# Math. 669: Combinatorial Theory of Polytopes with Applications to Optimization

## Winter term, 1995



**Contents.** 

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Monday, January 9

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## 1. Basics

**(1.1) Euclidean space.** We denote by $\mathbb{R}^d$ the $d$-dimensional Euclidean space. As a set, $\mathbb{R}^d$ consists of all $d$-tuples $x = (x_1, \ldots, x_d)$ of real numbers $x_1, \ldots, x_d$. An element $x \in \mathbb{R}^d$ is called a *point*. As usual, we define the *sum* $z = x + y$ of two points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ as the point $z = (z_1, \ldots, z_d)$, where $z_i = x_i + y_i$ for $i = 1, \ldots, d$. For a real number $\alpha$ and a point $x \in \mathbb{R}^d$ we define the *product* $y = \alpha x$ as the point $y = (y_1, \ldots, y_d)$, where $y_i = \alpha x_i$ for $i = 1, \ldots, d$. Together with linear operations we consider the *scalar product* on $\mathbb{R}^d$ given by the formula:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_d y_d, \quad \text{where} \quad x = (x_1, \ldots, x_d) \text{ and } y = (y_1, \ldots, y_d).$$

This scalar product defines the *Euclidean* norm $\|x\| = \sqrt{\langle x, x \rangle}$ and the corresponding Euclidean *distance*. The distance between two points $x$ and $y$ from $\mathbb{R}^d$ is equal to $\|x - y\|$.

A real-valued function $f$ on $\mathbb{R}^d$ is called *linear* if for any two points $x$ and $y$ from $\mathbb{R}^d$ and for any two real numbers $\alpha$ and $\beta$ one has $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

**(1.2) Convex sets.** Let $x$ and $y$ be two points in $\mathbb{R}^d$. The set $[x, y] = \big\{\alpha x + (1 - \alpha) y : 0 \leq \alpha \leq 1\big\}$ is called the *interval* between $x$ and $y$. The points $x$ and $y$ are called the *endpoints* of the interval.

We say that a set $A \in \mathbb{R}^d$ is *convex* if for any two points $x, y \in A$ we have $[x, y] \subset A$. We assume that the empty set $\emptyset$ is convex. It is immediate that the intersection of convex sets is convex.

Let $A \subset \mathbb{R}^d$ be a set. The *convex hull* of $A$ (denoted conv $A$) is the intersection of all convex sets containing $A$. In other words, conv $A$ is the smallest convex set containing $A$.

PROBLEMS.

1°. Describe convex sets in $\mathbb{R}^1$.

2°. Let $A \subset \mathbb{R}^d$ be a convex set and $T : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ be a linear map (that is, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for any $x, y \in \mathbb{R}^d$ and any $\alpha, \beta \in \mathbb{R}$). Then the image $T(A) \subset \mathbb{R}^k$ is convex.

3°. Let $A, B \subset \mathbb{R}^d$ be two sets. The set $A + B = \big\{x + y : \ x \in A, \ y \in B\big\}$ is called their *Minkowski sum*. Show that if $A$ and $B$ are convex then $A + B$ is also convex.

For a set $A \subset \mathbb{R}^d$ and a number $\alpha$ let us define the product $\alpha A = \big\{\alpha x : \ x \in A\big\}$. Show that $\alpha A$ is convex provided $A$ is convex.

Suppose that $A \subset \mathbb{R}^d$ is convex and $\alpha, \beta > 0$. Show that $(\alpha + \beta) A = \alpha A + \beta A$. Show that this is no longer true if $A$ is not convex.

4. Let $A \subset \mathbb{R}^d$ be a finite set consisting of at least $d + 2$ points. Then one can represent $A = B \cup C$ for some disjoint non-empty subsets $B$ and $C$ such that conv $C \cap$ conv $B \neq \emptyset$ (Radon's Theorem).

*Hint:* Let $A = \{x_1, \ldots, x_m\}$. Consider a non-zero solution to the linear system:

$$\lambda_1 x_1 + \ldots + \lambda_m x_m = 0, \quad \lambda_1 + \ldots + \lambda_m = 0.$$

Let $B = \{x_i : \ \lambda_i \geq 0\}$ and $C = \{x_i : \ \lambda_i < 0\}$.

5. Let $A_i \subset \mathbb{R}^d : \ i = 1, \ldots, m$ be a family of convex sets. Suppose that the intersection of any $d + 1$ of these sets is not empty. Then the intersection $\bigcap_{i=1}^m A_i$ is not empty (Helly's Theorem).

*Hint:* Use the induction on $m$ and Problem 4.

6*. Let $S = \{x : \ \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$. Consider a quadratic map $q : \mathbb{R}^d \longrightarrow \mathbb{R}^2$, $q(x) = \big(q_1(x), q_2(x)\big)$, where $q_1, q_2$ are quadratic forms. Prove that the image $q(S) \subset \mathbb{R}^2$ is convex provided $d > 2$ (Toeplitz-Hausdorff Theorem). Can you generalize this theorem? How?

## 2. Polytopes

**(2.1) Definition.** A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$.

**(2.2) Lemma.** *Let $A = \{a_1, \ldots, a_m\}$ be a finite set of points in $\mathbb{R}^d$. Then*

$$\text{conv } A = \Big\{\sum_{i=1}^{m} \gamma_i a_i : \ \gamma_i \geq 0 \text{ and } \gamma_1 + \ldots + \gamma_m = 1\Big\}.$$

**Proof.** Let

$$B = \Big\{\sum_{i=1}^{m} \gamma_i a_i : \ \gamma_i \geq 0 \text{ and } \gamma_1 + \ldots + \gamma_m = 1\Big\}.$$

Then $B$ is a convex set, since for any two points $x = \alpha_1 a_1 + \ldots + \alpha_m a_m$ and $y = \beta_1 a_1 + \ldots + \beta_m a_m$ and any $0 \leq \lambda \leq 1$ the point $(1-\lambda)x + \lambda y = \gamma_1 a_1 + \ldots + \gamma_m a_m$, where $\gamma_i = (1-\lambda)\alpha_i + \lambda \beta_i$ belongs to $B$. Therefore (see (1.2)) we must have conv $A \subset B$. Let us choose $x = \gamma_1 a_1 + \ldots + \gamma_m a_m$ from $B$. We must show that $x \in$ conv $A$. To do that, we use induction on the number $r$ of non-zero $\gamma$-s. If $r = 1$ then $x = a_i$ for some $i = 1, \ldots, m$ and obviously $x \in$ conv $A$. Suppose that there are $r + 1$ non-zero $\gamma$-s. Without loss of generality we may assume that $x = \gamma_1 a_1 + \ldots + \gamma_{r+1} a_{r+1}$ for some positive $\gamma_1, \ldots, \gamma_{r+1}$ such that $\gamma_1 + \ldots + \gamma_{r+1} = 1$. Let $\beta_i = \gamma_i/(\gamma_1 + \ldots + \gamma_r)$ for $i = 1, \ldots, r$. Then $\beta_1$ are positive and $\beta_1 + \ldots + \beta_r = 1$. Therefore, using the induction conjecture we conclude that the vector $y = \beta_1 a_1 + \ldots + \beta_r a_r$ belongs to conv $A$. On the other hand, we may write

$$x = (1 - \gamma_{r+1})y + \gamma_{r+1} a_{r+1}.$$

Since $y, a_{r+1} \in$ conv $A$ and conv $A$ is convex, we conclude that $x \in$ conv $A$. □

**(2.3) Examples.**

1. **Simplex.** Let $e_1, \ldots, e_{d+1}$ be the standard orthonormal basis in $\mathbb{R}^{d+1}$. The polytope $\Delta_d = \text{conv}\{e_1, \ldots, e_{d+1}\}$ is called the *standard* $d$-dimensional simplex.

In other words,

$$\Delta_d = \Big\{x = (x_1, \ldots, x_{d+1}) : \ x_i \geq 0 \text{ and } x_1 + \ldots + x_{d+1} = 1\Big\}.$$

2. **Cube.** Let $\{u_i : \ i \in I\}$ be the set of all $2^d$ vectors in $\mathbb{R}^d$ whose coordinates are either 1 or $-1$. The polytope $I_d = \text{conv}\{u_i : \ i \in I\}$ is called the standard $d$-dimensional cube. In other words,

$$I_d = \big\{x = (x_1, \ldots, x_d) : \ -1 \leq x_i \leq 1\big\}.$$

3. **Crosspolytope, also (hyper)octahedron** is the convex hull of $2d$ vectors $e_1, -e_1, \ldots, e_d, -e_d$ in $\mathbb{R}^d$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$:

$$O_d = \text{conv}\big\{e_1, -e_1, \ldots, e_d, -e_d\big\}.$$

In other words,

$$O_d = \big\{x = (x_1, \ldots, x_d) : |x_1| + \ldots + |x_d| \leq 1\big\}.$$

4. **Cyclic Polytope.** The curve $\gamma(t) = (t, t^2, \ldots, t^d) : -\infty \leq t \leq +\infty$ in $\mathbb{R}^d$ is called the *moment curve*. Let $t_1 < t_2 < \ldots < t_k$, $k > d+1$ be real numbers. The polytope

$$C_d(k) = \text{conv}\big\{\gamma(t_1), \ldots, \gamma(t_k)\big\}$$

is called *a Cyclic Polytope*.

5. **Birkhoff Polytope.** With a permutation $\sigma$ of the set $\{1, \ldots, d\}$ we associate the $d \times d$ matrix $X^\sigma$ as follows:

$$X^\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

The convex hull of all $d!$ permutation matrices

$$B_d = \text{conv}\big\{X^\sigma : \sigma \text{ is a permutation } \big\}$$

is called the *Birkhoff polytope*. In other words,

$$B_d = \left\{x = (x_{ij}) : x_{ij} \geq 0, \qquad \sum_{i=1}^d x_{ij} = 1, \ j = 1, \ldots, d, \qquad \sum_{j=1}^d x_{ij} = 1, \ i = 1, \ldots, d.\right\}$$

PROBLEM.

1°. Prove "in other words" part in Examples 1,2 and 3 above.

2*. Prove "in other words" part in Example 5 and find an "in other words" description in Example 4.

**(2.4) Theorem (Carathéodory's theorem).** *Let $A = \{a_1, \ldots, a_m\} \subset \mathbb{R}^d$ be a set. Then*

$$\text{conv } A = \left\{\sum_{i \in I} \gamma_i a_i : \gamma_i \geq 0, \quad \sum_{i \in I} \gamma_i = 1, \quad \text{where } I \subset \{1, \ldots, m\} : |I| = d+1\right\}.$$

**Proof.** Let us choose a point $x \in \operatorname{conv} A$. Then one can write $x$ in the form $x = \sum_{i=1}^{k} \lambda_i a_i$ for some positive $\lambda_i$ such that $\lambda_1 + \ldots + \lambda_k = 1$ (Lemma 2.2). We want to prove that if $k > d+1$ then we can find a representation with a smaller number of positive $\lambda$-s. Let us consider $k$ vectors $(a_1, 1), \ldots, (a_k, 1)$ in $\mathbb{R}^{d+1}$. If $k > d+1$ then they are linearly dependent and, therefore, there are numbers $\mu_1, \ldots, \mu_k$ (not all of them zeros) such that $\mu_1 a_1 + \ldots + \mu_k a_k = 0$ and $\mu_1 + \ldots + \mu_k = 0$. Let $t = \min\{\lambda_i/\mu_i : \ \mu_i > 0\}$ and let $\hat{\lambda}_i = \lambda_i - t\mu_i$. Then the representation

$$x = \sum_{i=1}^{k} \hat{\lambda}_i a_i$$

contains a smaller number of positive $\hat{\lambda}$-s. $\square$

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Wednesday, January 11

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## 3. Separation

**(3.1) Some topology.** For a point $a \in \mathbb{R}^d$ and a positive number $r$ by $B_r(a)$ we denote the ball with the center $a$ and radius $r$:

$$B_r(a) = \big\{x \in \mathbb{R}^d : \ \|x - a\| \leq r\big\}.$$

We say that a set $A \subset \mathbb{R}^d$ is *open* iff for any point $a \in A$ there exists $r > 0$ such that $B_r(a) \subset A$. A set $A \subset \mathbb{R}^d$ is said to be *closed* if the complement $\mathbb{R}^d \setminus A$ is open. We agree that the empty set $\emptyset$ is closed and open at the same time.

A closed bounded set in $\mathbb{R}^d$ is *compact*. If $f$ is a continuous map of $\mathbb{R}^d$ and $A$ is compact then the image $f(A)$ is compact, and, in particular, closed.

PROBLEMS.

1°. The standard simplex $\Delta_d \subset \mathbb{R}^{d+1}$ is compact (see Example 2.3, 1).

2°. The standard cube $I_d \subset \mathbb{R}^d$ is compact (see Example 2.3, 2).

**(3.2) Theorem.** *A polytope is compact.*

**Proof.** Let $P = \operatorname{conv}\{a_1, \ldots, a_m\}$, $a_i \in \mathbb{R}^d$ be a polytope. Let $\Delta_{m-1} = \big\{(\lambda_1, \ldots, \lambda_m) : \lambda_1 + \ldots + \lambda_m = 1, \lambda_i \geq 0\big\}$ be the standard $(m-1)$-dimensional simplex. Let us consider the map $f : \Delta_{m-1} \longrightarrow \mathbb{R}^d$ defined by the formula: $f(\lambda_1, \ldots, \lambda_m) = \lambda_1 a_1 + \ldots + \lambda_m a_m$. Lemma 2.2 implies then that $f(\Delta_{m-1}) = P$. Since $\Delta_{m-1}$ is compact (see Problem 1° above) and $f$ is continuous, the proof follows. $\square$

Along with Theorem 3.2 we proved an important fact: *every polytope is the image of a standard simplex under a linear map.*

PROBLEMS.

$1^\circ$. Every centrally symmetric polytope $P = \mathrm{conv}\{a_1, -a_1, \ldots, a_m, -a_m\}$ is the image of the octahedron $O_m$ (Examples 2.3, 3) under a linear map.

2. Let $A \subset \mathbb{R}^d$ be a compact set. Then conv $A$ is a compact set.

*Hint:* Let $\Delta_d = \{(\lambda_1, \ldots, \lambda_{d+1}) :\ \lambda_1 + \ldots + \lambda_{d+1} = 1,\ \lambda_i \geq 0\}$ be the standard $d$-dimensional simplex. Using Theorem 2.4 and a modification of Lemma 2.2 prove that conv $A$ is the image of the product $\Delta_d \times A \times \ldots \times A$ ($d+1$ factors $A$ altogether) under the map

$$(\lambda_1, \ldots, \lambda_{d+1}, a_1, \ldots, a_{d+1}) \longmapsto \lambda_1 a_1 + \ldots + \lambda_{d+1} a_{d+1}.$$

3. The convex hull of a closed set in $\mathbb{R}^d$ is not necessarily closed.

*Hint:* Consider the set consisting of a straight line and a point in $\mathbb{R}^2$.

4. If $A, B \subset \mathbb{R}^d$ are compact then their Minkowski sum $A + B$ (see Section 1.2, Problem $3^\circ$) is compact.

*Hint:* Consider the map $f : A \times B \longrightarrow \mathbb{R}^d,\ f(a, b) = a + b$.

5. The Minkowski sum (see Section 1.2, Problem $3^\circ$) of two closed convex sets in $\mathbb{R}^d$ is not necessarily closed.

*Hint:* Consider the sets $A = \{(x, y) \in \mathbb{R}^2 :\ xy \geq 1 \text{ and } x > 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 :\ xy \leq -1 \text{ and } x > 0\}$.

**(3.3) Hyperplanes and affine hyperplanes.** Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. As we know, $f(x) = \langle c, x\rangle$ for some vector $c \in \mathbb{R}^d$. If $c \neq 0$ then $f(x)$ is not identically zero (why?). Suppose that $f$ is not identically zero. Then the set $H = \{x \in \mathbb{R}^d :\ f(x) = 0\}$ is called a *linear hyperplane*. Equivalently, $H$ is a linear subspace of codimension 1. For a number $\alpha$ the set $H = \{x \in \mathbb{R}^d :\ f(x) = \alpha\}$ is called a *hyperplane* (or *affine hyperplane*). A hyperplane dissects $\mathbb{R}^d$ into two *halfspaces* $H_+ = \{x \in \mathbb{R}^d :\ f(x) \geq \alpha\}$ and $H_- = \{x \in \mathbb{R}^d :\ f(x) \leq \alpha\}$.

PROBLEMS.

$1^\circ$. Halfspaces and hyperplanes are convex closed sets.

$2^\circ$. Any hyperplane is a translation of a linear hyperplane.

**(3.4) Theorem.** *Let $A \subset \mathbb{R}^d$ be a (non-empty) convex closed set and $b \notin A$. Then there exists a linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and a number $\alpha$ such that $f(b) > \alpha > f(x)$ for every $x \in A$. In this case we say that the hyperplane $H = \{x \in \mathbb{R}^d : f(x) = \alpha\}$ strictly separates $A$ and $b$.*

**Proof.** Let us consider the function $\phi(x) = \|x - b\|^2$ for $x \in A$. We claim that $\phi$ attains its minimum on $A$, that is, there is $a \in A$ such that $\phi(a) \geq \phi(x)$ for every $x \in A$. Indeed, for a sufficiently large $r$ the intersection of the ball $B_r(b)$ and $A$ is a non-empty compact set, and, therefore there exists

$$\min\{f(x) :\ x \in B_r(b) \cap A\} = \min\{f(x) :\ x \in A\} = f(a) = \|b - a\|^2 > 0.$$

Now let $f(x) = \langle b - a, x \rangle$ and $\alpha = (\|b\|^2 - \|a\|^2)/2$.

First, we check that $f(b) > \alpha$. Indeed,

$$f(b) - \alpha = \|b\|^2 - \langle a, b \rangle - \frac{\|b\|^2}{2} + \frac{\|a\|^2}{2} = \frac{\|b\|^2 + \|a\|^2 - \langle a, b \rangle}{2} = \frac{\|a - b\|^2}{2} > 0.$$

Let us check that $f(x) < \alpha$ for any $x \in A$. We observe that

$$f(a) - \alpha = \langle a, b \rangle - \frac{\|a\|^2 + \|b\|^2}{2} = -\frac{\|a - b\|^2}{2} < 0$$

and we are going to prove that $f(x) \leq f(a)$ for any $x \in A$. Suppose that for some $y \in A$ we have $f(y) > f(a)$. Then we claim that for some $\lambda \in [0, 1]$ we have $\phi\big((1-\lambda)a + \lambda y\big) < \phi(a)$, which contradicts to our choice of $a$. Indeed, let us define a (quadratic) function $g(\lambda) = \phi\big((1 - \lambda)a + \lambda y\big)$. We claim that $g'(0) < 0$. The straightforward computation shows:

$$g'(0) = -2\|a\|^2 + 2\langle a, y \rangle + 2\langle a, b \rangle - 2\langle y, b \rangle = 2\big(f(a) - f(y)\big) < 0.$$

So $g(\lambda) < g(0)$ for some small positive $\lambda$ and the proof follows. □

## 4. Polarity

**(4.1) Definition.** Let $A \subset \mathbb{R}^d$ be a set. The *polar* of $A$ is the set $A^\circ \subset \mathbb{R}^d$ defined as follows

$$A = \big\{x \in \mathbb{R}^d : \langle x, a \rangle \leq 1 \text{ for every } a \in A\big\}.$$

In other words, the polar of $A$ consists of all linear functions whose values on $A$ is not greater than 1.

For every $a \in A$ let us consider the halfspace $H_a = \{x \in \mathbb{R}^d : \ \langle a, x \rangle \leq 1\}$. Then

$$A^\circ = \bigcap_{a \in A} H_a.$$

Therefore $A^\circ$ is closed and convex. It is always non-empty since $0 \in A^\circ$.

PROBLEMS.

$1^\circ$. Find $A^\circ$ for $A = \mathbb{R}^d$ and $A = \{0\}$.

$2^\circ$. Let $L \subset \mathbb{R}^d$ be a linear subspace. Then $L^\circ$ is the orthogonal complement to $L$ in $\mathbb{R}^d$.

3. Let $B = \{x \in \mathbb{R}^d : \|x\| = 1\}$ be the unit ball. Then $B^\circ = B$.

$4^\circ$. If $A \subset B$ then $B^\circ \subset A^\circ$.

$5^\circ$. $(A \cup B)^\circ = A^\circ \cap B^\circ$.

6°. A convex set $K \subset \mathbb{R}^d$ is called a *cone*, iff for any $x \in K$ and any $\lambda \geq 0$ one has $\lambda x \in K$. Prove that

$$K^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 0\}.$$

**(4.2) Theorem.** *If $A \subset \mathbb{R}^d$ is closed, convex and $0 \in A$ then*

$$(A^\circ)^\circ = A.$$

**Proof.** First we prove that $A \subset (A^\circ)^\circ$. Indeed, for every $x \in A$ and every $a \in A^\circ$ we have $\langle a, x \rangle \leq 1$, that is, $x \in (A^\circ)^\circ$. Next, we have to prove that $(A^\circ)^\circ \subset A$. Suppose there is $y \in (A^\circ)^\circ$ such that $y \notin A$. Then, by Theorem 3.4 there exists a linear function $f(x) = \langle c, x \rangle$ and a number $\alpha$ such that $f(y) > \alpha$ and $f(x) < \alpha$ for all $x \in A$. Since $0 \in A$ we must have $\alpha > 0$. Therefore, letting $b = \alpha^{-1}c$ we get $\langle b, x \rangle < 1$ for all $x \in A$, so $b \in A^\circ$. On the other hand, $\langle y, b \rangle > 1$ which contradicts to the assumption that $y \in (A^\circ)^\circ$. $\square$

PROBLEMS.

1. Prove that the polar to the cube $I_d$ (see Examples 2.3, 2) is the octahedron $O_d$ (Examples 2.3, 3) and vice versa.

2. The following set of questions concerns the so-called "projective duality". First, we construct the projective plane $P^2$. A *point* in $P^2$ is a straight line in $\mathbb{R}^3$ passing through the origin. A *line* in $P^2$ is a linear hyperplane in $\mathbb{R}^3$.

a) Prove that for any two distinct points in $P^2$ there is exactly one line which passes through these points and that any two distinct lines intersect exactly in one point.

Now we describe the projective duality, that is a correspondence between points and lines in $P^2$. Let $a \in P^2$ be a point (that is, a line in $\mathbb{R}^3$). The projective duality maps $a$ to the line $A$ (that is, a linear hyperplane in $\mathbb{R}^3$), such that the hyperplane $A$ is orthogonal to the line $a$. In what follows, we denote a point in $P^2$ by a small letter and the corresponding (under the projective duality) line by the corresponding capital letter.

b) Prove that the line $C \subset P^2$ contains points $a \in P^2$ and $b \in P^2$ if and only if the lines $A \subset P^2$ and $B \subset P^2$ intersect in the point $c \in P^2$.

Problems a) and b) allow one to translate any statement concerning lines and points in $P^2$ into the dual statement, formally replacing "points" by "lines", "intersecting in" by "passing through" and vice versa.

c) Translate the following "Pappus Theorem":

"suppose that three points $a_1, b_1, c_1$ lie on a line $X_1$ whereas other three points $a_2, b_2, c_2$ lie on a line $X_2$. Let $d$ be the intersection of the lines $Y_1 = (a_1, b_2)$ and $Y_2 = (b_1, a_2)$, $e$ be the intersection of the lines $Z_1 = (b_1, c_2)$ and $Z_2 = (c_1, b_2)$ and $f$ be the intersection of the lines $W_1 = (c_1, a_2)$ and $W_2 = (a_1, c_2)$. Then the three points $d, e$ and $f$ lie on the same line $T$".

d)*. Prove the Pappus Theorem.

3*. This problem concerns the "cone duality" (see Problem 6°, Section 4.1 for the definition).

Suppose that $K_0, K_1, \ldots, K_m \subset \mathbb{R}^d$ are convex *polyhedral* cones (that is, every $K_i$ is the intersection of finitely many halfspaces). We write $K_0 = \epsilon_1 K_1 + \ldots + \epsilon_m K_m$ for integer numbers $\epsilon_1, \ldots, \epsilon_m$ if the same identity holds for the characteristic functions

$$\chi_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{if } x \notin K_i. \end{cases}$$

Prove that if

$$K_0 = \epsilon_1 K_1 + \ldots + \epsilon_m K_m$$

then

$$K_0^\circ = \epsilon_1 K_1^\circ + \ldots + \epsilon_m K_m^\circ.$$

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Friday, January 13

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## 5. Extreme Points

Let $A \subset \mathbb{R}^d$ be a set. A point $a \in \mathbb{R}^d$ is called an *extreme point* of $A$ iff for any two points $b, c \in A$ such that $a = \dfrac{b+c}{2}$ we have $b = c = a$. In other words, $a$ is an extreme point of $A$ if and only if there is no non-trivial interval with the endpoints in $A$ whose center is $a$.

PROBLEMS.

1°. Find the extreme points of the unit ball $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$.

2°. Open set does not contain any extreme point.

3. Find the extreme points of the simplex, cube and octahedron (Examples 2.3 1-3).

4. Let $P = \text{conv}\{a_1, \ldots, a_m\}$ be a polytope and $x \in P$ be an extreme point of $P$. Then $x = a_i$ for some $i$.

