \mathbb{N}$ be fixed, and let one of the following two conditions be satisfied:*

(6.6.1) $\operatorname{rank} C = n/l + k$, *and the tensor C is represented in the form* (6.5.1).

(6.6.2) $\operatorname{rank}_2 C = k$, *and the tensor C is represented in the form* (6.5.2).

Then the value $P(C)$ may be computed in a number of arithmetic operations polynomial in n. The condition (6.6.1) *being replaced by the condition $\operatorname{rank} C = 2n/l$, C being represented in the form* (6.5.1), *the problem of computing $P(C)$ is polynomially equivalent to the problem of computing the invariant P for an arbitrary tensor of degree l.*

Proof. See [21].

Case (2.2.3). Let us define the charge μ on the vertices of the polytope P_τ by the formula

$$\mu(\tau(\sigma)) = \operatorname{sgn} \sigma, \quad \sigma \in S_n.$$

The linear functional c on the space $\operatorname{End}(V_\tau)$ of the representation $\tau = \rho^{\otimes 2}$ will be represented as a quadratic form on the space $\operatorname{End}(V_\rho)$ of operators of the representation ρ (see (2.1)).

(6.7) Theorem. *Suppose that the rank of the quadratic form c is fixed (i.e., it does not depend on n). Then* (6.1.2) *may be computed with an arbitrary precision ε using a number of arithmetic operations and applications of the exponential function polynomial in n, m.*

Proof. Let us expand the form c into the sum of k forms of rank 1,

$$c = \sum_{i=1}^{k} a^i \otimes b^i, \qquad k = \operatorname{rank} c,$$

where a^i, b^i are linear forms on the space $\operatorname{End}(V_\rho)$. Then

$$\sigma_\mu(c\,;\, m) = \sum_{\alpha = (\alpha_1, \ldots, \alpha_k)} \binom{m}{\alpha_1, \ldots, \alpha_k} \times \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \cdot \prod_{i=1}^{k} (\langle a^i, \rho(\sigma)\rangle \langle b^i, \rho(\sigma)\rangle)^{\alpha_i},$$

where the outer sum is taken over all the multi-indices α such that

$$\alpha_1 + \cdots + \alpha_k = m \quad \text{and} \quad \alpha_i \geqslant 0.$$

Note that each of the $\binom{m+k-1}{m}$ summands has the form

$$\frac{\partial^{2m}}{\partial t_1^{\alpha_1} \cdots \partial t_k^{\alpha_k} \partial y_1^{\alpha_1} \cdots \partial y_k^{\alpha_k}} \times \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \cdot \exp \left\{ \sum_{i=1}^{k} t_i \langle a^i, \rho(\sigma)\rangle + y_i \langle b^i, \rho(\sigma)\rangle \right\} \Bigg|_{t_1 = \cdots = t_k = y_1 = \cdots = y_k = 0}.$$

The expression whose derivative is taken is the statistical sum (6.1) for the polytope of the example (2.1.2) whose value at any point may be computed in $O(n^3)$ arithmetic operations and applications of the exponential function.

Let us replace the derivative by the difference operator

$$D(\Delta t) = \frac{1}{(\Delta t)^{2m}} \prod_{i=1}^{k} (Y_i - I)^{\alpha_i} (T_i - I)^{\alpha_i},$$

where Y_i, T_i are the operators shifting the arguments y_i, t_i, respectively, by Δt, and I is the identity operator. Thus, the application of the operator $D(\Delta t)$ requires computation of the sum under the derivative sign at

$$\prod_{i=1}^{k} (\alpha_i + 1)^2 \leqslant \left(\frac{m}{k} + 1\right)^{2k}$$

points. The formula for the remainder of the Taylor series allows one to estimate easily the value of Δt required to provide the given absolute error ε (see [19] for more details).

A similar result may be obtained for the other Weyl groups. The case of the symmetric group is considered here since it has essential applications in combinatorial optimization (see [19]).

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