5*. Find the extreme points of the set of all positive semidefinite $n \times n$ matrices with the trace 1.

*Hint:* Those are the positive semidefinite matrices with the trace 1 *and* rank 1.

**(5.1) Lemma.** *Let $A \subset \mathbb{R}^d$ be a non-empty compact set. Then there exists an extreme point of $A$.*

**Proof.** Let $\phi(x) = \|x\|^2$. Since $A$ is compact there is a point $a \in A$ such that $\phi(a) = \max\{\phi(x) : x \in A\}$. We claim that $a$ is an extreme point of $A$. Indeed, suppose that $a = \dfrac{b+c}{2}$ for some $b, c \in A$. Then

$$\phi(a) = \phi\Big(\frac{b+c}{2}\Big) = \frac{\|b\|^2}{4} + \frac{\|c\|^2}{4} + \frac{\langle b, c\rangle}{2} = \frac{\|b\|^2}{2} + \frac{\|c\|^2}{2} - \Big(\frac{\|b\|^2}{4} + \frac{\|c\|^2}{4} - \frac{\langle b, c\rangle}{2}\Big) =$$
$$= \frac{\phi(b) + \phi(c)}{2} - \frac{\|b-c\|^2}{4}.$$

So if $b \neq c$ then $\phi(a) < \dfrac{\phi(b)+\phi(c)}{2}$ and thus either $\phi(a) < \phi(b)$ or $\phi(a) < \phi(c)$ (or both) which contradicts to our choice of $a$. □

**(5.2) Theorem.** *Let $A \subset \mathbb{R}^d$ be a non-empty compact set and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. Then there exists an extreme point $a \in A$ such that $f(a) = \max\{f(x) : x \in A\}$.*

**Proof.** Let $t = \max\{f(x) : x \in A\}$. Then the set $E = \{x \in A : \ f(x) = t\}$ is a non-empty compact. Thus $E$ contains an extreme point, say, $a \in E$ (Lemma 5.1). We claim that $a$ is an extreme point of $A$. Indeed, if $a = \dfrac{b+c}{2}$ for $b, c \in A$ then we must have $f(a) = \dfrac{f(b)+f(c)}{2}$. Since $f(a) = t$ and $f(b), f(c) \leq t$ we must have $f(b) = f(c) = t$. This means that $b, c \in E$. So $b = c$ since $a$ is an extreme point of $E$. □

PROBLEMS.

1. a) Construct a convex compact set $A$ in $\mathbb{R}^3$ whose set of extreme points is not closed.

*Hint:* Consider the convex hull of the two points $(1, 0, 1)$, $(1, 0, -1)$ and the circle $x^2 + y^2 = 1, \ z = 0$.

b)* Is there such a set in $\mathbb{R}^2$?

2. A point $a \in A$ of a set $A \subset \mathbb{R}^d$ is called *exposed* iff there is a linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $f(a) > f(x)$ for every $x \neq a$ in $A$.

a)°. Any exposed point is an extreme point.

b) Construct a convex compact set $A \subset \mathbb{R}^2$ and an extreme point $a \in A$ such that $a$ is not an exposed point.

*Hint:* Consider the union of two circles and their common tangents.

## 6. Weyl - Minkowski Theorem

**(6.1) Polyhedra.** A *polyhedron* in $\mathbb{R}^d$ is the intersection of finitely many halfspaces. So, the general form of a polyhedron is $P = \{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \beta_i\}$ for some vectors $a_i \in \mathbb{R}^d$ and some real numbers $\beta_i$.

PROBLEMS.

1°. Polyhedron is a convex closed set.

2°. Simplex, Cube and Octahedron (Examples 2.3) are polyhedra.

3°. There exists a polyhedron which is not a polytope.

4°. An intersection of finitely many polyhedra is a polyhedron.

5. Any closed convex set in $\mathbb{R}^d$ is the intersection of halfspaces.

*Hint:* Consider the intersection of all halfspaces that contain the set and use Separation Theorem 3.4.

**(6.2) Theorem.** *Let $P \subset \mathbb{R}^d$ be a polytope. Then $P^\circ$ is a polyhedron.*

**Proof.** Let $P =$conv $\{a_1, \ldots, a_m\}$. We claim that $P^\circ = \big\{x \in \mathbb{R}^d : \ \langle x, a_i\rangle \leq 1, \ i = 1, \ldots, m\big\}$. Indeed, since $a_i \in P$ then for any $x \in P^\circ$ we must have $\langle x, a_i\rangle \leq 1$. On the other hand, for any $y \in P$ we have $y = \lambda_1 a_1 + \ldots + \lambda_m a_m$ where $\lambda_i \geq 0$ and

$\lambda_1 + \ldots + \lambda_m = 1$ (Lemma 2.2). Therefore, if $\langle x, a_i \rangle \le 1$ for $i = 1, \ldots, m$ then $\langle x, y \rangle = \lambda_1 \cdot \langle x, a_1 \rangle + \cdots + \lambda_m \cdot \langle x, a_m \rangle \le \lambda_1 + \ldots + \lambda_m = 1$. $\square$

PROBLEM.

1°. Is it true that every polyhedron is the polar of a polytope?

*Hint:* Of course, not.

2. Every polyhedron which contains the origin as its interior point is the polar of a polytope.

*Hint:* Use Theorem 4.2.

In this section we are going to prove the following classical result:

**(6.3) Theorem (Weyl - Minkowski Theorem).** *Every polytope is a polyhedron. Every bounded polyhedron is a polytope.*

**Proof.**

**First Part.** First we prove that a bounded polyhedron is a polytope. Let $P \subset \mathbb{R}^d$ be a bounded polyhedron, which is given by a system of linear inequalities

$$P = \big\{x \in \mathbb{R}^d : \ \langle c_i, x \rangle \le \beta_i : \ i = 1, \ldots, m\big\}.$$

We claim that a) the set of extreme points of $P$ is finite and b) $P$ is the convex hull of the set of its extreme points.

a) Suppose that $p \in P$ is an extreme point. Let $I = \{i : \ \langle c_i, p \rangle = \beta_i\}$. Thus $\langle c_i, p \rangle < \beta_i$ for $i \notin I$. Let us consider the affine subspace

$$A = \{y \in \mathbb{R}^d : \langle c_i, y \rangle = \beta_i : \ i \in I\}.$$

Obviously, $A$ is not empty since $p \in A$. If the dimension of $A$ is greater than 0 then we can find an interval $[q, r] \subset A$ such that $p = \dfrac{q + r}{2}$. Furthermore, if we choose $q$ and $r$ sufficiently close to $p$ they will still satisfy the inequalities $\langle c_i, \cdot \rangle < \beta_i : \ i \notin I$. Thus $p$ would not be an extreme point. So the dimension of $A$ should be equal to 0 (thus $A = p$) which indicates that the set $I$ contains not fewer than $d$ elements. Therefore the number of extreme points in $P$ does not exceed $\binom{m}{d}$.

b) Let $Q = \text{conv}\big\{p : \text{ is an extreme point of } P\big\}$. By a) it follows that $Q$ is a polytope. We want to prove that $Q = P$. Suppose that there exists a point $s \in P \setminus Q$. Since $Q$ is a compact set (see Theorem 3.2) there exists a linear function $f : \mathbb{R}^d$ such that $f(s) > f(x)$ for every $x \in Q$. Since $P$ is compact there exists an extreme point $p$ of $P$ where $f$ attains its maximum (Theorem 5.2). But then $f(p) \ge f(s) > f(x)$ for every $x \in Q$, so $p \notin Q$ what contradicts to the definition of $Q$.

**Second Part.** Now we prove that a polytope is a polyhedron. Without loss of generality we may assume that the polytope $P$ contains the origin as its interior point. So for a sufficiently small ball $B_r(0)$ we have $B_r(0) \subset P$. Then $P^\circ \subset B_{1/r}(0)$, and, in particular, $P^\circ$ is bounded. By Theorem 6.1 $P^\circ$ is a polyhedron. Since $P^\circ$ is a bounded polyhedron by First Part (which we proved already) we conclude that $P^\circ$ is a polytope. Since $(P^\circ)^\circ = P$ (Theorem 4.2) and the polar to a polytope is a polyhedron (Theorem 6.1) we conclude that $P$ is a polyhedron. $\square$

PROBLEMS.

1°. The intersection of two polytopes is a polytope.

2. a)° A Cyclic Polytope (Examples 2.3, 3) is a polyhedron.

b)* Could you find a system of linear inequalities defining this polyhedron?

The previous question indicates the major difficulty in computational convex geometry: find a polyhedral description of a polytope or a polytopal description of a polyhedron, even if we know that such descriptions do exist. A different approach to Weyl-Minkowski Theorem is suggested in the following problem.

3. Let $P \subset \mathbb{R}^d$ be a polyhedron and $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ be a linear map. Then the image $\phi(P)$ is a polyhedron.

*Hint:* This is easy if $\phi$ has trivial kernel. If $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^k : k < d$ has non-zero kernel, we may represent it as a sequence of elementary projections $\mathbb{R}^d \longrightarrow \mathbb{R}^{d-1} \ldots \longrightarrow \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^k$ (each time the dimension falls by 1). Prove that the image of a polyhedron under an elementary projection is a polyhedron. See the "Fourier-Motzkin Elimination" in Sections 1.2, 1.3 of G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1995, Graduate texts in mathematics, **152**.

4*. Since the standard simplex is a polyhedron, and every polytope is a projection of the simplex (see remark after Theorem 3.2), we get yet another proof that any polytope is a polyhedron. How many linear inequalities we get for a polytope $P = \text{conv}\ \{a_1, \ldots, a_m\}$, where $a_i \in \mathbb{R}^d$ using the method of Problem 3? Using our method of Theorem 6.3? Which estimate is better? When?

5. A polytope is the convex hull of the set of its extreme points.

6*. A convex compact set in $\mathbb{R}^d$ is the convex hull of the set of its extreme points (Krein - Milman Theorem).

7*. A convex compact set in $\mathbb{R}^d$ is the closure of the convex hull of the set of its exposed points (see Problem 2, Section 5.2) (Strashevich's Theorem).

Few general "polarity" problems:

8°. For any non-empty set $A \subset \mathbb{R}^d$ we have $((A^\circ)^\circ)^\circ = A^\circ$.

9. Suppose that $A^\circ = A$ for a non-empty set $A \subset \mathbb{R}^d$. Then $A = \{x : \mathbb{R}^d : \|x\| \leq 1\}$ is the unit ball.

10°. Let $A = \{x \in \mathbb{R}^d : \langle x, a_i \rangle \leq 1,\ i = 1, \ldots, m\}$ be a polyhedron. Then $A^\circ =$conv $\{0, a_1, \ldots, a_m\}$.

11*. Suppose that $p$ and $q$ are positive real numbers such that $1/p + 1/q = 1$. Then

$$\Big\{x \in \mathbb{R}^d : \sum_{i=1}^d |x_i|^p \leq 1\Big\}^\circ = \Big\{x \in \mathbb{R}^d : \sum_{i=1}^d |x_i|^q \leq 1\Big\}.$$

12°. Let $\mathbb{R}^d_+ = \{x = (x_1, \ldots, x_d) : \ x_i \geq 0, \ i = 1, \ldots, d\}$. Then $(\mathbb{R}^d_+)^\circ = -\mathbb{R}^d_+$, where $-\mathbb{R}^d_+ = \{-x : x \in \mathbb{R}^d_+\}$.

13. Let us consider the space $\mathbb{R}^{\frac{n(n+1)}{2}}$ of $n \times n$ symmetric matrices $A = (a_{ij})$ with the scalar product $\langle A, B \rangle = \sum\limits_{1 \leq i,j \leq n} A_{ij} B_{ij}$. Let $X_+ = \{A : \ A \text{ is positive semidefinite }\}$. Then $(X_+)^\circ = -X_+$, where $-X_+ = \{-A : A \in X_+\}$.

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Wednesday, January 18

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## 7. Faces of Polytopes

**(7.1) Definitions.** Let $P \subset \mathbb{R}^d$ be a set. A linear inequality $f(x) \leq \alpha$ is said to be *valid* on $P$ iff $f(x) \leq \alpha$ for every $x \in P$. A set $F \subset P$ is called a *face* of $P$ iff there exists a linear inequality $f(x) \leq \alpha$ which is valid on $P$ and such that $F = \{x \in P : f(x) = \alpha\}$. In this case $f$ is called a *supporting function* of $F$ and the hyperplane $H = \{x \in \mathbb{R}^d : \ f(x) = \alpha\}$ is called a *supporting hyperplane* of $F$. Let $F \subset P$ be a face. Consider the smallest affine subspace $A_F$ in $\mathbb{R}^d$ generated by $F$. Its dimension is called the *dimension* of $F$.

If $P$ is a polyhedron, then there are special names for faces of various dimensions. A face of dimension 0 is called a *vertex*. A face of dimension 1 is called an *edge*. A face $F$ such that $\dim F = \dim P - 1$ is called a *facet*. A face $F$ such that $\dim F = \dim P - 2$ is called (sometimes) a *ridge*. We agree that the empty set is a face of dimension $-1$ whereas $P$ itself is a face of dimension $\dim P$. Any face of $P$ except $\emptyset$ and $P$ itself is called *proper*. The set of vertices of a polytope $P$ is denoted by Vert $P$.

**(7.2) Theorem.** *Let $P = \operatorname{conv}\{a_1, \ldots, a_m\}, \ a_i \in \mathbb{R}^d$ be a polytope and $F \subset P$ be a face. Then $F = \operatorname{conv}\{a_i : \ a_i \in F\}$.*

**Proof.** Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function for $F$. So $f(x) \leq \alpha$ for any $x \in P$ and $F = \{x \in P : f(x) = \alpha\}$. Let $I = \{i : f(a_i) = \alpha\}$. Obviously, $\operatorname{conv}\{a_i : \ i \in I\} \subset F$, since for $x \in \operatorname{conv}\{a_i : \ i \in I\}$ we have $x = \sum\limits_{i \in I} \lambda_i a_i$, where $\lambda_i \geq 0$ and $\sum\limits_{i \in I} \lambda_i = 1$ and, therefore, $f(x) = \sum\limits_{i \in I} \lambda_i f(a_i) = \alpha \sum\limits_{i \in I} \lambda_i = \alpha$.

Now let us prove that $F \subset \operatorname{conv}\{a_i : \ i \in I\}$. Choose $x \in F, \ x = \lambda_1 a_1 + \ldots + \lambda_m a_m$, where $\lambda_1 + \ldots + \lambda_m = 1$ and $\lambda_i \geq 0$. Suppose that $\lambda_j > 0$ for some $j \notin I$. Thus $f(a_j) < \alpha$, $\lambda_j f(a_j) < \lambda_j \alpha$ and $f(x) = \lambda_1 f(a_1) + \ldots + \lambda_m f(a_m) < \lambda_1 \alpha + \ldots + \lambda_m \alpha = \alpha$. We arrived to the contradiction which shows that $\lambda_j = 0$ for any $j \notin I$. Therefore $x \in \operatorname{conv}\{a_i : \ i \in I\}$. $\square$

**(7.3) Corollary.** *The number of faces of a polytope is finite.*

□

The number of $k$-dimensional faces of a polytope $P$ is denoted by $f_k(P)$. The vector of integers $\big(f_{-1}(P), f_0(P), \dots, f_d(P)\big)$, where $d = \dim P$ is called the $f$-vector of $P$.

PROBLEMS.

1°. Find the $f$-vector of the standard $d$-dimensional simplex (Examples 2.3, 1).

2. Find the $f$-vectors of the cube and octahedron (Examples 2.3, 2 and 3).

3. Every non-empty polytope, which is not a point, has a facet.

4. If $P$ is a $d$-dimensional polytope then $f_k(P) > 0$ for any $0 \leq k \leq d$.

5*. Let $X_+$ be a set of positive semidefinite $n \times n$ matrices (cf. Problem 13 from Section 6.2). Then $X_+$ is a convex set and the dimensions of its faces are $0, 1, 3, \dots, \frac{r(r+1)}{2}, \dots, \frac{n\cdot(n+1)}{2}$.

## 8. Cyclic Polytopes

**(8.1) Theorem.** *Let $\gamma(t) = (t, t^2, \dots, t^d) :\ -\infty < t < +\infty$ be the moment curve in $\mathbb{R}^d$. Then every hyperplane intersects $\gamma(t)$ in not more than $d$ points.*

**Proof.** Let $H = \big\{x \in \mathbb{R}^d : \langle c, x\rangle = \alpha\big\}$ be a hyperplane. Let $c = (c_1, \dots, c_d)$. Then $\gamma(t) \in H$ iff $-\alpha + c_1 \cdot t + \ldots + c_d \cdot t^d = 0$. Since a (non-zero) polynomial of degree $d$ does not have more than $d$ roots, the proof follows. □

PROBLEMS.

1. Let $S^1 = \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 = 1\}$ be the unit circle.

a) Suppose that $d$ is even. Then there exists an embedding $\phi : S^1 \longrightarrow \mathbb{R}^d$ such that every hyperplane in $\mathbb{R}^d$ intersects the image $\phi(S)$ in not more than $d$ points.

*Hint:* Consider the map $[0, 2\pi] \longrightarrow \mathbb{R}^d$ :

$$x \longmapsto \big(\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin(d/2)x, \cos(d/2)x\big).$$

b) Suppose that $d$ is odd. Then there is no embedding $\phi : S \longrightarrow \mathbb{R}^d$ such that every hyperplane in $\mathbb{R}^d$ intersects the image $\phi(S)$ in not more than $d$ points.

*Hint:* Suppose that such an embedding exists. Choose $d+1$ distinct points (say, $p_1, \dots, p_{d+1}$) on the circle. Their image defines an orientation in $\mathbb{R}^d$. If we permute any two points $p_i$ and $p_j$, we get the reverse orientation. Apply $d$ permutations to get the sequence $p_2, p_3, \dots, p_{d+1}, p_1$. It must define the reverse orientation. On the other hand, we can continuously move $p_1$ in front of $p_2$ to get $p_1, p_2, \dots, p_{d+1}$ again. So it must be the same orientation.

2. a) Let us consider a Y-shaped curve in $\mathbb{R}^2$. There is no embedding of this curve in $\mathbb{R}^d$ such that every hyperplane in $\mathbb{R}^d$ intersects the image in not more than $d$ points.

*Hint:* Use Hint to Problem 1, b).

b)* Let $X \subset \mathbb{R}^2$ be a compact set. If there is an embedding $X \longrightarrow \mathbb{R}^d$ such that any hyperplane in $\mathbb{R}^d$ intersects the image of $X$ in not more than $d$ points then $X$ is homeomorphic to a subset of the unit circle $S^1$.

3*. Does there exists an embedding $\mathbb{R}^2 \longrightarrow \mathbb{R}^d$ such that any $(d-2)$-dimensional affine subspace in $\mathbb{R}^d$ intersects the image of $\mathbb{R}^2$ in not more than $d-1$ points?

5*. Construct an isometric embedding (that is, preserving the length) of the unit circle in $\mathbb{R}^d$ that maximizes the volume of the convex hull of the image. (If $d$ is even then such an embedding, up to a normalizing factor, is given by

$$x \longmapsto \Big(\sin x, \cos x, \frac{\sin 2x}{2}, \frac{\cos 2x}{2}, \ldots, \frac{\sin mx}{m}, \frac{\cos mx}{m}\Big), \quad \text{where} \quad m = d/2.$$

If $d$ is odd, it is not known, see S. Karlin and W.J. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Interscience John Wiley & Sons, 1966, Pure and Applied Mathematics, Vol. XV).

**(8.2) Theorem.** *Let us choose $t_1 < t_2 < \ldots < t_m$ and let $v_i = \gamma(t_i)$, where $\gamma(t) : \mathbb{R} \longrightarrow \mathbb{R}^d$ is the moment curve. Let*

$$C_d(m) = \operatorname{conv}\big\{v_1, \ldots, v_m\big\}$$

*be a cyclic polytope.*

*Then*

*a) For any $k \le [d/2]$ points $v_{i_1}, \ldots, v_{i_k}$ the convex hull $\operatorname{conv}\big\{v_{i_1}, \ldots, v_{i_k}\big\}$ is a face of $C_d(m)$;*

*b) Let $S \subset \{1, \ldots, m\}$. The set $\operatorname{conv}\big\{v_i : \ i \in S\big\}$ is a facet of $C_d(m)$ if and only if $S$ contains $d$ elements and the following "Gale's evenness condition" is satisfied: If $i < j$ are not in $S$, then the number of $k \in S$ between $i$ and $j$ is even.*

**Proof.** a) Let us choose any $k \le [d/2]$ points, say, $v_i : \ i \in I, |I| = k$. We must present a linear function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and a number $\alpha$ such that $f(v_i) = \alpha$ if $i \in I$ and $f(v_j) < \alpha$ if $i \notin I$. Let $r = d - 2k \ge 0$ and $T > t_m$ be a sufficiently large number. We consider a polynomial of degree $d$

$$p(t) = -(T-t)^r \cdot \prod_{i \in I}(t - t_i)^2 = c_0 + c_1 t^1 + \ldots + c_d t^d.$$

We observe that $p(t_i) = 0$ if $i \in I$ and $p(t_i) < 0$ if $i \notin I$. Let $f(x) = \langle c, x \rangle$, where $c = (c_1, \ldots, c_d)$ and $\alpha = -c_0$. Then $f(v_i) = \alpha$ for $i \in I$ and $f(v_i) < \alpha$ for $i \notin I$.

b) Suppose that $S$ contains $d$ elements and that "Gale's evenness condition" is satisfied.

Let us consider a polynomial

$$p(t) = \prod_{i \in S}(t - t_i) = c_0 + c_1 t^1 + \ldots + c_d t^d.$$

Then $p(t_i)=0$ if $i \in S$. Since $p(t)$ changes its sign when $t$ passes through any $t_i: i \in S$, the Gale's evenness condition implies that the values $p(t_i)$ for $i \notin S$ all have the same sign.

Therefore the hyperplane $H = \{x \in \mathbb{R}^d : \langle c, x\rangle = -c_0\}$, where $c=(c_1,\ldots,c_d)$, is a supporting hyperplane of the face $F = \operatorname{conv}\{v_i : i \in S\}$ (Theorem 7.2). We claim that $\dim F = d-1$. Indeed, if we had $\dim F = (d-1)-k$, then by choosing additional $k$ points on the moment curve, we would get a hyperplane which intersects $\gamma(t)$ in more than $d$ points. This, however, contradicts to Theorem 8.1.

Suppose now that $F = \operatorname{conv}\{v_i : i \in S\}$ is a facet of $C_d(m)$. Then the set $S$ must contain at least $d$ elements (otherwise $\dim F < d-1$). Because of Theorem 8.1 it can not contain more than $d$ elements, so $S$ contains exactly $d$ elements. Suppose that $H = \{x \in \mathbb{R}^d : \langle c, x\rangle = \alpha\}$ is a supporting hyperplane for $F$. Let us consider the polynomial

$$p(t) = -\alpha + c_1 \cdot t + \ldots c_d \cdot t^d, \quad \text{where } c=(c_1,\ldots,c_d).$$

We must have $p(t_i)=0$ for $i \in S$ and $p(t_j)$ for $j \notin S$ all have the same sign. From the first condition we derive that

$$p(t) = C\prod_{i\in S}(t-t_i),$$

where $C$ is a non-zero constant. Since $p(t)$ changes its sign when $t$ passes through $t_i$, the second condition is equivalent to the Gale's evenness condition. □

PROBLEMS.

1°. Prove that $f_k(C_d(m)) = \binom{m}{k}$ for $0 \le k \le [d/2]$.

2. Compute the $f$-vectors of $C_1(m), C_2(m), C_3(m)$, and $C_4(m)$.

3°. Prove that for any $d$ we have $f_{d-1}(C_d(m)) = O(m^{[d/2]})$ as $m$ tends to infinity.

4. Prove that

$$f_{d-1}(C_{d-1}(m)) = \begin{cases} \frac{m}{m-k}\binom{m-k}{k} & \text{if } d=2k \text{ even,} \\ 2\binom{m-k-1}{k} & \text{if } d=2k+1 \text{ odd.} \end{cases}$$

A permutation $g$ of vertices $v_1,\ldots,v_m$ of a cyclic polytope $C_d(m)$ is called a *combinatorial automorphism* of $C_d(m)$ iff $\operatorname{conv}\{v_i : i \in I\}$ is a facet of $C_d(m)$ whenever $\operatorname{conv}\{g(v_i) : i \in I\}$ is a facet of $C_d(m)$.

5. Prove that the permutation $v_i \longleftrightarrow v_{m-i+1}$ is a combinatorial automorphism of $C_d(m)$.

6. Suppose that $d$ is even. Prove that the cyclic permutation $v_1 \longmapsto v_2 \longmapsto \ldots \longmapsto v_m \longmapsto v_1$ is a combinatorial automorphism of $C_d(m)$. This justifies (only partially, however) the name "Cyclic Polytope". Prove that this is not true if $d>1$ is odd.

7*. For $i=1,\ldots,m$ let

$$v_i = \Big(\sin(\frac{2\pi i}{m}), \cos(\frac{2\pi i}{m}), \ldots, \sin(\frac{2\pi k i}{m}), \cos(\frac{2\pi k i}{m}), \ldots, \sin(\frac{2\pi d i}{m}), \cos(\frac{2\pi d i}{m})\Big) \in \mathbb{R}^{2d}.$$

Prove that the polytope $P = \mathrm{conv}\{v_i :\ i = 1, \ldots, m\}$ has the same face structure as the polytope $C_{2d}(m)$.

8*. Suppose that a 3-dimensional polytope $P$ has the property that the convex hull of any two distinct vertices of $P$ is an edge of $P$. Then $P$ has 4 vertices and it is a tetrahedron.

9*. What is the topology of the space of all $m$-tuples $(a_1, \ldots, a_m)$ of the points $a_i \in \mathbb{R}^d$ such that the set $\mathrm{conv}\{a_i :\ i \in I\}$ is a face of the polytope $P = \mathrm{conv}\{a_1, \ldots, a_m\}$ if and only if the set $\mathrm{conv}\{v_i :\ i \in I\}$ is a face of $C_d(m)$?

*Hint:* If $d$ is even or $d = 3$ then this space is known to be homeomorphic to the product $GL(n, \mathbb{R}) \times \mathbb{R}^{m-d}$, where $GL(n, \mathbb{R})$ is the general linear group.

See also A.I. Barvinok, On topological properties of spaces of polytopes, in: *Topology and Geometry - Rohlin Seminar*, 495-500, **1346**(1988), Springer, Berlin - New York and other papers in that volume.

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Friday, January 20

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## 9. Vertices

**(9.1) Theorem.** *Let $P \subset \mathbb{R}^d$ be a polytope and $F \subset P$ be a face. Let $G \subset F$ be a face of $F$. Then $G$ is a face of $P$.*

**Proof.** Suppose that $P = \mathrm{conv}\{a_1, \ldots, a_m\}$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function of $F$. Then by Theorem 7.2, the face $F$ can be represented in the form $F = \mathrm{conv}\{a_i :\ i \in I\}$, where $f(a_i) = \alpha$ if $i \in I$ and $f(a_i) < \alpha$ if $i \notin I$. Let $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function of $G$. Then by Theorem 7.2, we have $G = \mathrm{conv}\{a_j :\ j \in J \subset I\}$, where $g(a_j) = \beta$ for $j \in J$ and $g(a_j) < \beta$ for $j \in I \setminus J$. Let us choose an $\epsilon > 0$ and let $h = f + \epsilon \cdot g$. Then $h$ is a linear function and we claim that $h$ is a supporting function of $G$ in $P$ if $\epsilon$ is sufficiently small. Indeed, for any $j \in J$ we have $h(a_j) = \alpha + \epsilon \cdot \beta$. If $j \in I \setminus J$ then $h(a_j) < \alpha + \epsilon \cdot \beta$. Finally, if $j \notin I$ then $f(a_j) < \alpha$, so choosing $\epsilon$ smaller, say, than $\dfrac{\alpha - \max\{f(a_i) :\ i \notin I\}}{\max\{|g(a_i)| :\ i = 1, \ldots, m\}}$ we get that $f(a_i) < \alpha + \epsilon \cdot \beta$ for all $i \notin I$. □

PROBLEM.

1. Construct an example of a convex compact set $P \subset \mathbb{R}^2$, a face $F$ of $P$ and a face $G$ of $F$ such that $G$ is not a face of $P$ (cf. Problem 2, b) of Section 5.2).

**(9.2) Corollary.** *Every non-empty polytope has a vertex.*

**Proof.** Every non-empty polytope has a proper face (just choose a non-zero linear function on the polytope and consider the set where the function attains its maximum). If the polytope is not a point, this face has a smaller dimension. The face itself is a polytope, which, in turn, has a proper face. Going this way we come to a point, which is a vertex because of Theorem 9.1. □

**(9.3) Theorem.** *Every polytope is the convex hull of the set of its vertices.*

Let $P \subset \mathbb{R}^d$ be a non-empty polytope and let $Q$ be the convex hull of the set of vertices of $P$. Then $Q$ is a non-empty (Corollary 9.2) polytope (Corollary 7.3). We claim that $Q = P$. Indeed, if there exists a point $s \in P \setminus Q$ we can separate it from $Q$ by a linear function $f$ (Theorem 3.4). That is, there exists a linear function $f$ and a number $\alpha$ such that $f(s) > \alpha > f(x)$ for all $x \in Q$. Let $t = \max\{f(x) : \ x \in P\}$ and $F = \{x \in P : \ f(x) = t\}$. Then $F$ is a face of $P$ and by Corollary 9.2 it has a vertex, say, $p$. By Theorem 9.1 $p$ is a vertex of $P$. But $p \notin Q$ which contradicts to the definition of $Q$. $\square$

The significance of Theorem 9.3 is that it finally justifies our study of polytopes as combinatorial and geometric objects. A polytope may have plenty representations as the convex hull of a set of points. Now we singled out a canonical one: a polytope is the convex hull of the set of its vertices.

**(9.3) Definition.** We say that two $P$ and $Q$ are *combinatorially isomorphic* iff they have the same number of vertices and there is a one-to-one correspondence $p_i \longleftrightarrow q_i$ between the vertices of $P$ and the vertices of $Q$ such that $\operatorname{conv}\{p_i : \ i \in I\}$ is a face of $P$ if and only if $\operatorname{conv}\{q_i : \ i \in I\}$ is a face of $Q$. Such a correspondence $P \longleftrightarrow Q$ is called *combinatorial isomorphism.*

PROBLEMS.

1°. Let $P \subset \mathbb{R}^d$ be a polytope and $p \in P$ be the point with the maximal norm: $\|p\| = \max\{\|x\| : \ x \in P\}$. Then $p$ is a vertex of $P$.

2°. A $d$-dimensional polytope has at least $d+1$ vertices.

3°. Let $P \subset \mathbb{R}^d$ be a polytope and $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a non-degenerate linear transformation. Then the polytope $Q = T(P)$ is combinatorially isomorphic to $P$.

4. Let $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a non-degenerate linear transformation, $c \in \mathbb{R}^d$ be a non-zero vector and $\alpha \in \mathbb{R}$ be a number. Let $P \subset \mathbb{R}^d$ be a polytope which does not intersect the hyperplane $H = \{x \in \mathbb{R}^d : \ \langle c, x\rangle + \alpha = 0\}$. The map $x \longmapsto \frac{1}{\langle c,x\rangle + \alpha} \cdot T(x)$ is called a *projective transformation* of the polytope $P$. Prove that the image of the polytope $P$ under this transformation is a polytope which is combinatorially isomorphic to $P$.

## 10. Birkhoff Polytope

We defined the Birkhoff Polytope in Examples 2.3, 5 as the convex hull of the set of permutation matrices. Now we are going to justify "in other words" description of this polytope and thus to solve the remaining part of Problem 2*, Section 2.3.

**(10.1) Theorem.** *Let us define a polytope $B_d \subset \mathbb{R}^{d^2}$ as the set of non-negative $d \times d$ matrices with row and column sums equal to 1:*

$$B_d = \Big\{X = (x_{ij}) : \quad x_{ij} \geq 0 \quad i,j = 1, \ldots, d;$$

$$\text{(10.1.1)} \qquad \sum_{i=1}^{d} x_{ij} = 1 \quad i = 1, \ldots, d; \qquad \sum_{j=1}^{d} x_{ij} = 1 \quad j = 1, \ldots, d \Big\}.$$

*Then $B_d$ is a polytope and $X = (x_{ij})$ is a vertex of $B_d$ iff there exists a permutation $\sigma$ of the set $\{1, \ldots, d\}$ such that*

$$x_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let us denote by $X^\sigma$ the permutation matrix corresponding to $\sigma$. First, we prove that $X^\sigma$ is indeed a vertex. Let us consider a linear function $f(X) = \langle X, X^\sigma \rangle$. Then it is clear that $f(X) \leq d$ for any $X \in B_d$ and $f(X) = d$ for some $X \in B_d$ only if $X = X^\sigma$.

Next, we should prove that if $X$ is a vertex of $B_d$ then $X = X^\sigma$ for some $\sigma$. It is slightly more convenient to prove that if $X$ is an extreme point of $B_d$ then $X = X^\sigma$.

**First step.** Let us consider the affine subspace $L_d$ defined by the equations (10.1.1). We claim that dim $L = (d-1)^2$. Indeed, any $(d-1) \times (d-1)$ matrix $X$ we can append in the unique way to a $d \times d$ matrix from $L$. We must define

$$x_{id} = 1 - \sum_{j=1}^{d-1} x_{ij}, \ i = 1, \ldots, d-1, \quad x_{di} = 1 - \sum_{j=1}^{d-1} x_{ji}, \ i = 1, \ldots, d-1.$$

Then there is only (always open) possibility to let

$$x_{dd} = -d + 2 + \sum_{i,j=1}^{d-1} x_{ij}.$$

**Second step.** Let $A \in B_d$ be an extreme point. We claim that at least $(d-1)^2$ coordinates $a_{ij}$ of $A$ are zeros. Suppose that the number of zero coordinates of $A$ is less than $(d-1)^2$. Let us consider the linear space

$$M = \{(x_{ij}) : \ x_{ij} = 0 \text{ iff } a_{ij} = 0\}.$$

Then dim $M > d^2 - (d-1)^2 = 2d - 1$. Then dim $M+$ dim $L > 2d - 1 + (d-1)^2 > d^2$ and hence the intersection $\Pi = M \cap L$ must have dimension at least 1. We have that $A \in \Pi \cap B_d$ and it is easy to see that a sufficiently small neighborhood of $A$ in $\Pi$ still belongs to $B_d$. Since dim $\Pi > 1$ we conclude that $A$ can not be an extreme point.

**Third step.** We use induction on $d$. For $d = 1$ the result is obviously correct. Let us consider an extreme point $X$ of $B_d$. From the second step we conclude that at least $(d-1)^2$ entries of $X$ are zeros. Then $X$ should have at least $(d-1)$ zero entries in some row (otherwise the total number of zeros would be $(d-2) \cdot d$ which is short by 1 of the required number of zeros). Without loss of generality let us suppose that $x_{d1} = \ldots = x_{dd-1} = 0$, Then, necessarily $x_{dd} = 1$ and thus $x_{1d} = \ldots = x_{d-1d} = 0$. Let us consider the $(d-1)\times(d-1)$ matrix $Y$ formed by the first $(d-1)$ rows and the first $(d-1)$ columns. $Y \in B_{d-1}$ and must be a an extreme point there. We apply induction. $\square$

PROBLEM.

1. Let $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ be a point with pairwise different coordinates. The polytope $P(a) = \operatorname{conv}\{(a_{\sigma(1)}, \ldots, a_{\sigma(d)}) : \ \sigma \text{ is a permutation}\}$ is called the *permutation polytope*. Prove that the permutation polytope is a projection of the Birkhoff polytope $B_d$.

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Monday, January 23

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## 11. Duality of Polytopes

Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope, containing the origin as its interior point. So, $B_r(0) \subset P \subset B_R(0)$ for some balls centered in the origin. Therefore $B_{1/R}(0) \subset P^\circ \subset P_{1/r}(0)$ and hence $P^\circ$ is a $d$-dimensional polytope containing the origin as its interior point (cf. Weyl - Minkowski Theorem 6.3). We will established a bijection between the faces of $P$ and the faces of $P^\circ$.

**(11.1) Theorem.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope containing the origin as its interior point and $Q = P^\circ$. For a non-empty face $F \subset P$ let*

$$\hat{F} = \{x \in Q : \langle x, y \rangle = 1 \text{ for all } y \in F\}$$

*and for an empty face $F$ let $\hat{F} = Q$. Then*

1. *The set $\hat{F}$ is a face of $Q$ and $\dim F + \dim \hat{F} = d - 1$.*
2. *We have*

$$F = \{x \in P : \ \langle x, y \rangle = 1 \text{ for all } y \in \hat{F}\}.$$

*In other words, $\hat{\hat{F}} = F$.*

3. *If $F, G \subset P$ are faces such that $F \subset G$ then $\hat{G}, \hat{F}$ are faces of $Q$ such that $\hat{G} \subset \hat{F}$.*

**Proof.**

1. Let $P = \operatorname{conv}\{v_1, \ldots, v_m\}$, $v_i \in \mathbb{R}^d$ and $F = \operatorname{conv}\{v_i : \ i \in I\}$ (see Theorem 7.2). Clearly, it suffices to prove the statement only for a non-empty face. Let

$$v = \frac{1}{|I|} \sum_{i \in I} v_i.$$

Then $v \in F$ and we claim that

$$\hat{F} = \{x \in Q : \langle v, x \rangle = 1\}. \tag{11.1.1}$$

Indeed, since $v \in F$ then for any $x \in \hat{F}$ we must have $\langle x, v\rangle = 1$. On the other hand, if $x \in Q$ then $\langle v_i, x\rangle \leq 1$ for all $i \in I$. If, additionally, $\langle x, v\rangle = 1$ then we must have $\langle x, v_i\rangle = 1$ for all $i \in I$ and thus $\langle x, y\rangle = 1$ for all $y \in F$. So (11.1.1) is proven. Furthermore, since $\langle v, x\rangle \leq 1$ for all $x \in Q$ the equation (11.1.1) implies that $\hat{F}$ is a face of $Q$ (maybe empty so far) with the supporting function $f(x) = \langle v, x\rangle$.

Let us now prove that $\dim \hat{F} + \dim F = d - 1$. First, we prove that $\hat{F}$ is not empty. Since $F$ is a non-empty face of $P$ there exists a vector $a \in \mathbb{R}^d$ and a number $\alpha$ such that $\langle a, x\rangle \leq \alpha$ for all $x \in P$ and $F = \{x \in P : \ \langle a, x\rangle = \alpha\}$. Since 0 is in the interior of $P$ we must have $\alpha > 0$. Then the point $b = \alpha^{-1}a$ belongs to $\hat{F}$. Therefore, $\hat{F}$ is not empty. Furthermore, $\langle b, v_i\rangle = 1$ if $i \in I$ and $\langle b, v_i\rangle < 1$ if $i \notin I$. Let $\dim F = k$ and let $h_1, \ldots, h_{d-k-1} \in \mathbb{R}^d$ be linearly independent vectors orthogonal to the linear span of $F$. So $\langle h_j, v_i\rangle = 0$ for any $i \in I$ and any $j = 1, \ldots, d-k-1$. Therefore for all sufficiently small $\epsilon = (\epsilon_1, \ldots, \epsilon_{d-k+1})$ and $b_\epsilon = b + \epsilon_1 h_1 + \ldots + +\epsilon_{d-k-1}h_{d-k-1}$ we have $\langle b_\epsilon, v_i\rangle = 1$ if $i \in I$ and $\langle b_\epsilon, v_i\rangle < 1$ if $i \notin I$. Therefore $b_\epsilon \in \hat{F}$ and $\dim \hat{F} \geq d - 1 - \dim F$.

Finally, $\hat{F} \subset \{x \in \mathbb{R}^d : \ \langle x, v_i\rangle = 1 : \ i \in I\}$ and therefore $\dim \hat{F} \leq d - 1 - \dim F$. So $\dim \hat{F} + \dim F = d - 1$.

2. We observe that $Q^\circ = P$ (Theorem 4.2). Let $G = \{x \in P : \langle x, y\rangle = 1 \text{ for all } y \in \hat{F}\}$. Applying part 1 (which we have already proven) we get that $G$ is a face of $P$ and $\dim G = d - 1 - \dim \hat{F} = \dim F$. Furthermore, obviously we have $F \subset G$. Thus $F \subset G$ are two faces of the same dimension of the same polytope $P$. Hence we must have $F = G$ (why?).

3. Let us choose $x \in \hat{G}$. Thus $x \in Q$ and $\langle x, y\rangle = 1$ for all $y \in G$. Since $F \subset G$ we have $\langle x, y\rangle = 1$ for all $y \in F$. Therefore, $x \in \hat{F}$. □

**(11.2) Dual polytope.** Let $P \subset \mathbb{R}^d$ be a polytope. When we talk about the *dual polytope* we mean the following. Let us consider the smallest affine subspace containing $P$, so that $P$ has full dimension. Let us choose the origin in the interior of $P$ and let $Q = P^\circ$. As Theorem 11.1 shows, the combinatorial structure of $Q$ does not depend on the choice of the origin. This is why we may abuse terminology and talk about the dual polytope $Q$ without specifying which $Q$ we have chosen, provided we are talking about combinatorial (and not metric) properties. So, the polytope dual to $P$ is any polytope which is combinatorially isomorphic to $P^\circ$.

PROBLEMS.

1°. Let $P$ be a polytope dual to the $d$-dimensional simplex $\Delta_d$ (see Examples 2.3, 1). Show that $P$ is combinatorially isomorphic (Definition 9.3) to $\Delta_d$.

2°. Let $f = (f_{-1}, f_0, f_1, \ldots, f_d)$ be the $f$-vector of a polytope $P$. Find the $f$-vector of the dual polytope $Q$.

3°. Octahedron $O_d$ (Examples 2.3, 3) and Cube $I_d$ (Examples 2.3, 2) are dual to each other.

4°. A plane $n$-gone is dual to itself.

5. For any $n \geq 4$ construct a 3-dimensional polytope with $n$ vertices which is dual to itself.

6. Describe the polytope $P$ dual to the cyclic polytope $C_4(m)$. How many vertices and facets does $P$ have?

7. Let $P \subset \mathbb{R}^d$ be a polytope containing the origin in its interior and $p \in P$ is another point in its interior. Let $Q_0 = P^\circ$ and $Q_p$ be the polar of $P$ if we move the origin in $p$. Then $Q_p$ is the image of $Q$ under the projective transformation

$$x \longmapsto \frac{x}{1 - \langle x, p \rangle}.$$

8*. Let $e_1, e_2, e_3, e_4$ be the standard orthonormal basis in $\mathbb{R}^4$. Let

$$P = \operatorname{conv}\{e_i + e_j, -e_i + e_j, -e_i - e_j : i \neq j\}.$$

Then $P$ is dual to itself.

This is so-called *24-cell*, a regular 4-dimensional polytope with 24 vertices and 24 facets.

## 12. Face Lattice

Theorem 9.1 establishes, in particular, that the set of all faces of a polytope form a partially ordered set. Namely, if $F$ and $G$ are faces of a polytope $P$ then we write $F \leq G$ iff $F \subset G$. Theorem 11.1 says that if we reverse the order then we still get a partially ordered set of a polytope, namely that of the dual polytope. This is a useful property.

**(12.1) Corollary.**

*1. Every face $F \neq P$ of a polytope $P$ belongs to some facet of $P$. In particular, any non-empty polytope $P$ has a facet.*

*2. For any two faces $F, G \subset P$ there exists a unique smallest (under inclusion) face $H$ of $P$ which contains both $F$ and $G$.*

*3. Every proper face $F$ of $P$ is the intersection of the facets containing $F$.*

*4. Every ridge of $P$ belongs to exactly two facets.*

**Proof.**

Let $Q$ be polytope dual to $P$. Theorem 11.1 implies that the properties 1-4 for $P$ are equivalent to the following properties 1°- 4° for $Q$.

1°. Every face $\hat{F} \neq \emptyset$ of $Q$ contains a vertex of $P$ (Corollary 9.2 and Theorem 7.2).

2°. Intersection of any two faces $\hat{F}, \hat{G}$ of $Q$ is a face of $Q$. Indeed, if $f$ is a supporting function of $\hat{F}$ and $g$ is a supporting function of $\hat{G}$ then $f + g$ is a supporting function of the intersection $\hat{F} \cap \hat{G}$.

3°. Every non-empty face $\hat{F}$ of $Q$ is the convex hull of vertices which are contained in $\hat{F}$ (Theorem 7.2, Theorem 9.3).

4°. Every edge of $Q$ contains exactly two vertices (obvious). □

Properties 2 and 2° imply that the partially ordered set of a polytope is a *lattice*, which is called the *face lattice*. Since every face is the convex hull of a set of vertices, we conclude that the lattice is *atomic*. Since every face is the intersection of some facets, we conclude that the lattice is *coatomic*. The lattice has *grading*: to every face we assign a number that is its dimension.

PROBLEMS.

1°. Prove that the face lattice of the simplex $\Delta_d$ is isomorphic to the *Boolean algebra* $\mathcal{B}_{d+1}$, consisting of all subsets of the set $\{1, \ldots, d\}$.

2. Prove the following *Diamond Property*: let $P$ be a polytope and $F \subset G \subset P$ be its faces such that dim $F =$ dim $G - 2 \geq 0$. Then there are exactly two faces $H_i$ such that $F \subset H_i \subset G$.

3. Every $k$-dimensional face of a polytope $P$ contains at least $k+1$ $(k-1)$-dimensional faces.

4. Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope and $F \subset P$ be a $k$-dimensional face of $P$. We construct a polytope $[P : F]$ as follows: let $Q = P^\circ$ and $\hat{F} \subset Q$; then $[P : F]$ is the polytope which is dual to $\hat{F}$.

Prove that $[P : F]$ is $(d-k)$-dimensional and that its face lattice is isomorphic to the interval of the face lattice of $P$ consisting of all faces $G$ which contain $F$.

5. Let $\Lambda_1$ be a face lattice of $P_1$ and $\Lambda_2$ be a face lattice of $P_2$. Consider the lattice $\Lambda = \Lambda_1 \times \Lambda_2$ consisting of all pairs $(x_1, x_2) : \ x_1 \in \Lambda_1, x_2 \in \Lambda_2$ with the order $(x_1, x_2) \geq (y_1, y_2)$ iff $x_1 \geq y_1$ and $x_2 \geq y_2$. Then $\Lambda$ is the face lattice of a polytope.

*Hint:* Let $P_1 = \text{conv}\{a_1, \ldots, a_m\} \subset \mathbb{R}^{d_1}$ and $P_2 = \text{conv}\{b_1, \ldots, b_n\} \subset \mathbb{R}^{d_2}$. Consider $P_1 \times P_2 \subset \mathbb{R}^{d_1+d_2} = \text{conv}\{(a_i, b_j) : \ i = 1, \ldots, m; j = 1, \ldots n\}$.

6. Let $P \subset \mathbb{R}^d$ be a polytope. Consider an inclusion $\mathbb{R}^d \subset \mathbb{R}^{d+1}$ and a point $p \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d$. Let $Q = \text{conv}\{P, q\}$. Describe the face lattice of $Q$ in terms of the face lattice of $P$.

7*. Find a sufficient and necessary combinatorial condition for a partially ordered set to be the face lattice of a polytope.

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Wednesday, January 25

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## 13. Optimization

Let $P$ be a polytope (polyhedron) and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. We consider the following *Optimization Problem.*

Find $\max\{f(x) : x \in P\}$.

Sometimes this problem is interesting by itself.

**(13.1) Example. Birkhoff Polytope and the Assignment Problem.** Let $f : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}$ be a linear function. So

$$f(x) = \sum_{i,j=1}^{d} c_{ij}x_{ij}$$

for some $d\times d$ matrix $c = (c_{ij})$. Let $B_d \subset \mathbb{R}^{d\times d}$ be the Birkhoff polytope (see Section 10). Then

$$\max\{f(x) : \ x \in B_d\} = \max\Big\{\sum_{i=1}^{d} c_{i\sigma(i)}, \ \sigma \text{ is a permutation}\Big\}.$$

This is so-called *Assignment Problem.*

The following special case is of interest. Suppose that rank $c = 1$, that is, $c_{ij} = a_i b_j$ for some vectors $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$. Then

$$\sum_{i=1}^{d} c_{i\sigma(i)} = \sum_{i=1}^{d} a_i b_{\sigma(i)}.$$

Let us consider the polytope

$$P(b) = \operatorname{conv}\big\{(b_{\sigma(1)}, \ldots, b_{\sigma(d)}) : \ \sigma \text{ is a permutation } \big\}.$$

It is called *permutation polytope.* If all the coordinates $b_1, \ldots, b_d$ are distinct then we call $P(b)$ a *permutohedron.* We will assume that in this case $b_1 > b_2 > \ldots > b_d$. Our problem can be formulated as a problem of optimization of the linear function $f(x) = a_1x_1 + \ldots + a_dx_d$ on the permutation polytope $P(b)$.

PROBLEM.

1°. Permutohedron $P_{d-1}(a)$ is a $(d-1)$-dimensional polytope having $d!$ vertices.

2°. Let $f(x) = a_1x_1+\ldots+a_dx_d$ be a linear function on the permutohedron $P(b)$. Let $\sigma$ be a permutation corresponding to a vertex of $P(b)$ where the maximum of $f(x)$ is attained and $\tau = \sigma^{-1}$ (we assumed that $b_1 > b_2 > \ldots > b_d$). Then $a_{\tau(1)} \geq a_{\tau(2)} \geq \ldots \geq a_{\tau(d)}$.

*Hint:* This problem appeals to common sense.

3. Prove that every vertex of a permutohedron $P(b) \subset \mathbb{R}^d$ belongs exactly to $d-1$ edges. The vertex corresponding to the identity permutation $12\ldots d$ is joined by an edge with the vertices which correspond to the transpositions $(k, k+1)$ and only to them.

*Hint:* Use Problem 2°.

4*. Permutation polytope $P(b)$ is defined by the following system of linear equations and inequalities:

$$x_1 + \ldots + x_d = b_1 + \ldots + b_d,$$

$$\sum_{i\in I} x_i \leq \sum_{i=1}^{|I|} b_i \qquad \text{for any } I \subset \{1, \ldots, d\}$$

(Rado's Theorem).

See, for example, Theorem 3.1, Section 3, Chapter V of V.A. Emelichev, M.M. Kovalev, M.K. Kravtsov, *Polytopes, Graphs and Optimization*, Cambridge University Press, Cambridge - New York, 1984.

5*. Let $\lambda_1, \ldots \lambda_d$ be some real numbers. Consider the set of all diagonals of symmetric $d \times d$ matrices whose eigenvalues are $\lambda_1, \ldots, \lambda_d$. This set is a permutation polytope $P(\lambda)$ for $\lambda = (\lambda_1, \ldots, \lambda_d)$ (Schur's Theorem).

*Hint:* It is relatively easy to prove that all the diagonals belong to the permutation polytope. To do that, let us consider the set $\mathcal{A}$ of all symmetric $d \times d$ matrices $A$ having eigenvalues $\lambda_1, \ldots, \lambda_d$. Let $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_d\}$ be a diagonal $d \times d$ matrix. Then

$$\mathcal{A} = \big\{UDU^{-1} : \ U \text{ is an orthogonal matrix}\big\}.$$

Let us choose an arbitrary diagonal matrix $C = \operatorname{diag}\{c_1, \ldots, c_d\}$ with distinct $c_1, \ldots, c_d$. If we prove that $\max\big\{\langle C, A\rangle : \ A \in \mathcal{A}\big\}$ is attained at a matrix $D_\sigma$ of the type $D_\sigma = \operatorname{diag}\{\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(d)}\}$ then we are done. Now we observe that $\mathcal{A}$ is a smooth manifold and we can use Calculus to figure out critical points of the smooth function $f(A) = \langle C, A\rangle$. It turns out that if $A$ is a critical point then $A$ must commute with $C$, that is $CA = AC$, which, in turn implies that $A = D_\sigma$ for some permutation $\sigma$. This problem provides a peculiar example of a *moment map*.

See also Problem 40.2 in V.V. Prasolov, *Problems and Theorems in Linear Algebra*, Translations of Mathematical Monographs, **134**, AMS, Providence, RI, 1994.

6. Prove that the equations $x_{ij} = 0$ define supporting hyperplanes to the facets of the Birkhoff Polytope $B_d$.

7. Find the radius of the ball centered in $x_{ij} = 1/d$, inscribed in the Birkhoff Polytope $B_d$ and touching the facets $x_{ij} = 0$.

8*. Two permutation matrices $X^\sigma$ and $X^\tau$ are connected by an edge in $B_d$ iff the permutation $\sigma\tau^{-1}$ contains precisely one cycle of a non-zero length.

See, for example, Theorem 1.3, Section 1, Chapter V of V.A. Emelichev, M.M. Kovalev, M.K. Kravtsov, *Polytopes, Graphs and Optimization*, Cambridge University Press, Cambridge - New York, 1984.

## 14. The Ellipsoid Method

The *Ellipsoid Method* was designed to optimize linear functions on polyhedra. We describe the very basic idea of this method while omitting many relevant details.

A detailed description, see, for example, in C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Prentice-Hall, Englewood Cliffs, N.J., 1982.

An *ellipsoid* $E$ in $\mathbb{R}^d$ is a set of the form $E = T(B)+a$, where $B = \{x \in \mathbb{R}^d : \|x\| = 1\}$ is the unit ball, $T$ is a non-degenerate linear transformation, and $a \in R^d$ is a vector. The volume of $E$ is equal to $|\det T| \cdot \operatorname{vol} B$.

**(14.1) Problem.** Given a set of linear inequalities $f_i(x) \leq \alpha_i, i = 1, \ldots, m, \ x \in \mathbb{R}^d$ decide whether it is feasible.

We assume that we have a certain numerical guarantee:

a) we are given a number $R$ such that if a solution to the system exists then there exists a solution in the ball $B_R(0)$;

b) we are given a number $r$ such that if a solution to the system exists then the set of solutions contains a ball $B_r(y) \subset B_R(0)$ for some $y \in \mathbb{R}^d$.

**(14.2) Lemma.** *Let $E \subset \mathbb{R}^d$ be an ellipsoid:*

$$E = \Big\{ x \in \mathbb{R}^d : \frac{d^2-1}{d^2} x_1^2 + \ldots + \frac{d^2-1}{d^2} x_{d-1}^2 + \frac{(d+1)^2}{d^2} \Big( x_d - \frac{1}{d+1} \Big)^2 \Big\} \leq 1.$$

*Then*

*a) The ellipsoid $E$ contains "the upper half" of the ball $B$:*

$$B^+ \subset E : \ B^+ = \{ x \in B : \ x_d \geq 0 \}.$$

*b) The volume of $E$ is smaller than the volume of $B$:*

$$\frac{\text{vol } E}{\text{vol } B} < 2^{-1/2(d+1)}.$$

**Proof.** a) Let us choose $x \in B^+$. Then $x_1^2 + \ldots + x_{d-1}^2 \leq 1 - x_d^2$, so

$$\frac{d^2-1}{d^2} \big( x_1^2 + \ldots + x_{d-1}^2 \big) \leq \frac{d^2-1}{d^2} - \frac{d^2-1}{d^2} x_d^2. \tag{14.1.1}$$

Furthermore,

$$\frac{(d+1)^2}{d^2} \Big( x_d - \frac{1}{d+1} \Big)^2 = \frac{(d+1)^2}{d^2} x_d^2 - \frac{2d+2}{d^2} x_d + \frac{1}{d^2}. \tag{14.1.2}$$

Adding (14.1.1) and (14.1.2) we get

$$\frac{d^2-1}{d^2} x_1^2 + \ldots + \frac{d^2-1}{d^2} x_{d-1}^2 + \frac{(d+1)^2}{d^2} \Big( x_d - \frac{1}{d+1} \Big)^2 \leq 1 + \frac{2d+2}{d^2} \Big( x_d^2 - x_d \Big).$$

Since $0 \leq x_d \leq 1$ we have $x_d^2 - x_d < 0$ and a) is proven.

b) We have

$$\frac{\text{vol } E}{\text{vol } B} = \frac{d}{d+1} \Big( \frac{d^2}{d^2-1} \Big)^{(d-1)/2}.$$

Now use the inequalities $1 + x \leq e^x$ for $x = -1/(d+1)$ and $x = 1/(d^2-1)$. We get

$$\frac{\text{vol } E}{\text{vol } B} \leq e^{-1/(d+1)} \cdot e^{1/2(d+1)} = e^{-1/2(d+1)} < 2^{-1/2(d+1)}.$$

$\square$

**(14.3) The Algorithm.** On the $k$-th step of the algorithm we have an ellipsoid $E_k$ such that if there exists a solution to the system, it exists in the ellipsoid $E_k$. After every step the volume of the ellipsoid decreases at least by the factor $2^{-1/2(d+1)}$. If we get the volume less than the volume of the ball $B_r(y)$ we stop and conclude that there is no solution.

We start out with $E_0 = B_R(0)$. Suppose that $E_k$ is constructed. We try its center $c_k$. If it satisfies the system of linear inequalities then we are done. If not, we choose any inequality $f_i(x) \leq \alpha_i$ which is violated at the center $c_k$. If a solution to the system exists, it must belong to the "half" $E_k^+$ where $f_i(x) \leq f_i(c_k)$. Let us circumscribe the $E_{k+1}$ around $E_k^+$ as in Lemma 14.2 (applying a suitable affine transformation we are in the situation of Lemma 14.2).

After every $2(d+1)$ iterations of the algorithm, the volume of the ellipsoid decreases at least by factor 2. Since vol $B_R(0)/\text{vol } B_r(0) = (R/r)^d$ we will stop anyway after at most $2d(d+1) \cdot \log_2(R/r)$ iterations.

PROBLEMS.

1. The function $f(X) = -\ln \det X$ is convex on the set of all positive definite matrices $X$, that is $f(\frac{1}{2}A + \frac{1}{2}B) \leq \frac{1}{2}f(A) + \frac{1}{2}f(B)$ for any two positive definite $A$ and $B$ and the equality is attained only if $A = B$.

*Hint:* Use the representation

$$(\det A)^{-1/2} = (\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\langle Ax, x\rangle} dx$$

and apply the Cauchy-Schwartz inequality.

2. For any convex $d$-dimensional compact $C \subset \mathbb{R}^d$ where exists the *unique* ellipsoid of the minimal volume that contains $C$ (Löwner's Ellipsoid).

*Hint:* Suppose that $C \subset E_i,\ i = 1, 2$, where $E_i$ is an ellipsoid given by $E_i = \{x \in \mathbb{R}^d : q_i(x - x_i) \leq 1\}$, where $q_i$ are positive definite quadratic forms and $x_i \in \mathbb{R}^d$ are vectors. Define the ellipsoid $E$ by the formula $E = \{x \in \mathbb{R}^d : \frac{1}{2}q_1(x - x_1) + \frac{1}{2}q_2(x - x_2) \leq 1\}$.

3. Let $C \subset \mathbb{R}^d$ be a convex $d$-dimensional compact and $E$ be its Löwner's Ellipsoid (see Problem 2). Let $E'$ be the ellipsoid obtained from $E$ by contracting towards its center with the coefficient $1/d$. Then $E' \subset C$. If $C$ is centrally symmetric then we may use $1/\sqrt{d}$ instead of $d$.

4*. Let $C \subset \mathbb{R}^d$ be a convex $d$-dimensional compact. There exists the *unique* ellipsoid $E \subset C$ of the largest volume (John's Ellipsoid). Let $E'$ be the ellipsoid obtained from $E$ by dilating it from its center with the coefficient $d$. Then $C \subset E'$. If $C$ is centrally symmetric then we may use $\sqrt{d}$ instead of $d$.

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Friday, January 27

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## 15. Simple and Simplicial Polytopes

**(15.1) Simplex.** Points $a_1, \ldots, a_k \in \mathbb{R}^d$ are said to be *affinely independent* iff there are no $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 a_1 + \ldots + \lambda_k a_k = 0$, $\lambda_1 + \ldots + \lambda_k = 0$ and $\lambda_1^2 + \ldots + \lambda_k^2 > 0$. Let $a_1, \ldots, a_{k+1} \in \mathbb{R}^d$ be affinely independent points. The polytope $S = \operatorname{conv}\{a_1, \ldots, a_{k+1}\}$ is called a $k$-dimensional *simplex*. In other words, a $k$-dimensional simplex in $\mathbb{R}^d$ is the image of the standard $k$-dimensional simplex $\Delta_k \subset \mathbb{R}^{k+1}$ (see Examples 2.3, 1) under a non-degenerate affine map $T : H \longrightarrow \mathbb{R}^d$, where $H = \{x \in \mathbb{R}^{k+1} : \ x_1 + \ldots + x_{k+1} = 0\}$ is the hyperplane containing $\Delta_{k+1}$.

PROBLEMS.

1°. Let $S = \operatorname{conv}\{a_1, \ldots, a_{k+1}\}$ be a $k$-dimensional simplex. For any $I \subset \{1, \ldots, k+1\}$ the set $F = \operatorname{conv}\{a_i : \ i \in I\}$ is a face of $S$.

2°. A face of a simplex is a simplex.

3°. There is no $(d+1)$-dimensional simplex in $\mathbb{R}^d$.

4°. There is a $d$-dimensional simplex in $\mathbb{R}^d$.

5°. Suppose that a $d$-dimensional polytope in $\mathbb{R}^d$ is the intersection of $d+1$ halfspaces. Then this polytope is a $d$-dimensional simplex.

6. For $k \leq d+1$ let us consider the set $\mathbb{R}^{dk} = \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ of all $k$-tuples of $d$-vectors $(a_1, \ldots, a_k)$, $a_i \in \mathbb{R}^d$. Then the subset

$$\mathcal{S} = \Big\{(a_1, \ldots, a_k) \in \mathbb{R}^{dk} : \ \operatorname{conv}\{a_1, \ldots, a_k\} \text{ is a } (k-1)\text{−dimensional simplex}\Big\}$$

is open and dense in $\mathbb{R}^{dk}$.

**(15.2) Definition.** A polytope $P$ is called *simplicial* iff every facet of $P$ is a simplex. A polytope $P$ is called *simple* iff its dual is a simplicial polytope.

PROBLEMS.

1°. A $d$-dimensional polytope is simple iff every vertex of the polytope belongs to exactly $d$ facets.

2°. Octahedron (Examples 2.3, 3) is a simplicial polytope whereas Cube (Examples 2.3, 2) is a simple polytope.

3°. Simplices and plane polygons are simple and simplicial at the same time.

4. If a polytope is simple and simplicial at the same time then it is either a simplex or a plane polygon (or both).

5°. Cyclic polytope is simplicial.

6. The Birkhoff polytope $B_d$ is simplicial and simple for $d = 2$, simplicial but not simple for $d = 3$ and neither simplicial nor simple for $d > 3$.

7*. A permutohedron is simple.

**(15.3) Definition.** Let $P$ be a polytope (polyhedron). Two vertices $v$ and $u$ of $P$ are called *neighbors* iff $[u, v]$ is the edge of $P$.

**(15.4) Theorem.** *Let $P \subset \mathbb{R}^d$ be a simple $d$-dimensional polytope. For every vertex $v_0$ of $P$ there exist exactly $d$ neighbors $v_1, \ldots, v_d$ of $P$. Furthermore, for every $k$ neighbors*

$v_{i_1}, \ldots, v_{i_k}$ *of* $v_0$ *there exists exactly one* $k$*-dimensional face* $F$ *of* $P$ *that contains* $v_0$ *and* $v_{i_1}, \ldots, v_{i_k}$.

**Proof.** Let $P^\circ$ be the dual polytope. Theorem 11.1 implies that we have to prove that every facet $\hat{v_0}$ of $P^\circ$ has exactly $d$ facets and that the intersection of every $k$ facets of $\hat{v_0}$ is a $(d-k-1)$-dimensional face of $P^\circ$. Since $\hat{v_0}$ is a $(d-1)$-dimensional simplex, this follows by Problem 1° from Section 12.1, say. □

**(15.5) Theorem.** *Every face of a simple polytope is a simple polytope.*

**Proof.** It is enough to prove that a facet $F$ of a simple $d$-dimensional polytope $P$ is a simple polytope. Let us consider a vertex $v$ of $F$. We must prove that $v$ belongs exactly to $(d-1)$ facets of $F$ (Problem 1° Section 15.2). Every face of $F$ is a ridge of $P$. Every ridge of $P$ belongs exactly to two facets of $P$ (Corollary 12.1, 4) and one of this facets is $F$. Furthermore, since $P$ is simple, there exists precisely $d$ facets of $P$ that contain $v$. Therefore, there exist at most $d-1$ facets of $F$ that contain $v$. Since $F$ is a $(d-1)$-dimensional, there should be exactly $(d-1)$ facets containing $v$ (consider the dual polytope: every facet there should have at least $(d-1)$ vertices). □

**(15.6) General Position.** Let $P \subset \mathbb{R}^d$ be a convex polytope, given as the convex hull of its vertices $v_1, \ldots, v_m \in \mathbb{R}^d$. If we slightly perturb the vertices $v_i \longmapsto \overline{v_i}$ $i = 1, \ldots, m$ then the polytope $\overline{P} = \operatorname{conv}\{\overline{v_1}, \ldots, \overline{v_m}\}$ is simplicial "almost for sure". On the other hand, if $P$ is a simple polytope, given by a family of linear inequalities $f_i(x) \leq \alpha_i,\ i = 1, \ldots, m$ and we slightly perturb $\alpha_i \longmapsto \overline{\alpha_i}$ the the polytope $\overline{P} = \{x \in \mathbb{R}^d : \ f_i(x) \leq \overline{\alpha_i}\}$ is simple "almost for sure".

PROBLEMS.

1°. Let us consider the space $\mathbb{R}^{dn} = \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ ($n$ times) consisting of all $n$-tuples $(a_1, \ldots, a_n)$ of points $a_i \in \mathbb{R}^d$. The set

$$S = \big\{(a_1, \ldots, a_n) : \ \operatorname{conv}\{a_1, \ldots, a_n\} \text{ is a simplicial polytope } \big\}$$

is open in $\mathbb{R}^{dn}$.

2*. Let $P = \operatorname{conv}\{a_1, \ldots, a_m\}$ be a $d$-dimensional polytope in $\mathbb{R}^d$. One can move the points $a_i \longmapsto \overline{a_i}$ so that the polytope $\overline{P} = \operatorname{conv}\{\overline{a_1}, \ldots, \overline{a_m}\}$ is simplicial and $f_i(P) \leq f_i(\overline{P})$ for $i = -1, \ldots, d$, where $f_i$ is the number of $i$-dimensional faces.

## 16. h-vector

**(16.1) Definitions.** Let $P$ be a $d$-dimensional simplicial polytope. Let

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} \quad k = 0, \ldots, d,$$

where $f_i = f_i(P)$ is the number of $i$-dimensional faces of $P$. The integer vector $h = (h_0, h_1, \ldots, h_d)$ is called the $h$-vector of $P$.

Let $P$ be a $d$-dimensional simple polytope. We define its $h$-vector as the $h$-vector of the dual polytope $P^\circ$. So we can write

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{d-i}$$

where $f_j = f_j(P)$.

PROBLEMS.

1°. If $P$ is a 3-dimensional simplicial polytope then $h_0 = 1, h_1 = f_0 - 3, h_2 = 3 - 2f_0 + f_1, h_3 = -1 + f_0 - f_1 + f_2$.

2°. If $P$ is a 4-dimensional simplicial polytope then $h_0 = 1, h_1 = f_0 - 4, h_2 = 6 - 3f_0 + f_1, h_3 = f_2 - 2f_1 + 3f_0 - 4, h_4 = 1 - f_0 + f_1 - f_2 + f_3$.

3°. For every simplicial $d$-dimensional polytope $P$ we have $h_0(P) = 1$, $h_1(P) = f_0(P) - d$ and $h_d(P) = f_{d-1} - f_{d-2} + \ldots + (-1)^d$.

**(16.2) Theorem.** *Let $P$ be a $d$-dimensional simplicial polytope. Then*

$$f_{k-1}(P) = \sum_{i=0}^{k} \binom{d-i}{k-i} \cdot h_i(P), \quad k = 0, \ldots, d.$$

*Furthermore, the numbers $h_i(P)$ are uniquely determined by these equations Similarly, if $P$ is a $d$-dimensional simple polytope then*

$$f_{d-k}(P) = \sum_{i=0}^{k} \binom{d-i}{k-i} \cdot h_i(P), \quad k = 0, \ldots, d.$$

*Furthermore, the numbers $h_i(P)$ are uniquely determined by these equations.*

**Proof.** Obviously, it is enough to prove this formula only for simplicial polytopes. Let

$$\mathbf{f}(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i}$$

be what is called $f$-polynomial and

$$\mathbf{h}(x) = \sum_{i=0}^{d} h_i x^{d-i}$$

be what is called $h$-polynomial. Definition 16.1 implies that $\mathbf{h}(x) = \mathbf{f}(x-1)$ (just compare the coefficients of $x^{d-k}$). Therefore $\mathbf{f}(x) = \mathbf{h}(x+1)$. Comparing the coefficients of $x^{d-k}$ we get the formula. □

Some technical results (preparation for greater achievements).

**(16.3) Definition.** The *relative interior* of a convex set $C$ is the interior of $C$ considered as a subset of the smallest affine subspace that contains $C$.

**(16.4) Theorem.** *Let $P$ be a polyhedron and $p \in P$ be a point. Then $p$ belongs to the relative interior of a face of $P$.*

**Proof.** Let $P = \big\{x \in \mathbb{R}^d : f_i(x) \leq \alpha_i,\ i = 1, \ldots, m\big\}$ be a polyhedron. Let $I = \{i : f_i(p) = \alpha_i\}$ and $F_I = \{x \in P :\ f_i(x) = \alpha_i,\ i \in I\}$. Then $F_I$ is a face of $P$ with the supporting function $\sum\limits_{i \in I} f_i$ and $p \in F_I$ belongs to the relative interior of $F_I$. ☐

**(16.5) Lemma.** *Let $P \subset \mathbb{R}^d$ be a polytope, $v \in P$ be a vertex and $v_i,\ i = 1, \ldots, k$ be all the neighbors of $v$. Let $u_i = v_i - v$. Then every point $x \in P$ can be represented in the form*

$$x = v_0 + \sum_{i=1}^{k} \lambda_i u_i \quad \text{for some } \lambda_i \geq 0.$$

Translating the polytope $P$, if necessary, we may assume that $v_0 = 0$. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function of $v_0$, so $f(y) < f(v_0) = 0$ for every $y \in P$. Let us choose a sufficiently small $\epsilon > 0$ so that there are no vertices $v$ of $P$ such that $0 > f(v) \geq -\epsilon$. Let $H = \{x \in \mathbb{R}^d :\ f(x) = -\epsilon\}$ be a hyperplane and $Q = P \cap H$. So $Q$ is a bounded polyhedron, that is, a polytope, that is the convex hull of its vertices $w_1, \ldots, w_m$ (Theorem 9.3). We claim that every $w_i$ is the intersection of some edge $[v_0, v_i]$ and $H$ (so, in particular, $m \leq k$ and $w_i = \alpha_i v_i$ for some $\alpha_i > 0$). Indeed, let $F$ be a face of $P$ containing $w$ in its relative interior (Theorem 16.4). Then we must have $\dim F = 1$, otherwise, $w$ can not be a vertex of $Q$. Now let $x \in P$ be a point. Since $f(x) < 0$ there exists some $t > 0$ such that $tx \in Q$. Then

$$tx = \sum_{i=1}^{m} \beta_i w_i \quad \text{where } \beta_i \geq 0 \quad \text{and } \sum_{i=1}^{m} \beta_i = 1.$$

Let $\lambda_i = (\beta_i \cdot \alpha_i)/t$. ☐

**(16.6) Corollary.** *Let $P \subset \mathbb{R}^d$ be a polytope, $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function and $v_0 \in P$ be a vertex. If $f(v_i) > f(v_0)$ for every neighbor $v_i$ of $v_0$ then $f(v_i) > x$ for every $x \in P$.*

**Proof.** Follows by Lemma 16.5. ☐

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Monday, January 30

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## 17. h-vector, Continued

This is the main theorem about $h$-vector.

**(17.1) Theorem.** *Let $P \subset \mathbb{R}^d$ be a simple $d$-dimensional polytope with the vertices $v_1, \ldots, v_m$. Let $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function such that $l(v_i) \neq l(v_j)$ whenever $[v_i, v_j]$ is an edge of $P$. For a vertex $v$ of $P$ let us define the index* ind $v$ *of $v$ as the number of neighbors $v_i$ of $v$ such that $l(v_i) < l(v)$. Then for every $k : \ 0 \leq k \leq d$ the number of vertices of $P$ which have index $k$ is equal to $h_k(P)$. In particular, it does not depend on $l$.*

**Proof.** Let us count $(d-k)$-dimensional faces $F$ of $P$. We know that this number is $f_{d-k}(P)$. Let us correspond to every $(d-k)$-dimensional face $F$ its only vertex $v \in F$ where the minimum of $l$ on $F$ is attained: $l(v) = \min\{l(x) : \ x \in F\}$. Let us see how often we get a given vertex $v \in P$. Suppose the index of $v$ is equal to $i$. Then there are exactly $d-i$ neighbors $v_i$ of $v$ for which $l(v_i) > l(v)$. For every $(d-k)$ of these neighbors there exists a unique $(d-k)$-dimensional face $F$ containing them (Theorem 15.4). Furthermore, Corollary 16.6 implies that $l$ attains its minimum on $F$ in $v$. On the other hand, if $F$ is a $(d-k)$-dimensional face such that the minimum of $l$ on $F$ is attained in $v$, then $v$ should have $(d-k)$ neighbors $v_1, \ldots, v_{d-k}$ in $F$ (Theorem 15.4, 15.5) and we must have $l(v_i) > l(v)$.

Thus every vertex $v$ of index $k$ corresponds to $\binom{d-i}{d-k}$ faces $F$. Therefore

$$f_{d-k}(P) = \sum_{i=0}^{k} \binom{d-i}{k-i} g_i(P, l),$$

where $g_i(P, l)$ is the number of vertices having index $i$. By Theorem 16.2 we must have $g_i(P, l) = h_i(P)$. □

**(17.2) Theorem (Dehn - Sommerville Equations).** *Let $P$ be either simple or simplicial $d$-dimensional polytope. Then*

$$h_k(P) = h_{d-k}(P) \quad \text{for } k = 0, \ldots, d.$$

**Proof.** Let us choose a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $l(v_i) \neq l(v_j)$ for any two neighboring vertices $v_i, v_j$. Then $h_k(P)$ is the number of vertices of the index $k$ (Theorem 17.1). Let us instead of $l$ choose $-l$. Then every vertex of index $k$ for $l$ is a vertex of index $d-k$ for $-l$ and vice versa. Therefore, the number of vertices of the index $k$ for $l$ is equal to $h_{d-k}$. □

PROBLEMS.

1°. For a simplicial 3-dimensional polytope $P$ the Dehn-Sommerville equations are equivalent to $f_0 - f_1 + f_2 = 2, 3f_2 - 2f_1 = 0$.

2°. For a simplicial 4-dimensional polytope $P$ the Dehn-Sommerville equations are equivalent to $f_0 - f_1 + f_2 - f_3 = 0$, $f_3 = f_1 - f_0$.

## 18. Example: the Permutohedron

Let us consider the standard permutohedron

$$P = \operatorname{conv}\{(\sigma(1), \dots, \sigma(d)) : \ \sigma \in S_d\} \subset \mathbb{R}^d,$$

where $S_d$ is the symmetric group of all $d!$ permutations.

**(18.1) Theorem.**

*a) The permutohedron is a $(d-1)$-dimensional simple polytope having $d!$ vertices.*

*b) The neighbors of a vertex $(a_1, \dots, a_k)$ are the vertices obtained by a transposition of any two adjacent values $a_i, a_j$ : $|a_i - a_j| = 1$.*

*c) $h_k(P) = E_{d,k+1}$, where $E_{d,n}$ is the number of permutations having $n-1$ descends (the $n$-th Eulerian number).*

**Proof.** It is clear that $\dim P \leq d-1$ since $P$ belongs to the hyperplane $x_1 + \dots + x_d = 1 + \dots + d$. Suppose there exists another hyperplane, say $a_1x_1 + \dots + a_dx_d = \alpha$ which contains $P$. Suppose that $a_i \neq a_j$ for some pair $i \neq j$. Let us consider two points of $P$: one corresponds to the identity permutation $(1, 2, 3, \dots, d)$ and the other one to the transposition $(i, j)$. Then we should have $a_1 + 2a_2 + 3a_3 + \dots + ia_i + \dots + ja_j + da_d = \alpha$ and $a_1 + 2a_2 + 3a_3 + \dots + ja_i + \dots + ia_j + da_d = \alpha$. Subtracting the second equation from the first one we get $a_i(i-j) + a_j(j-i) = 0$, that is $(a_i - a_j)(i-j) = 0$ which is a contradiction. So all $a$-s should be equal and hence $\dim P = (d-1)$.

It is clear that $P$ has at most $d!$ vertices. Furthermore, since $P$ is non-empty it must have at least one vertex, say, $(i_1, \dots, i_d)$. Since the permutation of the coordinates is a linear automorphism of $\mathbb{R}^d$ which maps $P$ onto itself, all other points $(i_{\sigma(1)}, \dots, i_{\sigma(d)})$ should be the vertices of $P$. They are all different, and, therefore, $P$ has $d!$ vertices.

Let us prove that the neighbors of the vertex $(1, 2, \dots, d)$ are exactly the vertices obtained by permutation of any two $(k, k+1)$. If we prove this, then we prove b) and, therefore, it would follow that $P$ is simple (see Problem 1 below).

First, we demonstrate that every vertex $u = (1, \dots, k-1, k+1, k, k+2, \dots, d)$ is a neighbor of $v = (1, \dots, d)$. We present a linear function which attains its maximum exactly on these two vertices. This is $\langle l, x \rangle$, where $l = (1, 2, \dots, k-1, k, k, k+2, \dots, d)$. Indeed, let us consider any vertex $w = (a_1, \dots, a_d)$ of $P$ where the maximum of $\langle l, x \rangle$ is attained. We claim that if for some pair $\{i < j\} \neq \{k, k+1\}$ we have $a_i > a_j$ then by switching $a_i$ and $a_j$ we get a bigger value of $l$. Indeed, the difference that we get is $l_ia_j + l_ja_i - l_ia_i - l_ja_j = l_i(a_j - a_i) + l_j(a_i - a_j) = (l_j - l_i)(a_i - a_j) > 0$. So the only two possibilities for $w$ are $(1, \dots, k-1, k, k+1, d)$ and $(1, \dots, k-1, k+1, k, k+2, \dots)$.

Next, let us prove that there are no other neighbors of $v = (1, \dots, d)$. Suppose that $l = (l_1, \dots, l_d)$ is the normal vector of a supporting hyperplane for an edge $[v, u]$ of $P$.

That is, the maximum value of $\langle l, x\rangle$ on $P$ is attained precisely at $v$ and $u$. First, we must have $l_1 \le l_2 \ldots \le l_d$ (otherwise, we may employ the same trick: if $l_i > l_j$ for some $i < j$ then switching $i$ and $j$ in $v$ we get a bigger value). Next, we claim that only two of $l_i$ could be equal. If we had an interval $l_i = l_{i+1} = l_{i+2}$ of equal $l_i$-s then the function $\langle l, x\rangle$ would have the same value on all 6 permutations of $i, i+1, i+2$.

Finally, we have to prove c). Let us take a linear function $l(x) = x_1 + 2x_2 + 3x_3 + \ldots + dx_d$ on the permutohedron $P$. Let us take a vertex $v = (a_1, \ldots, a_d)$ of $P$, corresponding to a permutation $\sigma$. We get a neighbor $w$ if we switch two adjacent values $a_i, a_j$ $(i < j)$. The difference $l(v) - l(w) = ia_i + ja_j - ia_j - ja_i = (i-j)(a_i - a_j)$ is going to be negative if and only if $a_i > a_j$. Therefore, the permutation $\sigma^{-1}$ has a descend. Therefore, the index of a vertex is $k$ if and only if the permutation $\sigma^{-1}$ has exactly $k$ descends. □

PROBLEMS.

1. Prove that a $d$-dimensional polytope $P$ is simple iff every its vertex has exactly $d$ neighbors.

2°. What do the Dehn - Sommerville equations $h_k = h_{d-k}$ tell us about Eulerian numbers?

3. This is how the $i$-dimensional faces of the permutohedron $P$ with $d!$ vertices look like:

list the numbers $1, \ldots, d$ in any order, say, $a_1, \ldots, a_d$. Subdivide the string $a_1, \ldots, a_d$ into $d - i$ substrings. Take all possible permutations $\sigma$ of $a_1, \ldots, a_d$ that leave every $a_i$ in its substring.

Then the set of $\sigma^{-1}$ is the set of vertices of an $i$-dimensional face, and every $i$-dimensional face you can get this way.

4. The $f$-vector of the permutohedron $P$ with $d!$ vertices is given by

$$f_i(P) = \sum \frac{d!}{t_1! \ldots t_{d-i}!},$$

where the sum is taken over all positive integral solutions of the equation $t_1 + t_2 + \ldots + t_{d-i} = d$.

5°. Find the $f$-vector of the permutohedron in terms of the Eulerian numbers.

6*. Describe the simplicial polytope dual to the $d$-dimensional permutohedron.

*Hint:* This is what is called the *first barycentric subdivision* of the $d$-dimensional simplex $\Delta$. Take a $d$-dimensional simplex. Choose a point in the relative interior of every non-empty face $F$ of $\Delta$ and consider a cone over $F$ with the apex at this point. To get the actual polytope, move slightly every such point out of the face, so that the points from faces of higher dimensions go farther. The face lattice of this polytope is a poset of flags in the Boolean algebra. Maximal flags there are naturally identified with permutations.

7*. For two $d$-vectors $x$ and $y$ with non-negative coordinates let us write $x \le y$ iff

$$\sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i$$

and there are permutations $\tau$ and $\sigma$ such that

$$\sum_{i=1}^{k} x_{\tau(i)} \leq \sum_{i=1}^{k} y_{\sigma(i)} \quad \text{for } k = 1, \ldots d.$$

Prove that $x \leq y$ iff there exists a doubly stochastic $d \times d$ matrix $A$ (that is, a point in the Birkhoff polytope $B_d$) such that $x = Ay$. Then prove that $x \in P$, where $P$ is the standard permutohedron, iff $x \leq (1, 2, \ldots, d)$.

Regarding the permutohedron, see also Section 3 of Chapter V of V.A. Emelichev, M.M. Kovalev, M.K. Kravtsov, *Polytopes, Graphs and Optimization*, Cambridge University Press, Cambridge - New York, 1984.

## 19. The Upper Bound Conjecture

**(19.1) Theorem.** *Let $P$ be a $d$-dimensional simple polytope and $F \subset P$ be its facet. Then*

$$h_k(F) \leq h_k(P) \quad \text{for all } k = 0, \ldots, d-1.$$

**Proof.** First, we note that a facet of a simple polytope is simple (Theorem 15.5). Of course, we are going to apply Theorem 17.1 (the main Theorem concerning $h$-vector). Let $-l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function of $F$, so $l(x) \geq \alpha$ for any $x \in P$ and $l(x) = \alpha$ iff $x \in F$. In particular, $l(v) > \alpha$ for any vertex $v \notin F$. Let us perturb $l \longmapsto \hat{l}$ so that we still have $\hat{l}(v) > \hat{l}(u)$ for any two vertices $u \in F$ and $v \notin F$. Suppose a vertex $u \in F$ has index $k$, that is for exactly $k$ neighbors $v$ of $u$ in $F$ we have $\hat{l}(v) < \hat{l}(u)$. Then $u$ has index $k$ in $P$ since all other neighbors (not in the facet $F$) may have only bigger values of $\hat{l}$. $\square$

**(19.2) Lemma.** *Let $P$ be a $d$-dimensional simple polytope. Then*

$$\sum_F h_k(F) = (d-k)h_k(P) + (k+1)h_{k+1}(P), \quad k = 0, \ldots, d-1,$$

*where the first sum is taken over all facets $F$ of $P$.*

**Proof.** Let us take a generic linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ on $P$. For a vertex $v$ we consider its usual index ind $(v)$ and the relative index ind $(v, F)$ as of a vertex in the face $F$. Let us call a neighbor $u$ of $v$ a *smaller* neighbor iff $l(u) < l(v)$. Now let us count all vertices in all facets which have relative index $k$. It is the sum in the left hand side. How many times do we count a given vertex $v \in P$? If ind $(v) < k$ then we don't count it because there are fewer than $k$ smaller neighbors of $v$. If ind $(v) > k+1$ then we don't count it since there are at least $(k+2)$ smaller neighbors of $v$ and since every facet $F$ contains all but one neighbors of $v$ then on every facet there will be at least $(k+1)$ of these smaller neighbors.

If ind $v=k$ then it has a relative index $k$ on exactly $(d-k)$ facets, namely the facets that contain all $k$ smaller neighbors and any $(d-k-1)$ out of the remaining $(d-k)$ neighbors.

If ind $v=k+1$ then it has relative index $k$ on exactly $k+1$ facets, namely the facets that contain some $k$ of $k+1$ smaller neighbors and all other $d-k-1$ neighbors. $\square$

**(19.3) Theorem.** *Let $P$ be a $d$-dimensional simple polytope with $n$ facets. Then*

$$h_k(P) \leq \binom{n-d+k-1}{k} \quad \text{for } k=0,\ldots,d.$$

**Proof.** From Lemma 19.2 and Theorem 19.1 we get

$$(i+1)h_{i+1}(P) \leq nh_i(P)-(d-i)h_i(P)=(n-d+i)h_i(P),$$

that is

$$h_{i+1}(P) \leq \frac{n-d+i}{i+1}h_i(P).$$

Since $h_0(P)=1$ we get what we need. $\square$

PROBLEMS.

1°. Check that for a simplicial 4-dimensional polytope with $n$ vertices we have $f_3 \leq \binom{n}{2}-n$.

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Friday, February 3

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## 20. The Upper Bound Conjecture, Continued

Let $P$ be a simple $d$-dimensional polytope. We know that

$$f_{d-k}(P)=\sum_{i=0}^{k}\binom{d-i}{k-i}h_i(P), \quad k=0,\ldots,d. \tag{20.1}$$

We know that

$$h_i(P)=h_{d-i}(P) \quad i=0,\ldots,d$$

and that

$$h_i(P) \leq \binom{n-d+i-1}{i} \quad \text{for } i=0,\ldots,d, \tag{20.2}$$

where $n$ is the number of facets of $P$. This inequality is strong when $i$ is small and not that strong if $i$ is large. So what we should do is to use the inequality (20.1) if $i \leq [d/2]$ and to use the inequality

$$h_i(P) = h_{d-i}(P) \leq \binom{n-i-1}{d-i} \tag{20.3}$$

when $i$ is large. If we substitute (20.2) and (20.3) into (20.1) then we get some inequality for $f_{d-k}$ which is called the Upper Bound Theorem.

Similarly, if $P$ is a simplicial $d$-dimensional polytope then

$$f_{k-1} = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i(P), \quad k = 0, \ldots, d.$$

We know that

$$h_i(P) = h_{d-i}(P) \quad i = 0, \ldots, d$$

and that

$$h_i(P) \leq \binom{n-d+i-1}{i} \quad \text{for } i = 0, \ldots, d,$$

where $n$ is the number of vertices of $P$. If we do the same trick we get some inequality which is called the Upper Bound Theorem.

Problem 2* from Section 15.6 then implies that the inequality we get is valid for any $d$-dimensional polytope, not necessarily simplicial.

## 21. Neighborly Polytopes

A $d$-dimensional polytope $P$ is called *$l$-neighborly* iff the convex hull of any of $k \leq l$ of its vertices is a $(k-1)$-dimensional face of $P$. If $l = [d/2]$ then the polytope is called *neighborly*. The main example is the Cyclic Polytope (see Theorem 8.2).

PROBLEM.

1. A neighborly polytope is simplicial.

At least we know that the Cyclic Polytope is simplicial (Problem 5°), Section 15.2.

2°. A polytope $P$ is dual to a $d$-dimensional $l$-neighborly polytope iff for any $k \leq l$ the intersection of any $k$ facets of $P$ is a $(d-k)$-dimensional face of $P$.

3°. Let $P$ be a $l$-neighborly polytope, $P^\circ$ be its dual and $F \subset P$ is a facet of $P$. Then $F$ is dual to a $(l-1)$ neighborly polytope.

4. A $d$-dimensional polytope is $[d/2]+1$ neighborly iff it is a simplex.

**(21.1) Theorem.** *Let $P$ be a neighborly $d$-dimensional polytope with $n$ vertices. Then*

$$h_k(P) = \binom{n-d+k-1}{k} \quad \text{for } k = 0, \ldots, [d/2].$$

**Proof.** (Sketch) We are going to prove that if a simple polytope $P$ has $n$ facets and it is dual to a neighborly polytope then

$$h_k(P) = \binom{n-d+k-1}{k} \quad \text{for } k = 0, \ldots, [d/2].$$

If we analyze how we get the inequality

$$h_k(P) \leq \binom{n-d+k-1}{k}$$

in Theorem 19.3 we see that the only place we could lose equality was the inequality $h_k(F) \leq h_k(P)$ for a facet $F$ of $P$ in Lemma 19.1. Let us prove that if $F$ is a facet of $P$ then

$$h_k(F) = h_k(P) \quad \text{for } k = 0, \ldots, [d/2] - 1.$$

As in the proof of Lemma 19.1 let us choose a supporting function $-l$ of $F$, so that $l$ has the minimal value on $F$ and let us consider the perturbed $\hat{l}$. What we have to prove is that there are no vertices of index $k$ which do not belong to $F$. Suppose there is a vertex $v$ of $P$, $v \notin F$. Let us consider $d-k$ neighbors of $v$ with the bigger value of $\hat{l}$. They, together with $v$ belong to the unique $(d-k)$-dimensional face $G$ of $P$ (Theorem 15.4), so what $\hat{l}$ attains its minimum on $G$ in $v$. Now, $G$ is the intersection of $k$ facets containing it (because $P$ is simple). Then the intersection of these $k$ facets and $F$ should have a common vertex, say, $u$. But $\hat{l}(u) < \hat{l}(v)$, which contradicts to the choice of $v$.

PROBLEMS.

1. Check and complete the proof of Theorem 21.1.

□

**(21.2) Corollary.** *If $P$ is a simplicial $d$-dimensional polytope with $n$vertices and $C_d(n)$ is the cyclic $d$-dimensional polytope with $n$ vertices then $f_i(P) \leq f_i\big(C_d(n)\big)$ for all $i$.*

**Proof.** Simply because $h_k(P) \leq h_k(C_d(n))$ for $k = 0, \ldots, [d/2]$. □

By Problem 2* from Section 15.6 we get

**(21.3) Corollary.** *If $P$ is a $d$-dimensional polytope with $n$ vertices and $C_d(n)$ is the cyclic $d$-dimensional polytope with $n$-vertices then $f_i(P) \leq f_i\big(C_d(n)\big)$ for all $i$.*

□

Details of this approach and other results can be found, for example, in A. Brøndsted, *An Introduction to Convex Polytopes*, Graduate Texts in Mathematics **90**, Springer-Verlag, New York - Berlin, 1983.

PROBLEMS

1. This is a way to construct an $l$-neighborly polytope. Consider the Euclidean space $\mathbb{R}^l$ as a hyperplane in $\mathbb{R}^{l+1}$ of the form $(x, 1)$, where $x \in \mathbb{R}^l$. Consider the map

$$\phi : \mathbb{R}^{l+1} \longrightarrow \mathbb{R}^{(l+1)^2} \qquad x \longmapsto x \otimes x.$$

In other words, to every vector $x \in \mathbb{R}^{l+1}$ we correspond the matrix $a = (a_{ij}) : \ a_{ij} = x_i \cdot x_j$. Then the image of $\mathbb{R}^l$ is contained in a $d$-dimensional affine subspace in $\mathbb{R}^{(l+1)^2}$, where $d = \binom{l+1}{2} + l$. Finally, iff $v_1, \ldots, v_m$ are some points in $\mathbb{R}^l$ such that no $l+1$ of them belong to the same hyperplane in $\mathbb{R}^l$ then the convex hull of their images $\phi(v_i)$ is an $l$-neighborly $d$-dimensional polytope.

2*. Construct a neighborly polytope which is not isomorphic to a cyclic polytope.

3°. Draw a picture of the 2-dimensional standard permutohedron (that, is the permutohedron whose vertices are the permutations of the row $(1, 2, 3)$).

4*. The standard $(d-1)$-dimensional permutohedron $P = \operatorname{conv}\big\{(\sigma(1), \ldots, \sigma(d)) : \sigma \in S_d\big\}$ is an affine image of the $m$-dimensional cube for $m = \binom{d}{2}$.

*Hint:* See Example 7.15 in Section 7.3 of G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, Berlin-New York, 1995.

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Monday, February 6

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## 22. Some Additional Insight into h-Vector

In this section we don't prove much.

**(22.1) Theorem ("Unimodality Conjecture")** . *Let $P$ be a $d$-dimensional simple or simplicial polytope. Then*

$$h_0(P) \leq h_1(P) \leq \ldots \leq h_i(P) \leq h_{i+1}(P) \leq h_{[d/2]}(P).$$

A combinatorial proof of Theorem 22.1 appeared only recently, see P. McMullen, *On simple polytopes*, Inventiones Math. 113(1993), 419–444 and P. McMullen, Weights on polytopes, *Discrete & Computational Geometry*, **15**(1996), 363–388.

PROBLEMS.

1°. Compute the $h$-vector of a $d$-dimensional simplex. Check that the Unimodality Conjecture is satisfied.

2. Compute the $h$-vector of a $d$-dimensional cube and $d$-dimensional octahedron. Check that the Unimodality Conjecture is satisfied.

3. Check that the Unimodality Conjecture is satisfied for the $h$-vector of a cyclic polytope.

4. Let $P$ be a $d$-dimensional $l$-neighborly polytope with $n$ vertices. Then

$$h_k(P) = \binom{n-d+k-1}{k} \quad \text{for } k = 0, 1, \ldots, l.$$

5°. Let $P$ be a $d$-dimensional simplicial polytope with $n$ vertices. Using the Unimodality Conjecture prove what is called the Lower Bound Theorem:

$$f_{k-1}(P) \geq \binom{d}{k} + (n-d) \sum_{i=1}^{k} \binom{d-i}{k-i}, \ k = 0, \ldots, d-1.$$

6. Construct a $d$-dimensional simplicial polytope $P$ with $n$ vertices such that $h_1(P) = h_2(P) = \ldots = h_{d-1}(P) = n - d$.

*Hint:* You start out with a $d$-dimensional simplex and then at every step add the next vertex only slightly above a facet of the polytope.

7. For a simplicial $d$-dimensional polytope $P$ prove that $h_2(P) \geq h_1(P)$.

8°. What does the Unimodality Conjecture tell us about the Eulerian numbers? (see Theorem 18.1).

9. For a simplicial polytope $P$ one has

$$f_{k-1}(P) = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1}, \ k = 0, 1, \ldots, d.$$

*Hint:* Use the Dehn - Sommerville Equations.

10*. Prove Theorem 22.1.

**(22.2) Weights.** Let $P$ be a simple $d$-dimensional polytope. Let $\mathcal{F}_r$ denote the set of all $r$-dimensional faces of $P$. A real function $\omega : \mathcal{F}_r \longrightarrow \mathbb{R}$ is called *r-weight* on $P$ iff for any $(r+1)$-dimensional face $G$ of $P$ we have

$$\sum_{F \subset G} \omega(F) \cdot n_F = 0,$$

where $F$ ranges over all $r$-dimensional faces of $G$ and $n_F$ denotes the unit outer normal to $F$ in $G$.

PROBLEMS.

1. Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope and $n_F$ be the unit outer normal to the facet $F$. Then

$$\sum_F (\mathrm{vol}_{d-1} F) \cdot n_F = 0,$$

where the sum is taken over all facet of $P$ and $\mathrm{vol}_{d-1}(F)$ is the $(d-1)$-dimensional volume of $F$. This is the so-called *Minkowski relation.* One can call it Stokes' formula too.

*Hint:* Suppose this sum is a non-zero vector, say $a$. Let us choose a hyperplane $H \subset \mathbb{R}^d$ orthogonal to $a$. Let us project $P$ onto $H$ and see what happens. Almost every point in the projection of $P$ is covered by the projection of a facet exactly twice: once for a facet with $\langle n_F, a\rangle > 0$ and once for a facet with $\langle n_F, a\rangle < 0$. Therefore

$$0 = \sum_F \langle a, n_F\rangle \cdot \mathrm{vol}_{d-1} F = \|a\|^2,$$

which is a contradiction.

2. The function $\omega : \mathcal{F}_r \longrightarrow \mathbb{R}$, where $\omega(F) = \mathrm{vol}_r(F)$ is an $r$-weight on $P$.

3°. How does a 0-weight look like?

4°. The set of all $r$-weights on a polytope is a linear space over $\mathbb{R}$.

Let us denote by $\Omega_r(P)$ the linear space of all $r$-weights on a simple polytope $P$.

**(22.3) Theorem.** *Suppose that $P$ is a $d$-dimensional simple polytope. Then*

$$\dim \Omega_r(P) = h_r(P) \quad \text{for } r = 0, \ldots, d.$$

PROBLEM.

1*. Prove Theorem 22.3.

*Hint:* Consider a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ on the polytope $P$ such that $l(v) \neq l(u)$ for any two vertices $u$ and $v$ on $P$. Let us order the vertices $v_1, \ldots, v_m$ of $P$ so that $l(v_i) > l(v_j)$ for any pair $i > j$. Let us correspond to any face $F \subset P$ the unique vertex $v \in F$ where $l(v) = \max\{l(x) : \ x \in F\}$. We enumerate the vertices $v_1, \ldots, v_m$ and do the following. If $\mathrm{ind}(v) < r$ we do nothing. If $\mathrm{ind}(v) = r$ then there is a unique $r$-dimensional face $F$ in $P$ such that $l(v) = \max\{l(x) : \ x \in P\}$ and we assign $\omega(F)$ arbitrarily. If $\mathrm{ind}(v) > r$ we have a unique choice of $\omega(F)$ for all $F$ containing $v$ such that $l(v) = \max\{l(x) : \ x \in F\}$.

**(22.4) Outer multiplication of weights.** Suppose $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^d$ are polytopes. By $P \times Q$ we denote the set in $\mathbb{R}^{2d}$

$$P \times Q = \big\{(x, y) : \ x \in \mathbb{R}^d, y \in \mathbb{R}^d\big\}.$$

PROBLEMS

1. If $P$ is a polytope with $n$ vertices $v_1, \ldots, v_n$ and $Q$ is a polytope with $m$ vertices $u_1, \ldots, u_m$ then $P \times Q$ is a polytope whose vertices are $\{(v_i, u_j) : \ i = 1, \ldots, n, j = 1, \ldots, m\}$.

2. If $P$ and $Q$ are simple polytopes then $P \times Q$ is a simple polytope.

3. Let $R = P \times Q$. Then the faces $F$ of $R$ are the products $F = G \times H$, where $G$ is a face of $P$ and $H$ is a face of $Q$.

4. Let $P$ and $Q$ be simple polytopes. Suppose that $\omega_r$ is an $r$-weight on $P$ and $\omega_s$ is an $s$-weight on $Q$. Let us define a function on the $(r+s)$-dimensional faces $F$ of $R$:

$$\omega_{r+s}(F) = \omega_r(G)\omega_s(H),$$

where $F = G \times H$. Then $\omega_{r+s} = \omega_r \times \omega_s$ is a weight on $R$.

Problem 4 allows us to define a bilinear form $\Omega_r(P) \times \Omega_s(Q) \longrightarrow \Omega_{r+s}(P \times Q)$

**(22.5) Projections of polytopes.**

PROBLEM

1. Let $P \subset \mathbb{R}^d$ be a polytope, $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a linear isomorphism and $Q = \phi(P)$. Let $F$ be an $r$-dimensional face of $P$ and $G = \phi(F)$ be a face of $Q$. We define $\gamma(F, \phi)$ by $\gamma(F, \phi)\mathrm{vol}_r(F) = \mathrm{vol}_r(G)$. Suppose that $\omega \in \Omega_r(P)$ is a weight on $P$. Let us define $\phi(\omega)$: $\phi(\omega)(G) = \gamma(F, \phi)\omega(F)$. Then $\phi(\omega) \in \Omega_r(Q)$ is a weight on $Q$.

2*. Let $P \subset \mathbb{R}^d$ be a polytope, let $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ be a projection (onto) and let $Q = \phi(P)$. Let $\omega \in \Omega_r(P)$ be a weight on $P$. We define $\phi(\omega)$ as follows: let us choose an $r$-face $G$ of $Q$. We can find a family of $r$-faces $\{F_i\}$ of $P$ such that $\phi(F_i)$ subdivide $G$ and the restriction of $\phi$ onto the affine hull of every $F_i$ is an affine isomorphism. For an $r$-face $G$ of $Q$ let

$$\phi(\omega)(G) = \sum_{F_i} \gamma(F_i, \phi)\omega(F_i),$$

where $\gamma(F_i, \phi)$ are defined as in Problem 1. Then $\phi(\omega) \in \Omega_r(Q)$ is independent on the choice of the family $\{F_i\}$.

So, a linear map $\phi : P \longrightarrow Q$ induces a linear map $\phi_* : \Omega_r(P) \longrightarrow \Omega_r(Q)$.

**(22.6) Inner multiplication of weights.** Let $P$ be a simple $d$-dimensional polytope in $\mathbb{R}^d$. Let us consider its direct product onto itself $P \times P \subset \mathbb{R}^{2d}$ and the map $\phi : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d : \ (x, y) \longmapsto x + y$. Problem 3° from Section 1 implies that $\phi(P \times P) = 2P = \{2x : x \in P\}$. Naturally, there is a canonical isomorphism $\Omega_r(P) = \Omega_r(2P)$. Let $\omega_1 \in \Omega_r(P)$ and $\omega_2 \in \Omega_s(P)$. We define the *product* $\omega_1\omega_2 \in \Omega_{r+s}(P)$ as $\phi_*\big(\omega_r(P) \times \omega_r(P)\big)$.

PROBLEM.

1*. This multiplication $\Omega_r(P) \times \Omega_s(P) \longrightarrow \Omega_{r+s}(P)$ is correctly defined.

**(22.7) Theorem.** *Let $P$ be a simple $d$-dimensional polytope and $\nu \in \Omega_1(P)$ be a 1-weight:*

$$\nu(\text{edge}) = \text{length of the edge}.$$

*Then the linear map*

$$\Omega_r(P) \longrightarrow \Omega_{d-r}(P) : \quad \omega \longmapsto \omega\nu^{d-2r}$$

*is an isomorphism of linear spaces for $r = 0, \ldots, [d/2]$.*

**(22.8) Corollary.** *The linear map*

$$\Omega_{r-1}(P) \longrightarrow \Omega_r(P): \ \omega \longmapsto \omega\nu$$

*has the trivial kernel for* $r = 1, \ldots, [d/2]$.

PROBLEM.
1°. Deduce Corollary 22.8 from Theorem 22.7.
2°. Deduce the Dehn-Sommerville equations from Theorem 22.7.
3°. Deduce Theorem 22.1 (the "Unimodality Conjecture") from Theorem 22.7.

This is how Theorem 22.1 ("Unimodality Conjecture") can be proven.

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Wednesday, February 8

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## 23. Euler-Poincaré Formula

**(23.1) Theorem.** *Let $P$ be a $d$-dimensional polytope ($d > -1$). Then*

$$\sum_{i=-1}^{d} (-1)^i f_i(P) = 0.$$

**Proof.** We use induction on the dimension $d$. If $d = 0$ the formula is clearly true. Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope. Choose a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ so that every hyperplane $H = \{x \in \mathbb{R}^d : l(x) = \alpha\}$ contains at most one vertex of $P$. Let $v_1, \ldots, v_m$ be the vertices of $P$ arranged in the decreasing order with respect to the function $l$ (so that $l(v_i) > l(v_j)$ for any two $i < j$). Let $\alpha_i = l(v_i)$ and let

$$H_{2i-1} = \big\{x \in \mathbb{R}^d : \ l(x) = \alpha_i\big\}.$$

Let us choose $\alpha_i < \beta_i < \alpha_{i+1}$ and let

$$H_{2i} = \big\{x \in \mathbb{R}^d : \ l(x) = \beta_i\big\}.$$

Hence we constructed $2m-1$ hyperplanes $H_1, \ldots, H_{2m-1}$ such that every odd-numbered hyperplane contains precisely one vertex and every even-numbered hyperplane does not contain any vertex. Now let us define

$$Q_i = P \cap H_i \quad i = 2, 3 \ldots, 2m-2.$$

Then $Q_i$ is a family of $2m-3$ polytopes of the dimension $(d-1)$. For a face $F$ of $P$ and a polytope $Q_i$ let us define

$$\delta(F, Q_i) = \begin{cases} 1 & \text{if } Q_i \text{ intersects with the relative interior of } F, \\ 0 & \text{otherwise.} \end{cases}$$

**Claim:** For every non-empty face $F$ of $P$ we have

$$\sum_{i=2}^{2m-2} (-1)^i \delta(F, Q_i) = 1.$$

**Proof.** Let $v_s$ be the vertex of $F$ where the maximum of $l$ is attained and $v_r$ be the vertex of $F$ where the minimum of $l$ is attained. Then every hyperplane $H_i$ with $2s-2 \leq i \leq 2r-2$ intersects the relative interior of $F$ (and only those intersect). But in any interval between two even numbers the number of odd numbers is less by 1 than the number of even numbers. □

Let us denote by $\mathcal{F}_j$ the set of faces of $P$ of the dimension $j$

$$\sum_{F \in \mathcal{F}_j} \sum_{i=2}^{2m-2} (-1)^i \delta(F, Q_i) = f_j(P),$$

and, therefore

$$\sum_{j=1}^{d-1} (-1)^j \sum_{F \in \mathcal{F}_j} \sum_{i=2}^{2m-2} (-1)^i \delta(F, Q_i) = \sum_{j=1}^{d-1} (-1)^j f_j(P).$$

Therefore

$$\sum_{j=1}^{d-1} (-1)^j f_j(P) = \sum_{i=2}^{2m-2} (-1)^i \sum_{j=1}^{d-1} (-1)^j \sum_{F \in \mathcal{F}_j} \delta(F, Q_i). \tag{23.1.1}$$

**Claim:**

$$\sum_{F \in \mathcal{F}_j} \delta(F, Q_i) = \begin{cases} f_0(Q_i) - 1 & \text{if } j = 1 \text{ and } i \text{ is odd,} \\ f_{j-1}(Q_i) & \text{otherwise.} \end{cases}$$

**Proof.** If $i$ is even or $j > 1$ then every $(j-1)$-dimensional face of $Q_i$ is the intersection of a $j$-dimensional face $F$ and $H_i$. Otherwise, if $i$ is odd and $j = 1$ then exactly one vertex of $Q_i$ is a vertex of $P$ and all other vertices of $Q_i$ are the intersections of $H_i$ and edges of $P$. □

This is why

$$\sum_{j=1}^{d-1} (-1)^j \sum_{F \in \mathcal{F}_j} \delta(F, Q_i) = \begin{cases} \sum_{j=1}^{d-1} (-1)^j f_{j-1}(Q_i) + 1 & i \text{ is odd,} \\ \sum_{j=1}^{d-1} (-1)^j f_{j-1}(Q_i) & i \text{ is even.} \end{cases}$$

Now it is time to apply the induction hypothesis. We get

$$\sum_{j=1}^{d-1}(-1)^j f_{j-1}(Q_i) = (-1)^{d-1} - 1.$$

Therefore the right hand side of (23.1.1) is $(-1)^{d-1} - 1 - (m-2)$. Since $f_0(P) = m$ we get

$$\sum_{j=1}^{d-1}(-1)^j f_j(P) = 1 + (-1)^{d-1} - f_0(P) \quad \text{or}$$

$$\sum_{i=0}^{d-1}(-1)^i f_i(P) = 1 + (-1)^{d-1}.$$

□

PROBLEMS.

1°. For simple or simplicial polytopes the Euler formula is equivalent to the Dehn-Sommerville equation $h_0 = h_d$.

2. Let $P \subset \mathbb{R}^d$ be a polytope and $H \subset \mathbb{R}^d$ be a hyperplane which contains some interior point of $P$ and does not contain any vertex of $P$. Let us denote by $f_i^+$ the number of $i$-dimensional faces of $P$ which belong to the halfspace $H^+$ bounded by $H$. Then

$$\sum_{i=0}^{d-1}(-1)^i f_i^+ = 1.$$

3. Let $P$ be a $d$-dimensional polytope and $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function with different values at the vertices of $P$. For a vertex $v \in P$ let us denote by $f_i^v$ the number of $i$-dimensional faces $F$ such that $l(v) = \max\{f(x) :\ x \in F\}$. Then

$$\sum_{i=0}^{d-1}(-1)^i f_i^v = \begin{cases} 1 & \text{if } l(v) \text{ is the minimum of } l \text{ on } P, \\ (-1)^{d-1} & \text{if } l(v) \text{ is the maximum of } l \text{ on } P, \\ 0 & \text{otherwise.} \end{cases}$$

4. Let $P$ be a 3-dimensional polytope. We define the *curvature* $\kappa(v)$ at every vertex $v$ of $P$ as follows: let $F_1, \ldots, F_m$ be the facets containing $v$. Then $F_i$ is a plane polygon and we can measure the plane angle $\alpha_i$ of $F_i$ that contains $v$. Let

$$\kappa(v) = 2\pi - \sum_{i=1}^{m}\alpha_i$$

(in particular, $\kappa(v) > 0$ for every vertex $v$). Then

$$\sum_{v \text{ is a vertex of } P} \kappa(v) = 4\pi$$

(Gauss-Bonnet formula).

5. Prove that the Euler equation is the only linear relation for the $f$-vector of a polytope in dimension $d$.

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Friday, February 10

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## 24. Euler-Poincaré Formula, Continued

**Going slowly over the proof.** The following sequence of results helps to understand what is going on in the proof of Theorem 23.1.

**(24.1) Lemma.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope, $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function such that $l(u) \neq l(v)$ for any two vertices $u$ and $v$ of $P$. Let $H = \{x \in \mathbb{R}^d : l(x) = \alpha\}$ be a hyperplane and $Q = P \cap H$ be a non-empty polytope. Then either $\alpha$ is an extreme value of $l$ on $P$ and $Q$ is a point or $Q$ is a $(d-1)$-dimensional polytope. In the latter case every $k$-dimensional face of $Q$ is the intersection of a $(k+1)$-dimensional face of $P$ and $H$ if $k > 0$ or if $k = 0$, every vertex of $Q$ is either the intersection of an edge of $P$ and $H$ or a vertex of $P$ (in the latter case $H$ contains precisely one vertex of $P$).*

**Proof.** Let $m = \min\{l(x) : x \in P\}$ and $M = \max\{l(x) : \ P\}$. Since $P$ is convex (and, therefore, connected) we have that $\{l(x) : x \in P\} = [m, M]$. If $\alpha = m$ or $\alpha = M$ then $Q$ is a face of $P$ but since $l$ has different values at different vertices of $P$, the face $Q$ must contain precisely one vertex (cf. Theorem 7.2), that is $Q$ must be a point. If $m < \alpha < M$ then the hyperplane $H$ cuts the polytope $P$ into two non-empty pieces $P^+ = \{x \in P : l(x) > \alpha\}$ and $P^- = \{x \in P : \ l(x) < \alpha\}$. Since $P$ is $d$-dimensional, one of these pieces contains a small ball $B_r(x_0)$. Another contains at least a point, say $y$. Then the ball $B_{\lambda r}\big(\lambda x_0 + (1-\lambda)y\big)$ for every $0 \leq \lambda \leq 1$ is in $P$ since $P$ is convex. There exists $0 < \lambda < 1$ such that $l(\lambda x_0 + (1-\lambda)y) = \alpha$. Thus $Q$ contains a central hyperplane section of a $d$-dimensional ball. So $Q$ must be $(d-1)$-dimensional.

First, we observe that the intersection of $H$ and a face of $P$ is a face of $Q$ (we can use the same supporting function). Let $F$ be a $k$-dimensional face of $P$. let us choose a point $p$ in the relative interior of $F$. Then $p \in P$ and there is a face $G$ of $P$ that contains $p$ in its relative interior (cf. Theorem 16.4). If dim $G \geq k+2$ then $p$ is contained in a $(k+2)$-dimensional ball in $G$, and, therefore, the intersection $P \cap H$ must contain a $(k+1)$-dimensional ball centered at $p$, which contradicts to dim $F = k$. Suppose that dim $G \leq k$. Then either the intersection $H \cap G$ is a face of dimension less than $k$ (that can not happen since a point in a relative interior of a face can not belong to a face of a smaller dimension) or $G \subset H$. This could happen only if $G$ is a vertex (since $l$ takes different values at different vertices). Then $F$ is a vertex. Otherwise we must have dim $G = k+1$. □

**(24.2) Lemma.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope and $H \subset \mathbb{R}^d$ is a hyperplane containing exactly one vertex of $P$ and intersecting the interior of $P$. Let $H^+, H^-$ be halfspaces bounded by $H$. Let $P^+ = P \cap H^+$ and $P^- = P \cap H^-$ and $P_0 = P \cap H$. If the Euler relation is true for $P^+, P^-$ and $P_0$ then it is true for $P$.*

**Proof.** Using Lemma 24.1 we get:

$$f_0(P) = f_0(P^+) + f_0(P^-) - 2f_0(P_0) + 1$$

$$f_1(P) = f_1(P^+) + f_1(P^-) - 2f_1(P_0) - f_0(P_0) + 1$$

$$f_k(P) = f_k(P^+) + f_k(P^-) - 2f_k(P_0) - f_{k-1}(P_0), \quad k = 2, \ldots, d-2.$$

$$f_{d-1}(P) = f_{d-1}(P^+) - 1 + f_{d-1}(P^-) - 1 - f_{d-2}(P_0).$$

Now we check straightforwardly:

$$\sum_{k=0}^{d-1}(-1)^k f_k(P) = f_0(P) - f_1(P) + \sum_{k=2}^{d-2}(-1)^k f_k(P) + (-1)^{(d-1)} f_{d-1}(P) =$$

$$f_0(P^+) + f_0(P^-) - 2f_0(P_0) + 1 - f_1(P^+) - f_1(P^-) + 2f_1(P_0) + f_0(P_0) - 1 +$$

$$\sum_{k=2}^{d-2}(-1)^k \big(f_k(P^+) + f_k(P^-) - 2f_k(P_0) - f_{k-1}(P_0)\big) +$$

$$(-1)^{(d-1)}(f_{d-1}(P^+) + f_{d-1}(P^-) - f_{d-2}(P_0) - 2) =$$

$$f_0(P^+) - f_1(P^+) + \sum_{k=2}^{d-2}(-1)^k f_k(P^+) + (-1)^{d-1} f_{d-1}(P^+) +$$

$$f_0(P^-) - f_1(P^-) + \sum_{k=2}^{d-2}(-1)^k f_k(P^-) + (-1)^{d-1} f_{d-1}(P^-) -$$

$$2\Big(f_0(P_0) - f_1(P_0) + \sum_{k=2}^{d-2}(-1)^k f_k(P_0) + (-1)^{d-1}\Big) +$$

$$f_0(P_0) - \sum_{k=2}^{d-2}(-1)^k f_{k-1}(P_0) - (-1)^{d-1} f_{d-2}(P_0) =$$

$$1 + (-1)^{d-1} + 1 + (-1)^{d-1} - 2(1 + (-1)^{d-2} + (-1)^{d-1}) + (1 + (-1)^{d-2}) =$$

$$1 + (-1)^{d-1}.$$

$\square$

**(24.3) Lemma.** *Let $H_{-1}$ and $H_1$ be two parallel hyperplanes in $\mathbb{R}^d$. Suppose that $P_{-1} \subset H_{-1}$ and $P_1 \subset H_1$ are two polytopes and $P = \text{conv}\{P_{-1} \cup P_1\}$ is a $d$-dimensional polytope (which is called $d$-prismoid, by the way). Suppose that $H_0$ is a hyperplane which is parallel to $H_i$ and which contains an interior point of $P$. Let $P_0 = H_0 \cap P$. If the Euler relation is true for $P_{-1}, P_1$ and $P_0$ then it is true for $P$.*

**Proof.** Using Lemma 24.1 we get

$$f_0(P) = f_0(P_{-1}) + f_0(P_1)$$

$$f_k(P) = f_k(P_{-1}) + f_k(P_1) + f_{k-1}(P_0) \quad k = 1, \ldots, d-1.$$

Now we check straightforwardly:

$$\sum_{k=0}^{d-1}(-1)^k f_k(P) = f_0(P) + \sum_{k=1}^{d-1}(-1)^k f_k(P) =$$

$$f_0(P_{-1}) + f_0(P_1) + \sum_{k=1}^{d-1}(-1)^k \Big(f_k(P_{-1}) + f_k(P_1) + f_{k-1}(P_0)\Big) =$$

$$\sum_{k=0}^{d-1}(-1)^k f_k(P_{-1}) + \sum_{k=0}^{d-1}(-1)^k f_k(P_1) + \sum_{k=1}^{d-1}(-1)^k f_{k-1}(P_0) =$$

$$1 + 1 - (1 + (-1)^{d-2}) = 1 + (-1)^{d-1}.$$

□

**Proof of the Euler Formula** Let us choose a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$, which has different values at different vertices, let us order the vertices $v_1, \ldots, v_m$ according to the values of the linear function, so that $l(v_i) = \alpha_i > \alpha_j = l(v_j)$ for any pair $i < j$. Then dissect $P$ by the hyperplanes $H_i = \{x \in \mathbb{R}^d : \ l(x_i) = \alpha_i\}$ into $d$-prizmoids. Use induction on $d$ and Lemma 24.3 to show that the Euler relation is satisfied for every prismoid and then Lemma 24.2 and the induction on the number of prismoids to show that it holds for the polytope $P$. □

PROBLEMS.

1°. What kind of identity we get if we apply the Euler formula to a simplex?

2. A sequence of non-negative integers $p_3, p_4, \ldots,$ is called *3-realizable* iff there exists a simple 3-polytope which has as its facets $p_3$ triangles, $p_4$ 4-gons, $p_5$ 5-gons, etc. A necessary condition for 3-realizability is given by the equation

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7}(k-6)p_k.$$

3*. If we look carefully at Problem 2, we observe that the number of hexagons $p_6$ does not participate in the equation. Prove the following Eberhard Theorem: for every sequence $p_3, p_4, p_5, p_7, p_8, \ldots$ such that the equation of Problem 2 is satisfied there exists a value of $p_6$ such that the sequence $p_3, p_4, p_5, p_6, p_7, p_8, \ldots$ is 3-realizable.

See B. Grünbaum, *Convex Polytopes*, Interscience, London, New York, 1967.

4. Let $P$ be a $d$-dimensional polytope and $F \subset P$ be its $k$-dimensional face. By $g_j(F)$ for $j \geq k$ we denote the number of $j$-dimensional faces of $P$ which contain $F$. Then we have

$$\sum_{j=k}^{d-1} (-1)^j g_j(F) = (-1)^{d-1}.$$

*Hint:* Use Problem 4 from Section 12.1.

5. Suppose that $P$ is a polyhedron. We denote by $f_k^0(P)$ the number of bounded $k$-dimensional faces of $P$, by $f_k^\infty$ the number of unbounded $k$-dimensional faces of $P$ and by $f_k(P) = f_k^0(P) + f_k^\infty(P)$ the total number of $k$-dimensional faces of $P$. If $P$ does not contain straight lines then

$$\sum_{i=0}^{d-1} (-1)^i f_i^0(P) = 1, \quad \sum_{i=1}^{d} (-1)^{i+1} f_i^\infty(P) = 1, \quad \sum_{i=0}^{d} (-1)^i f_i(P) = 0.$$

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Monday, February 13

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## 25. Euler Characteristic

A *polyhedral set* is a set in $\mathbb{R}^d$ that can be represented as a union of finitely many polyhedra. The set of all polyhedral sets in $\mathbb{R}^d$ we denote by $\mathcal{P}_d$.

**(25.1) Theorem.** *There exists a unique function $\mu : \mathcal{P}_d \longrightarrow \mathbb{R}$ which is additive*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \quad \textit{for all } A, B \in \mathcal{P}_d$$

*and satisfies the initial condition*

$$\mu(A) = \begin{cases} 1 & \textit{if } A \textit{ is a non-empty polyhedron;} \\ 0 & \textit{if } A \textit{ is an empty polyhedron.} \end{cases}$$

This unique function is called the *Euler characteristic*.

PROBLEMS.

1°. Find the Euler characteristic of the "hollow" triangle, that is of the union of three intervals $[a, b]$, $[b, c]$ and $[c, a]$ with pairwise non-intersecting interiors.

2. The Euler characteristic of a hollow polygon is equal to 0.

3. Find the Euler characteristic of the hollow 3-dimensional cube, that is, of the union of all the 6 facets of the cube.

4. Find the Euler characteristic of the union of five plane pentagons having non-empty intersection.

5. Prove Theorem 25.1 in dimension 1.

**(25.2) Proof of Theorem 25.2.** First, we observe that if a function $\mu$ exists, it must be unique. Indeed, if $A \subset \mathbb{R}^d$ is a polyhedral set, then $A$ can be represented as a union of finitely many polyhedra

$$A = \bigcup_{i=1,\ldots,m} P_i.$$

Therefore, by the "inclusion-exclusion" principle we must have

(25.2.1)

$$\mu(A) = \sum_{i=1}^{m} \mu(P_i) - \sum_{i_1<i_2} \mu(P_i \cap P_{i_2}) + \ldots + (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \mu(P_{i_1} \cap \ldots \cap P_{i_k}) + \ldots + (-1)^{m-1} \mu(P_1 \cap \ldots \cap P_m).$$

Since the intersection of polyhedra is a polyhedron with $\mu = 0$ when the intersection is empty and $\mu = 1$ when the intersection is non-empty, the value of $\mu(A)$ is uniquely determined (if it exists).

Let us prove now that $\mu$ exists. First, we prove that $\mu$ exists on a subset $\overline{\mathcal{P}}_d$ of bounded polyhedral sets in $\mathbb{R}^d$. We prove it by induction on the dimension $d$. If $d = 0$ then every polyhedral set is either a point or empty. Thus we may define $\mu_0(\text{point}) = 1$ and $\mu_0(\emptyset) = 0$. Suppose that we have constructed $\mu_{d-1}$ in $\mathbb{R}^{d-1}$. Let us construct $\mu_d$ in $\mathbb{R}^d$. Let us choose a non-zero linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$. Let $A \subset \mathbb{R}^d$ be a bounded polyhedral set. For every $\alpha \in \mathbb{R}$ the intersection $A^\alpha = A \cap H^\alpha$, where $H^\alpha = \{x : \ l(x) = \alpha\}$ is a bounded polyhedral set in $\mathbb{R}^{d-1}$ and by the induction hypothesis we can define $\mu_{d-1}(A^\alpha)$.

Let us call a value $\alpha \in \mathbb{R}$ of *regular* iff there exists a $\delta > 0$ such that $\mu(A^\alpha) = \mu(A^{\alpha-\epsilon})$ for any $0 \leq \epsilon \leq \delta$. Otherwise, we say that $\alpha$ is *critical* value. We claim that there exists finitely many critical values of $\alpha \in \mathbb{R}$ (once we fixed a polyhedral set $A$ and a function $l$). Indeed, let us represent $A$ as a union of polytopes

$$A = \bigcup_{i=1,\ldots,m} P_i.$$

Furthermore, for any subset $I \subset \{1, \ldots, m\}$ let us define a polytope

$$P_I = \bigcap_{i \in I} P_i.$$

Finally, let $\beta_I = \min\{l(x) : \ x \in P_I\}$. We claim that the set of critical points of $l$ on $A$ is contained in the finite set $\mathcal{B} = \big\{\beta_I : \ I \subset \{1, \ldots, m\}\big\}$. Indeed, if $\alpha \notin \mathcal{B}$ then the hyperplane $H^\alpha$ intersects every polytope $P_I$ if and only if for all sufficiently small $\epsilon > 0$ the hyperplane $H^{\alpha-\epsilon}$ intersects $P_I$. Therefore the intersection $P_I^{\alpha-\epsilon} = H^{\alpha-\epsilon} \cap P_I$ is non-empty for all sufficiently small $\epsilon > 0$ if and only if it is not empty for $\epsilon = 0$. Since $\mu_{d-1}(A^{\alpha-\epsilon})$ is uniquely determined by $\mu_{d-1}(P_I^{\alpha-\epsilon})$ because of the inclusion-exclusion principle (see (25.2.1)) we get that $\mu(A^{\alpha-\epsilon}) = \mu(A^\alpha)$ for all sufficiently small $\epsilon > 0$ whenever $\alpha \notin \mathcal{B}$.

Now we are ready to define $\mu_d(A)$. Let $\mathcal{C} \subset \mathbb{R}$ be the set of critical values. We let

$$\mu_d(A) = \sum_{\alpha \in \mathcal{C}} \Big( \mu_{d-1}(A^\alpha) - \mu_{d-1}(A^{\alpha-\epsilon}) \Big).$$

Here $\epsilon$ is a sufficiently small positive real number. Note that we get the same expression if we consider summation over a larger set of values since for a regular $\alpha$ we have

$$\mu_{d-1}(A^\alpha) - \mu_{d-1}(A^{\alpha-\epsilon}) = 0$$

for all sufficiently small $\epsilon$. Thus we may write formally

$$\mu_d(A) = \sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(A^\alpha) - \mu_{d-1}(A^{\alpha-\epsilon}) \Big), \tag{25.2.2}$$

since only finitely many summands are non-zero. Let us prove that the function $\mu_d$ is additive. Suppose that $A = B \cup C$, where $B$ and $C$ are bounded polyhedral sets and let $D = B \cap C$. Let $A^\alpha, B^\alpha, C^\alpha$ and $D^\alpha$ be the intersections of $A, B, C$ and $D$ with the hyperplane $H_\alpha$. Then $A^\alpha = B^\alpha \cup C^\alpha$ and $D^\alpha = B^\alpha \cap C^\alpha$ for all $\alpha \in \mathbb{R}$. Thus we have

$$\mu_{d-1}(A^\alpha) = \mu_{d-1}(B^\alpha) + \mu_{d-1}(C^\alpha) - \mu_{d-1}(D^\alpha)$$

for every $\alpha \in \mathbb{R}$. Applying (25.2.2) we get

$$\mu_d(A) = \sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(A^\alpha) - \mu_{d-1}(A^{\alpha-\epsilon}) \Big) =$$

$$\sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(B^\alpha) + \mu_{d-1}(C^\alpha) - \mu_{d-1}(D^\alpha) - \mu_{d-1}(B^{\alpha-\epsilon}) - \mu_{d-1}(C^{\alpha-\epsilon}) + \mu_{d-1}(D^{\alpha-\epsilon}) \Big) =$$

$$\sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(B^\alpha) - \mu_{d-1}(B^{\alpha-\epsilon}) \Big) + \sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(C^\alpha) - \mu_{d-1}(C^{\alpha-\epsilon}) \Big) -$$

$$\sum_{\alpha \in \mathbb{R}} \Big( \mu_{d-1}(D^\alpha) - \mu_{d-1}(D^{\alpha-\epsilon}) \Big) = \mu_d(B) + \mu_d(C) - \mu_d(D).$$

It is clear that $\mu_d(\emptyset) = 0$. Let us check that $\mu_d(P) = 1$ if $P$ is a non-empty polytope. Let $\beta = \min\{l(x) : x \in P\}$. Then $\beta$ is the only critical point of $l$ on $P$. Furthermore, $P^{\beta-\epsilon}$ is empty so $\mu_{d-1}(P^{\beta-\epsilon}) = 0$ and $P^\beta$ is a face so $\mu_{d-1}(P^\beta) = 1$. Therefore by (25.2.2) $\mu_d(P) = 1$.

Now it is time to extend $\mu$ from bounded polyhedral sets to general polyhedral sets in $\mathbb{R}^d$. Let $\Delta$ be a $d$-dimensional polytope containing the origin as its interior point. For $\lambda > 0$ let $\lambda\Delta = \{\lambda x : \ x \in \Delta\}$. If $A \subset \mathbb{R}^d$ is a polyhedral set then for any $\lambda > 0$ the set $A \cap \lambda\Delta$ is a bounded polyhedral set and hence we can define $\mu(A \cap \lambda\Delta)$. We claim that for some sufficiently large $\Lambda = \Lambda(A)$ we have $\mu(A \cap \lambda\Delta) = \mu(A \cap \Lambda\Delta)$ for all $\lambda \geq \Lambda$. Indeed, let

$$A = \bigcup_{i=1,\ldots,m} P_i$$

be a representation of $A$ as a union of finitely many polyhedra $P_i$. For a subset $I \subset \{1, \ldots, m\}$ let us define $P_I = \bigcup_{i \in I} P_i$. Then we can define

$$\Lambda = \min\Big\{\lambda > 0 : P_I \cap \lambda\Delta \neq \emptyset \text{ whenever } P_I \neq \emptyset\Big\}.$$

Then for any $\lambda > \Lambda$ $\mu(\lambda\Delta \cap A) = \mu(\Lambda\Delta \cap A)$ because of the inclusion-exclusion formula (see (25.2.1)).

Let us define $\mu(A) = \mu(\Lambda\Delta \cap A)$ for $\Lambda = \Lambda(A)$. The desired additivity and initial conditions are checked immediately. □

This construction belongs to H. Hadwiger.

**(25.3) Corollary.** *Let $A \subset \mathbb{R}^d$ be a bounded polyhedral set, $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function and $A^\alpha = \{x \in A : \ l(x) = \alpha\}$. Then*

$$\mu(A) = \sum_{\alpha \in \mathbb{R}} \Big(\mu(A^\alpha) - \mu(A^{\alpha-\epsilon})\Big)$$

*for sufficiently small $\epsilon > 0$.*

□

PROBLEM.

1. A *hollow* polytope is the union of all proper faces of a polytope. Prove that the Euler characteristic of a hollow $d$-dimensional polytope is equal to $1 + (-1)^{d-1}$.

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Wednesday, February 15

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## 26. Polytopal Subdivisions

**(26.1) Definition.** Let $A \subset \mathbb{R}^d$ be a bounded polyhedral set. A *polytopal subdivision* of $P$ is a representation of $A$ as a union of convex polytopes

$$A = \bigcup_{i=1}^{m} P_i$$

such that the intersection $P_i \cap P_j$ of any two polytopes is a common face of both polytopes $P_i$ and $P_j$. A polyhedral set may have several polytopal subdivisions. To distinguish between a polyhedral set $A$ and a polyhedral set with a polytopal subdivision we write $(A, S)$ meaning some particular subdivision.

A subset $F \subset A$ is called a face of the polytopal subdivision iff it is a face of some polytope $P_i$. The number of $k$-dimensional faces of a subdivision $(A, S)$ we denote by $f_k(A, S)$

PROBLEMS.

1°. In a polytopal subdivision the intersection of any family of polytopes is a common face of all the polytopes from the family.

2. Any bounded polyhedral set admits a polytopal subdivision.

**(26.2) Theorem.** *Let $A \subset \mathbb{R}^d$ be a bounded polyhedral set and*

$$A = \bigcup_{i=1}^{m} P_i$$

*be a polytopal subdivision $S$ of $A$. Then*

$$\mu(A) = \sum_{k=0}^{d} (-1)^k f_k(A, S),$$

*where $\mu$ is the Euler characteristic.*

**Proof.** We prove by induction on the number $m$ of polytopes in the subdivision of $A$. If $m = 0$ then $A = \emptyset$, $\mu(A) = 0$ and the theorem is obviously correct. Suppose that the theorem is proven for all subdivisions containing at most $m-1$ polytopes. Let us consider a subdivision of $A$ with $m$ polytopes and let

$$B = \bigcup_{i=1}^{m-1} P_i, \quad C = B \cap P_m = \bigcup_{i=1}^{m-1} (P_i \cap P_m)$$

be two subdivisions $(B, S_1)$ and $(C, S_2)$ correspondingly. Then $A = B \cup P_m$ and

$$\mu(A) = \mu(B) + 1 - \mu(C). \tag{26.2.1}$$

Since $B$ and $C$ both can be represented as a union of $m-1$ polytopes we have

$$\mu(B)=\sum_{k=0}^{d}(-1)^k f_k(B,S_1), \quad \mu(C)=\sum_{k=0}^{d}(-1)^k f_k(C,S_2). \tag{26.2.2}$$

Let us consider non-empty faces of $P_m$. They can be divided into two classes: the first class $\mathcal{F}_1$ consists of the faces which belong to $B$ and the second $\mathcal{F}_2$ consists of the faces which do not belong to $B$. Therefore

$$\sum_{k=0}^{d}(-1)^k f_k(A,S)=\sum_{k=0}^{d}(-1)^k f_k(B,S_1)+\sum_{F\in\mathcal{F}_2}(-1)^{\dim F}. \tag{26.2.3}$$

Applying the Euler-Poincaré formula to $P_m$ we get that

$$\sum_{F\in\mathcal{F}_2}(-1)^{\dim F}+\sum_{F\in\mathcal{F}_1}(-1)^{\dim F}=1.$$

Therefore from (26.2.3) we get that

$$\sum_{k=0}^{d}(-1)^k f_k(A,S)=1+\sum_{k=0}^{d}(-1)^k f_k(B,S_1)-\sum_{F\in\mathcal{F}_1}(-1)^{\dim F}. \tag{26.2.4}$$

By the definition of subdivision the first class consists of faces of the subdivision $(C,S_2)$ and only of them. Therefore, by (26.2.2)

$$\sum_{F\in\mathcal{F}_1}(-1)^{\dim F}=\sum_{k=0}^{d}(-1)^k f_k(C,S_2)=\mu(C,S_2). \tag{26.2.5}$$

Combining (26.2.2), (26.2.4) and (26.2.5) we get

$$\sum_{k=0}^{d}(-1)^k f_k(A,S)=1+\mu(B)-\mu(C).$$

Therefore by (26.2.1)

$$\mu(A)=\sum_{k=0}^{d}(-1)^k f_k(A,S).$$

□

PROBLEM.

1. Let $A\subset \mathbb{R}^p$ be a polyhedral set and $B\subset\mathbb{R}^q$ be another polyhedral set. Let $C=A\times B=\{(x,y): x\in A, y\in B\}\subset\mathbb{R}^{p+q}$ be its product. Then $C$ is a polyhedral set and $\mu(C)=\mu(A)\mu(B)$.

**(26.3) The third proof of the Euler-Poincaré formula.** It looks like we used the Euler-Poincaré formula to prove Theorem 26.2. However, the following set of problems allows us to get around it.

PROBLEMS.

1. Let us define the *dimension* of a polytopal subdivision $A = \bigcup_{i=1}^m P_i$ as the largest dimension of the polytopes $P_i$. Prove Theorem 26.2 for 0-dimensional subdivisions. Prove the Euler-Poincaré formula in dimension 0.

2. Suppose that the Euler-Poincaré formula is proven up to dimension $d$. Prove Theorem 26.2 for subdivisions up to dimension $d$.

3. Using Problem 1 of Section 25.3 and and Theorem 26.2 in dimension $d$ prove the Euler-Poincaré formula in dimension $d+1$.

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Friday, February 17

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## 27. Generalized Euler Characteristic

**(27.1) Characteristic functions.** Let $S \subset \mathbb{R}^d$ be a set. The function

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

is called the *characteristic function* of $S$. We consider the real vector space $\mathcal{P}(\mathbb{R}^d)$ generated by the characteristic functions of polyhedra. It is called *Polyhedral Algebra*. Thus a typical function $f \in \mathcal{P}(\mathbb{R}^d)$ is a linear combination

$$f = \sum_{i=1}^m \gamma_i \chi_{P_i},$$

where $\gamma_i$ are real numbers and $P_i$ are (non-empty) polyhedra. A function $f \in \mathcal{P}(\mathbb{R}^d)$ we call a *polyhedral* function.

PROBLEMS.

1°. Do the functions $\chi_P$ where $P$ is a polyhedron constitute a basis of $\mathcal{P}(\mathbb{R}^d)$ for $d > 0$?

*Hint:* Of course not.

2*. Find a "reasonable" basis of $\mathcal{P}(\mathbb{R}^d)$.

3°. Prove that $\mathcal{P}(\mathbb{R}^d)$ is closed under multiplication of functions.

**(27.2) Theorem.** *There exists the unique linear function $\mu : \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ that satisfies the condition*

$$\mu(\chi_P) = 1,$$

*for every non-empty polyhedron $P$.*

**Proof.** Our proof closely follows the proof of Theorem 25.1. It is easy to see that if the function $\mu$ exists it must be unique. Indeed, if

$$f = \sum_{i=1}^{d} \gamma_i \chi_{P_i}$$

is a function from $\mathcal{P}(\mathbb{R}^d)$, where $P_i$ are non-empty polyhedra, then we must have

$$\mu(f) = \sum_{i=1}^{d} \gamma_i \mu(\chi_{P_i}) = \sum_{i=1}^{d} \gamma_i.$$

Let us prove that $\mu$ indeed exists. First, we prove that $\mu$ exists on the linear subspace $\overline{\mathcal{P}}(\mathbb{R}^d)$ generated by the characteristic functions of polytopes (it is called the *Polytope Algebra*). We prove it by induction on $d$. For $d = 0$ the theorem is more than obvious. Suppose that the result is true in dimension $d-1$ and let us prove it in dimension $d$. Let us choose a non-zero linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$. For $\alpha \in \mathbb{R}$ let $H^\alpha = \{x \in \mathbb{R}^d : l(x) = \alpha\}$ be a hyperplane. Let us choose a function $f \in \overline{\mathcal{P}}(\mathbb{R}^d)$. Then for any $\alpha \in \mathbb{R}$ the restriction $f^\alpha$ of $f$ onto $H^\alpha$ is a function from $\overline{\mathcal{P}}(\mathbb{R}^{d-1})$ and there exists a value $\mu_{d-1}(f^\alpha)$. Let us call a value $\alpha \in \mathbb{R}$ regular iff $\mu_{d-1}(f^\alpha) = \mu_{d-1}(f^{\alpha-\epsilon})$ for all sufficiently small $\epsilon > 0$. Otherwise we call $\alpha$ a *critical* value. We claim that there exists finitely many critical values (once we fixed a polyhedral function $f$ and a linear function $l$). Indeed, let

$$f = \sum_{i=1}^{m} \gamma_i \chi_{P_i},$$

where $P_i \subset \mathbb{R}^d$ are non-empty polytopes. Let $\beta_i = \min\{l(x) : x \in P_i\}$. We claim that the set of critical values of $f$ is contained in the set $\mathcal{B} = \{\beta_i : i = 1, \ldots, m\}$. Let $P_i^\alpha = P_i \cap H^\alpha$. Then

$$f^\alpha = \sum_{i=1}^{m} \gamma_i \chi_{P_i^\alpha}.$$

Applying the induction conjecture we get that

$$\mu_{d-1}(f^\alpha) = \sum_{i : P_i^\alpha \neq \emptyset} \gamma_i.$$

If $\alpha \notin \mathcal{B}$ then $P_i^\alpha$ is non-empty if and only if $P_i^{\alpha-\epsilon}$ is non-empty for all sufficiently small $\epsilon > 0$. Now we are ready to define $\mu_d(f)$. Let

$$\mu_d(f) = \sum_{\alpha \in R} \Big( \mu_{d-1}(f^\alpha) - \mu_{d-1}(f^{\alpha-\epsilon}) \Big)$$

for a sufficiently small $\epsilon > 0$. Since the set of critical values is finite the above sum contains only finitely many non-zero summands. It is immediate from the induction conjecture that $\mu_d$ is a linear function on $\overline{\mathcal{P}}(\mathbb{R}^{d-1})$. Let us check that $\mu_d(f) = 1$, where $f = \chi_P$ is the characteristic function of a non-empty polytope. It is easy to see that there is only one critical value $\alpha = \min\{l(x) : x \in P\}$. Thus we have $f^\alpha = \chi_{P^\alpha}$, where $P^\alpha$ is a face of $P$ and hence $\mu_{d-1}(f^\alpha) = 1$. For any $\epsilon > 0$ we have $f^{\alpha-\epsilon} = 0$ and, therefore, $\mu_{d-1}(f^{\alpha-\epsilon}) = 0$. Thus $\mu_d(f) = 1$.

Now we extend $\mu$ to a linear function on $\mathcal{P}(\mathbb{R}^d)$. As in the proof of Theorem 25.1 we choose a simplex $\Delta$ containing the origin in its interior and define

$$\mu_d(f) = \lim_{\lambda \longrightarrow +\infty} \mu\Big(f \cdot \chi_{\lambda\Delta}\Big).$$

It is straightforward that $\mu$ satisfies the desired properties. □

The linear function $\mu$ that we constructed we shall call the *Generalized Euler Characteristic*. The reason is explained below.

PROBLEMS.

1°. Let $A \subset \mathbb{R}^d$ be a polyhedral set. Prove that $\chi_A \in \mathcal{P}(\mathbb{R}^d)$ and that $\mu(\chi_A) = \mu(A)$ is the Euler characteristic of $A$.

2°. Construct an example of a set $A \subset \mathbb{R}^d$ such that $A$ is not a polyhedral set but $\chi_A \in \mathcal{P}(\mathbb{R}^d)$.

So we can define the Euler characteristic even for non-polyhedral sets.

3. This problem deals with the polytope algebra $\overline{\mathcal{P}}(\mathbb{R}^d)$ generated by the characteristic functions of polytopes in $\mathbb{R}^d$. Show that if $P \subset \mathbb{R}^d$ is a polytope and $\hat{P}$ is the relative interior of $P$ then

$$\chi_{\hat{P}} = (-1)^{\dim P} \sum_F (-1)^{\dim F} \chi_F,$$

where the sum is taken over all faces $F$ of $P$. In particular, $\chi_{\hat{P}} \in \overline{\mathcal{P}}(\mathbb{R}^d)$.

Show then that the map

$$\hat{}:\ \chi_P \longmapsto \sum_F (-1)^{\dim F} \chi_F$$

can be extended to an involutory isomorphism of $\overline{\mathcal{P}}(\mathbb{R}^d)$.

4. This is a general problem involving the Euler characteristic. Let $P, P_1, \ldots, P_m \subset \mathbb{R}^d$ be polyhedra such that

$$P = \bigcup_{i=1}^m P_i.$$

Show that if every $k$ of them have a non-empty intersection then some $k+1$ of them have a non-empty intersection.

## 28. Identities

**(28.1) Theorem.** *Let $P_1, \ldots, P_m \subset \mathbb{R}^d$ be polyhedra and $\alpha_1, \ldots, \alpha_m$ be real numbers such that*

$$\alpha_1 \chi_{P_1} + \ldots + \alpha_m \chi_{P_m} = 0.$$

*Suppose that $\mathcal{L} : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ is a linear map and $Q_i = \mathcal{L}(P_i)$ are (necessarily) polyhedra in $\mathbb{R}^k$. Then*

$$\alpha_1 \chi_{Q_1} + \ldots + \alpha_m \chi_{Q_m} = 0.$$

**Proof.** Let us consider the function $K(x, y) : \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}$:

$$K(x,y) = \begin{cases} 1 & \text{if } \mathcal{L}(x) = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $y \in \mathbb{R}^k$ the product $K(x,y)\chi_{P_i}(x) \in \mathcal{P}(\mathbb{R}^d)$ is the characteristic function of the intersection $P_i \cap \mathcal{L}^{-1}(y)$ and we may apply the function $\mu : \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ which we denote $\mu_x$ to distinguish the variables. Hence

$$\alpha_1 \mu_x \Big( K(x,y)\chi_{Q_1}(x) \Big) + \ldots + \alpha_m \mu_x \Big( K(x,y)\chi_{Q_m}(x) \Big) = 0.$$

Then we observe that

$$\mu_x \Big( K(x,y)\chi_{P_i}(x) \Big) = \chi_{Q_i}(y).$$

□

PROBLEMS.

1. Let $P, Q \subset \mathbb{R}^d$ be polyhedra. Their Minkowski sum $P+Q = \{x+y : \ x \in P, y \in Q\}$ (see Problem 3° from Section 1.2) is a polyhedron. Prove that if we have two identities

$$\alpha_1 \chi_{P_1} + \ldots + \alpha_m \chi_{P_m} = 0, \quad \beta_1 \chi_{Q_1} + \ldots + \beta_k \chi_{Q_k} = 0,$$

where $P_i, Q_i \subset \mathbb{R}^d$ are polyhedra then we have the identity

$$\sum_{i,j} (\alpha_i + \beta_j) \chi_{P_i + Q_j} = 0.$$

*Hint:* Consider the product $P \times Q = \{(x,y) : \ x \in P, y \in Q\}$ in $\mathbb{R}^{2d}$ and the projection $\mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d : \ (x,y) \longmapsto x + y$.

2. There exists a map $\oplus : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathcal{P}(\mathbb{R}^d)$ such that $\chi_P \oplus \chi_Q = \chi_{P+Q}$ for any two polyhedra $P$ and $Q$ and such that $(\alpha f + \beta g) \oplus h = \alpha(f \oplus h) + \beta(g \oplus h)$ for all $f, g, h \in \mathcal{P}(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{R}$. Note that generally $\chi_A \oplus \chi_B \neq \chi_{A+B}$ if $A, B$ are not convex.

3*. This problems deals with the polytope algebra $\mathcal{P}(\mathbb{R}^d)$ generated by the characteristic functions of polytopes. Let $P \subset \mathbb{R}^d$ be a polytope, $-P = \{-x : x \in P\}$ be its centrally-symmetric image and $\widehat{-P}$ be the relative interior of $-P$. Then

$$\chi_P \oplus (-\chi_{\widehat{-P}}) = \chi_0.$$

For polytope algebra and theory of valuations, see P. McMullen and R. Schneider, Valuations on convex bodies, in: *Convexity and its Applications*, Boston, Birkhäuser, 1983, 170–247, P. McMullen, Valuations and dissections, in: *Handbook of Convex Geometry*, Amsterdam, North-Holland, 1993, 938–988 and A.G. Khovanskii and A.V. Pukhlikov, Finitely additive measures of virtual polyhedra, *St. Petersburg Math. Journal*, **4**(1993), 337–356.

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Monday, February 27

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## 29. Cones

**(29.1) Definition.** A polyhedron $P \subset \mathbb{R}^d$ is called a *cone* iff for any $x \in P$ and any $\lambda \geq 0$ we have $\lambda x \in P$.

PROBLEMS.
1°. If $K \subset \mathbb{R}^d$ is a non-empty cone then $0 \in K$.
2°. Intersection of finitely many cones is a cone.
3. Minkowski sum (see Problem 3° from Section 1.1) of two cones is a cone.

**(29.2) Lemma.** *Let $K \subset \mathbb{R}^d$ be a cone. Then*

$$K^\circ = \big\{x \in \mathbb{R}^d : \ \langle x, y \rangle \leq 0 \text{ for all } y \in K\big\}.$$

**Proof.** Obviously, if $\langle x, y \rangle \leq 0$ for all $y \in K$ then $\langle x, y \rangle \leq 1$ for all $y \in K$ and hence $x \in K^\circ$. On the other hand, if for some $x \in K^\circ$ and for some $y \in K$ we have $\langle x, y \rangle > 0$ then for a sufficiently big $\lambda > 0$ we shall have $\langle x, \lambda y \rangle > 1$ and $\lambda y \in K$ which contradicts to the assumption that $x \in K^\circ$. □

Let $a_1, \ldots, a_m \in \mathbb{R}^d$ be a finite set of points. The set

$$\operatorname{co}\{a_1, \ldots, a_m\} = \Big\{\sum_{i=0}^{m} \lambda_i a_i : \ \lambda_i \geq 0, \ i = 1, \ldots, m\Big\}$$

is called the *conic hull* of $a_1, \ldots, a_m$.

PROBLEM.
1. If a cone $K \subset \mathbb{R}^d$ contains some set $a_1, \ldots, a_m$ then $K$ contains $\operatorname{co}\{a_1, \ldots, a_m\}$.

**(29.3) Theorem.** *Let $K \subset \mathbb{R}^d$ be a set. Then*

(29.3.1) *The set $K$ is a cone iff it can be represented in the form $K = \{x \in \mathbb{R}^d : l_i(x) \leq 0 : i = 1, \ldots, m\}$ for some linear functions $l_1, \ldots, l_m : \mathbb{R}^d \longrightarrow \mathbb{R}$;*

(29.3.2) *The set $K$ is a cone iff it is the conic hull of finitely many points: $K = \operatorname{co}\{a_1, \ldots, a_m\}$.*

**Proof.** Let us prove (29.3.1). It is clear that the set $\{x \in \mathbb{R}^d : \ l_i(x) \leq 0 : i = 1, \ldots, m\}$ is a cone. Let us consider a cone $K = \{x \in \mathbb{R}^d : \ l_i(x) \leq \alpha_i : i = 1, \ldots, m\}$. Since $0 \in K$ we must have $\alpha_i \geq 0$ for $i = 1, \ldots, m$. Therefore, any $x \in \mathbb{R}^d$ which satisfies the inequalities $l_i(x) \leq 0$ must belong to $K$. Suppose that $x \in K$. If $l_i(x) > 0$ for some $i$ then for a sufficiently large $\lambda > 0$ we would have $l_i(\lambda x) > \alpha_i$ which contradicts to the condition $\lambda x \in K$. So $K = \{x \in \mathbb{R}^d : \ l_i(x) \leq 0 : i = 1, \ldots, m\}$ and (29.3.1) is proven.

Let us prove (29.3.2). Suppose that $K \subset \mathbb{R}^d$ is a cone. Let us choose a polytope $\Delta \subset \mathbb{R}^d$ which contains the origin in its interior and let $Q = K \cap \Delta$. Then $Q$ is a polytope and as such it is a convex hull of finitely many points, say, $a_1, \ldots, a_m$. We claim that $K = \operatorname{co}\{a_1, \ldots, a_m\}$. Indeed, it is immediate that $\operatorname{co}\{a_1, \ldots, a_m\} \subset K$. Furthermore, for any $x \in K$ there exists a sufficiently small $\alpha > 0$ such that $\alpha x \in Q$. Then

$$\alpha x = \sum_{i=1}^{m} \lambda_i a_i \quad \text{for some } \lambda_i \geq 0 \quad \text{such that } \sum_{i=1}^{m} \lambda_i = 1$$

and hence

$$x = \sum_{i=1}^{m} (\lambda_i / \alpha) a_i \in \operatorname{co}\{a_1, \ldots, a_m\}.$$

Now we have to prove that the conic hull $K = \operatorname{co}\{a_1, \ldots, a_m\}$ of finitely many points is a cone. Clearly, we only have to prove that $K$ is a polyhedron. A possible way to do it is to consider the polyhedron

$$\mathbb{R}^n_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \ \lambda_i \geq 0\}$$

and its projection

$$\mathbb{R}^n_+ \longrightarrow K : \lambda \longmapsto \lambda_1 a_1 + \ldots + \lambda_m a_m$$

and use Problem 3 from Section 6.3 Another possibility is to apply the same trick as in the proof of Theorem 6.3. Let us consider $K^\circ$. Using the same argument as in Lemma 29.2 we prove that

$$K^\circ = \{x \in \mathbb{R}^d : \ \langle x, a_i \rangle \leq 0 : \ i = 1, \ldots, m\}.$$

Then by (29.3.1) $K^\circ$ is a cone and as we have already proven we have $K^\circ = \operatorname{co}\{b_1, \ldots, b_n\}$ for some $b_1, \ldots, b_n \in \mathbb{R}^d$. Then

$$K = (K^\circ)^\circ = \{x \in \mathbb{R}^d : \ \langle x, b_i \rangle \leq 0\}$$

(see Theorem 4.2) and hence $K$ is a cone. □

PROBLEMS.

1. If $K_1, K_2 \subset \mathbb{R}^d$ are cones then $(K_1 \cap K_2)^\circ = K_1^\circ + K_2^\circ$ and $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$ where "+" denotes Minkowski summation.

2. Let us consider $\mathbb{R}^d \subset \mathbb{R}^{d+1}$ as the hyperplane $x_{d+1} = 1$. We say that a cone $K \subset \mathbb{R}^{d+1}$ is positively oriented iff $\langle u, x \rangle < 0$ for any $x \in K$ where $u = (0, \ldots, 0, -1) \in \mathbb{R}^d$. If $P \subset \mathbb{R}^d$ is a polytope then $K = \mathrm{co}\{P, 0\}$ is a positively oriented cone. Furthermore, if $K \subset \mathbb{R}^{d+1}$ is a positively oriented cone then $P = K \cap \mathbb{R}^d$ is a polytope. Finally, If $K_1, K_2 \subset \mathbb{R}^{d+1}$ are positively oriented then both $K_1 \cap K_2$ and $K_1 + K_2$ are positively oriented. If $P_i = K_i \cap \mathbb{R}^d$ then $(K_1 \cap K_2) \cap \mathbb{R}^d = P_1 \cap P_2$ and $(K_1 + K_2) \cap \mathbb{R}^d = \mathrm{conv}\{P_1, P_2\}$.

## 30. Polarity and Cones

The real vector space generated by characteristic functions $\chi_K$ of cones $K \subset \mathbb{R}^d$ we denote by $\mathcal{K}(\mathbb{R}^d)$ and call the space of *conic functions.*

**(30.1) Theorem.** *There exists a linear map $T : \mathcal{K}(\mathbb{R}^d) \longrightarrow \mathcal{K}(\mathbb{R}^d)$ such that*

$$T(\chi_K) = 1 - \chi_{K^\circ}.$$

**Proof.** Let us consider a function on $\mathbb{R}^d \times \mathbb{R}^d$:

$$J(x,y) = \begin{cases} 1 & \text{if } \langle x, y \rangle = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu : \mathcal{K}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ be the restriction of the generalized Euler characteristic on the subspace $\mathcal{K}(\mathbb{R}^d)$ (see Section 27). We formally let

$$T(f)(y) = \mu_x\Big(J(x,y) f(x)\Big) \quad \text{for any } f \in \mathcal{K}(\mathbb{R}^d).$$

It is enough to prove that $T$ is correctly defined on the generators $\chi_K$ and that $T(\chi_K) = 1 - \chi_{K^\circ}$. Let $K \subset \mathbb{R}^d$ be a cone. Then

$$J(x,y)\chi_K(x) = \begin{cases} 1 & \text{if } \langle x, y \rangle = 1 \text{ and } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for any $y \in \mathbb{R}^d$ $J(x,y)\chi_K(x) = \chi_{K \cap H_y}$, where $H_y = \{x : \langle x, y \rangle = 1\}$ is a hyperplane. Thus $\mu_x\Big(J(x,y)\chi_K(x)\Big)$ is well-defined for any $y$ and

$$\mu_x\Big(J(x,y)\chi_K(x)\Big) = \begin{cases} 1 & \text{if there exists } x \in K \text{ such that } \langle x, y \rangle = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $y \in K^\circ$. Then $\langle y, x \rangle \le 0$ for every $x \in K$ (see Lemma 29.2) and therefore $\mu_x\Big(J(x,y)\chi_K(x)\Big) = 0$. Suppose that $y \notin K^\circ$. Then there exists $x \in K$ such that $\langle x, y \rangle > 0$ (see Lemma 29.2). Therefore for some $\lambda > 0$ we shall have $\langle \lambda x, y \rangle = 1$. Hence $\mu_x\Big(J(x,y)\chi_K(x)\Big) = 1$. Thus we have proven that $T(\chi_K) = 1 - \chi_{K^\circ}$. So the operator $T$ is correctly defined as a linear operator $T : \mathcal{K}(\mathbb{R}^d) \longrightarrow \mathcal{K}(\mathbb{R}^d)$. □

**(30.2) Corollary.** *Let $K_1, \ldots, K_m \subset \mathbb{R}^d$ be cones and $\alpha_1, \ldots, \alpha_m$ be real numbers. If*

$$\alpha_1 \chi_{K_1} + \ldots + \alpha_m \chi_{K_m} = 0$$

*then*

$$\alpha_1 \chi_{K_1^\circ} + \ldots + \alpha_m \chi_{K_m^\circ} = 0.$$

**Proof.** We may assume that all the cones $K_i$ are non-empty. Applying the generalized Euler characteristic $\mu$ we get that $\alpha_1 + \ldots + \alpha_m = 0$. Applying the operator $T$ to the first equality we get $\alpha_1(1 - \chi_{K_1^\circ}) + \ldots + \alpha_m(1 - \chi_{K_m^\circ}) = 0$, that is $(\alpha_1 + \ldots + \alpha_m) - \alpha_1 \chi_{K_1^\circ} - \ldots - \alpha_m \chi_{K_m^\circ} = 0$, that is $\alpha_1 \chi_{K_1^\circ} + \ldots + \alpha_m \chi_{K_m^\circ} = 0$. □

PROBLEMS.

1. A function $\nu$ on the cones in $\mathbb{R}^d$ is called a *valuation* iff $\nu(K_1 \cup K_2) = \nu(K_1) + \nu(K_2) - \nu(K_1 \cap K_2)$, whenever $K_1 \cup K_2$ is a cone. Prove that if $\nu$ is a valuation then $\nu^\circ(K) = \nu(K^\circ)$ is a valuation.

2. Prove that $T^2(\chi_K) = \chi_K - \chi_0$.

3. Prove that $T^3 = T$. Find the eigenvalues of $T$.

4*. Let $P \subset \mathbb{R}^d$ be a simple $d$-dimensional polytope with the vertices $v_1, \ldots, v_m$. Let $A_1, \ldots, A_n$ be the supporting hyperplanes of the facets of $P$. Let $A_0$ be a "generic" hyperplane which does not intersect $P$ and does not pass through any point of intersection of any $d$ hyperplanes $A_{i_1}, \ldots, A_{i_d}$: $1 \leq i_1 < i_2 \ldots < i_d \leq n$. For every hyperplane $A_i$ let $A_i^+$ be the closed halfspace that contains $P$ and $A_i^-$ be the complementary open halfspace. Now, for every vertex $v_k$ let us define a "semi-open simplex " $\Delta_k$ as the only bounded region of the type $A_{i_1}^{\epsilon_1} \cap \ldots A_{i_d}^{\epsilon_d} \cap A_0^+$, where $A_{i_1}, \ldots A_{i_d}$ are the hyperplanes containing $v_k$. Finally, let us define the sign $\epsilon_k$ as $(-1)$ to the parity of the number of halfspaces with "$-$" sign.

Then

$$\chi_P = \sum_{k=1}^{m} \epsilon_k \chi_{\Delta_k}.$$

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Wednesday, March 1

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## Solutions to Some of the Problems from the First Set

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Friday, March 3

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## 31. Canonical Projective Form of a Polytope

Let $\Delta \subset \mathbb{R}^n$ be the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$ (see Example 2.3, 1) and let $L$ be a subspace in $\mathbb{R}^n$. Let $p_L : \mathbb{R}^n \longrightarrow L$ be the orthogonal projection onto $L$. Then $p(\Delta)$ is a polytope. The following result shows that actually any polytope looks like the orthogonal projection of a simplex.

**(31.1) Theorem.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope with the vertices $v_1, \ldots, v_n$. Let $\Delta \subset \mathbb{R}^n$ be the standard simplex. Then there exists a $d$-dimensional subspace $L \subset \mathbb{R}^n$ and a linear isomorphism $\phi : L \longrightarrow \mathbb{R}^d$ such that $\phi\big(p_L(\Delta)\big) = P$, where $p_L$ is the orthogonal projection of $\mathbb{R}^n$ onto $L$.*

**Proof.** Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, so $\Delta = \operatorname{conv}\{e_1, \ldots, e_n\}$. Let us consider a linear map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^d$, $\psi(x) = x_1 v_1 + \ldots + x_n v_n$. Then $\psi(\Delta) = \operatorname{conv}\{v_1, \ldots, v_n\} = P$. Now let $L$ be the orthogonal complement to $\operatorname{Ker} \psi$ and $\phi$ be the restriction of $\psi$ on $L$. □

PROBLEMS.

1. This is how it looks in coordinates. Let $v_{ij}$ be the $j$-th coordinate of the vector $v_i$. Then $L \subset \mathbb{R}^n$ is generated by the $d$ vectors $u_1 = (v_{11}, v_{21}, \ldots, v_{n1}), \ldots, u_k = (v_{1k}, v_{2k}, \ldots, v_{nk}), \ldots, u_d = (v_{1d}, v_{2d}, \ldots, v_{nd})$.

2. For every centrally symmetric polytope $P \subset \mathbb{R}^d$ with $n$ vertices there exists a subspace $L \subset \mathbb{R}^n$ and a linear isomorphism $\phi : L \longrightarrow \mathbb{R}^d$ such that $\phi(p_L(O)) = P$, where $O \subset \mathbb{R}^n$ is the standard octahedron.

3. Let $P$ be a polytope with $n$ vertices $v_1, \ldots, v_n$. Let $v = (v_1 + \ldots + v_n)/n$. There exist two ellipsoids $E_1$ and $E_2$ centered at $v$ such that $E_1 \subset P \subset E_2$ and $E_1$ and $E_2$ are homothetic with the coefficient $n$ (cf. Problem 3 from Section 14.3)

**(31.2) Definitions.** Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let $e = (e_1 + \ldots + e_n)/n$. Let $L \subset \mathbb{R}^n$ be a $d$-dimensional subspace, orthogonal to $e$. Let $p_L(\Delta) \subset L$ be the orthogonal projection of $\Delta$ onto $L$. We say that the polytope $p_L(\Delta)$ is in the *canonical projective form.*

**(31.3) Corollary.** *Let $P \subset \mathbb{R}^d$ be a polytope with $n$ vertices $v_1, \ldots, v_n$. Then there exists a polytope $Q \subset L \subset \mathbb{R}^n$ in the canonical projective form and an affine isomorphism $\phi : L \longrightarrow \mathbb{R}^d$ such that $\phi(Q) = P$.*

**Proof.** We can move the polytope $P$ so that $v_1 + \ldots + v_n = 0$ and then apply Theorem 31.1. □

PROBLEM.

1. Let $L \subset \mathbb{R}^n$ be a subspace containing $e$. We say that the polytope $Q = \Delta \cap L$ is in the *canonical injective form.* Prove that for any polytope $P \subset \mathbb{R}^d$ with $n$ facets there exists a polytope $Q \subset L \subset \mathbb{R}^n$ in the canonical injective form and an affine isomorphism $\phi : L \longrightarrow \mathbb{R}^d$ such that $\phi(Q) = P$.

## 32. Gale Diagrams

**(32.1) Definition.** Let $L$ be a linear space. We consider a *multiset* $V = \{v_1, \ldots, v_n\}$ of points from $L$ (that is, geometrically coinciding points we treat as different). A submultiset

$U \subset V$ is called a *coface* of $V$ iff for every $v \in U$ there exists $\lambda_v > 0$ such that

$$\sum_{v \in U} \lambda_v v = 0.$$

PROBLEMS.

$1^\circ$. Union of cofaces is a coface.

2. $U \subset V$ is a coface iff 0 belongs to the relative interior of the convex hull of $\{v \in U\}$.

$3^\circ$. Let $V \subset \mathbb{R}^d$ be a finite multiset, $V = \{v_1, \ldots, v_n\}$. Let us construct the multiset $W \subset L$ as follows: $w_i = v_i / \|v_i\|$, if $v_i \neq 0$ and $w_i = v_i$ if $v_i = 0$. Then $\{v_i : \ i \in I\}$ is a coface of $V$ iff $\{w_i : \ i \in I\}$ is a coface of $W$.

Let $L \subset \mathbb{R}^n$ be a linear subspace orthogonal to $e = (e_1 + \ldots + e_n)/n$. Let $v_i = p_L(e_i)$ be the orthogonal projection of the $i$-th basis vector onto $L$. We consider the multiset $V = \{v_1, \ldots, v_n\}$. It is seen that $v_1 + \ldots + v_n = 0$. The *Gale Transform* of the multiset $V$ is the multiset $\widehat{V}$, constructed as follows. Let $\widehat{L}$ be the orthogonal complement to $L \oplus \mathbb{R}e$. Then $\widehat{V} = \{\widehat{v_1}, \ldots, \widehat{v_n}\}$, where $\widehat{v_i} = p_{\widehat{L}}(e_i)$. In particular, $\widehat{v_1} + \ldots + \widehat{v_n} = 0$. It is immediate that $\dim L + \dim \widehat{L} = n - 1$ and that $\widehat{\widehat{V}} = V$.

**(32.2) Lemma.** *Let $L \subset \mathbb{R}^n$ be a subspace, orthogonal to $e$, $v_i = p_L(e_i)$ and $\widehat{v_i}$ be the Gale Transform of $v_i$. Then*

$$\sum_{i=1}^{n} \lambda_i \widehat{v_i} = 0, \quad \sum_{i=1}^{n} \lambda_i = 0 \tag{32.2.1}$$

*iff there exists a linear function $l : L \longrightarrow \mathbb{R}$ such that $\lambda_i = l(v_i)$.*

**Proof.** Let $\dim L = d$. Suppose that for some $\lambda_1, \ldots, \lambda_n$ the equations (32.2.1) are satisfied. Let us consider the vector $\lambda = \lambda_1 e_1 + \ldots + \lambda_d e_d$. Then $p_{\widehat{L}}(\lambda) = 0$ and $\lambda$ is orthogonal to $e$. Therefore, $\lambda$ is orthogonal to $\widehat{L} \oplus \mathbb{R}e$, so $\lambda \in L$. On the other hand, suppose that $\lambda \in L$. Then $\lambda$ is orthogonal to $e$, so the second relation of (32.2.1) is satisfied. Since $\lambda \in L$ we have that $p_{\widehat{L}}(\lambda) = 0$, that is the first relation from (32.2.1) is satisfied. Since $v_i$ is the orthogonal projection of $e_i$ onto $L$, the function $l(x) = \langle \lambda, x \rangle$ is the unique linear function on $L$ that satisfies $l(v_i) = \lambda_i$. $\square$

**(32.3) Theorem.** *Let $V = \{v_1, \ldots, v_n\} \subset L \subset \mathbb{R}^n$ be a multiset, $v_1 + \ldots + v_n = 0$ and $\widehat{V} \subset \widehat{L} \subset \mathbb{R}^n$ be its Gale transform.*

*A set $I \subset \{1, \ldots, n\}$ is the set of indices of the points on a face of $P = \operatorname{conv}\{v_1, \ldots, v_n\}$ iff the complement $J = \{1, \ldots, n\} \setminus I$ is the set of indices of the points on a coface in the Gale Transform $\widehat{V}$.*

**Proof.** Let $F$ be a face of $P$ and $l : L \longrightarrow \mathbb{R}$ be its supporting function. That is, there exists $\alpha$ such that $l(v_i) = \alpha$ for all $i \in I$ and $l(v_j) < \alpha$ for all $j \in J$. Let $\lambda_j = l(v_i)$. Then by Lemma 32.2 we have

$$\sum_{i=1}^{n} l(v_i) \widehat{v_i} = 0.$$

Since $\widehat{v_1} + \ldots + \widehat{v_n} = 0$ we have that

$$\sum_{i=1}^{n} (\alpha - l(v_i))\widehat{v_i} = 0.$$

On the other hand, $\alpha - l(v_i) = 0$ for $i \in I$ and $\alpha - l(v_j) > 0$ for $j \in J$, so $\{v_j : j \in J\}$ is a coface of $\widehat{V}$.

Now, let $\{\widehat{v_j}, j \in J\}$ be a coface of $\widehat{V}$. That is, there exist $\lambda_j > 0$ such that

$$\sum_{j \in J} \lambda_j \widehat{v_j} = 0.$$

Let $\alpha = n^{-1} \sum_{j \in J} \lambda_j$. Since $\widehat{v_1} + \ldots + \widehat{v_n} = 0$ we have

$$\sum_{j \in J} (\lambda_j - \alpha)\widehat{v_j} + \sum_{i \in I} (-\alpha)\widehat{v_i} = 0.$$

By Lemma 32.2 there exists a linear function $l : L \longrightarrow \mathbb{R}$ such that $l(v_j) = \lambda_j - \alpha_j$ for $j \in J$ and $l(v_i) = -\alpha$ for $i \in I$. Since $\lambda_j > 0$ for every $j \in J$ we have that $\{v_i : \ i \in I\}$ are the points on a face of $P$ with the supporting function $l$. $\square$

Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope with $n$ vertices $v_1, \ldots, v_n$. To construct the *Gale diagram* of $P$ we do the following. Using Corollary 31.3 we construct the polytope $Q \subset L \subset \mathbb{R}^d$ in the canonical projective form. Then we construct the Gale transform $\widehat{V} = \widehat{v_1}, \ldots, \widehat{v_n}$ of the vertex set of $Q$. So $\widehat{V} \subset \widehat{L}$, where $\widehat{L} \subset \mathbb{R}^n$ is a $(n-d-1)$-subspace of $\mathbb{R}^n$. Now a *Gale diagram* of $P$ is the image of $\widehat{V}$ under a linear isomorphism $\widehat{L} \longrightarrow \mathbb{R}^{n-d-1}$ and arbitrary multiplication of the resulted points onto positive numbers. Often a Gale diagram is defined as our Gale diagram whose non-zero points are projected onto the unit sphere (see Problem 3° from Section 32.1).

**(32.4) Corollary.** *A multiset $W = \{w_1, \ldots, w_n\} \subset \mathbb{R}^{n-d-1}$ is a Gale diagram of a $d$-dimensional polytope $P$ with $n$ vertices, say, $v_1, \ldots, v_n$ iff $W$ generates $\mathbb{R}^{n-d-1}$ and for every $i$ the set $W \setminus \{w_i\}$ is a coface of $W$.*

*In this case $\{v_i : \ i \in I\}$ are the vertices of a face of $P$ iff $\{w_i : i \notin I\}$ is a coface of $W$.*

**Proof.** First, we observe that by Problem 1° from Section 32.1, $W$ is a coface. Therefore, multiplying $w_i$ onto positive numbers we can bring them in the position $w_1 + \ldots + w_n = 0$. Now, using the construction of Theorem 31.1 let us bring $W$ into the canonical projective form. Then the convex hull of the the Gale transform $\widehat{W}$ is the desired $d$-dimensional polytope $P$ with $n$ vertices. The rest follows by Theorem 32.3. $\square$

PROBLEMS.

1. $W \subset \mathbb{R}^{n-d-1}$ is a Gale diagram of a $d$-dimensional polytope $P$ with $n$ vertices iff $W$ spans $\mathbb{R}^{n-d-1}$ and for every linear hyperplane $H \subset \mathbb{R}^{n-d-1}$ there exist at least two points of $W$ in each open halfspace bounded by $H$.

2. Let $W \subset \mathbb{R}^{n-d-1}$ be a set which spans $\mathbb{R}^{n-d-1}$ and such that for every linear hyperplane $H$ in $\mathbb{R}^{n-d-1}$ there exist at least $k+1$ points of $W$ in every open halfspace bounded by $H$. Then $W$ is a Gale diagram of a polytope $P$ such that the convex hull of every $k$ vertices of $P$ is a face of $P$.

3. A polytope $P$ is simplicial iff the linear span of every coface of its Gale diagram $W$ coincides with the linear span of $W$.

5°. Construct the Gale diagram of a simplex.

6. Describe all possible Gale diagrams with $n$ points in $\mathbb{R}^1$ and then describe combinatorial types of all possible $d$-dimensional polytopes with $d+2$ vertices.

7. Describe the combinatorial structure of a polytope whose Gale diagram is a regular hexagon centered at the origin.

8. Let $P$ be a polytope and $W$ be its Gale diagram. Add a new vertex to $W$ placing it in the origin. What you get is a Gale diagram of a polytope $P_1$. How to construct $P_1$ from $P$?

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Monday, March 6

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## 33. Constructing Weird Polytopes

**(33.1) Definition.** We recall that two polytopes are called *combinatorially isomorphic* iff there exists a bijection between their vertex sets which induces an isomorphism of their face lattices. In other words, if $P$ and $Q$ are combinatorially isomorphic polytopes then one can order the vertices $v_1, \ldots, v_n$ of $P$ and vertices $u_1, \ldots, u_n$ of $Q$ so that the convex hull of the set $\{v_i : i \in i\}$ is a face of $P$ iff the convex hull of the set $\{u_i : i \in I\}$ is a face of $Q$. A polytope $P \subset \mathbb{R}^d$ is called *rational* iff its vertices have rational coordinates, that is, $P = \operatorname{conv}\{v_1, \ldots, v_n\}$, where $v_i \in \mathbb{Q}^d$ are the vertices of $P$.

PROBLEMS.

1. Combinatorially isomorphic polytopes have equal dimension.

2°. For every 2-dimensional polytope there exists a combinatorially isomorphic rational polytope.

3*. For every 3-dimensional polytope there exists a combinatorially isomorphic rational polytope.

4*. For every 3-dimensional polytope with $n$ vertices there exists a combinatorially isomorphic polytope with integral vertices so that the absolute values of the coordinates of the vertices do not exceed $n^{169n^3}$. Does there exist a polynomial in $n$ bound?

5*. There exists a 4-dimensional polytope which is not combinatorially isomorphic to any rational polytope.

6. Consider a 4-dimensional polytope whose Gale diagram is a regular 7-gon centered at the origin. This polytope is combinatorially isomorphic to a Cyclic polytope.

7°. Every simplicial polytope is combinatorially isomorphic to a rational polytope.

8. Every simple polytope is combinatorially isomorphic to a rational polytope.

**(33.2) Theorem ("Existence of weird polytopes").** *Let $\mathcal{A} \subset \mathbb{R}$ be the field of all algebraic numbers and let $\mathcal{F} \subset \mathcal{A}$ be its proper subfield. Then there exists a dimension $d$ and a $d$-dimensional polytope $P \subset \mathbb{R}^d$ with $d+4$ vertices such that $P$ is not combinatorially isomorphic to any polytope whose vertices belong to $\mathcal{F}^d$.*

**Sketch of proof.** First, we observe that if the $n$ vertices of a polytope belong to $\mathcal{F}^d$ then the polytope has a Gale diagram whose $n$ points belong to $\mathcal{F}^{n-d-1}$. This follows since a Gale diagram can be constructed by using rational operations only (addition, subtraction, multiplication and division). Thus we have to construct a multiset $W$ of points in $\mathbb{R}^3$ whose cofacial structure can not be represented by a multiset in $\mathcal{F}^3$. Our set $W$ will be centrally symmetric, so $W = \{w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n}\}$, where $w_i = -w_{2n+1-i}$. Thus whenever we have a linear dependence $\sum_{i \in I} \lambda_i w_i = 0$ with non-zero $\lambda$-s, we get a coface replacing $w_i$ with $\lambda_i < 0$ by $w_{2n+1-i}$. Hence our ultimate goal is to construct a set of vectors from $\mathbb{R}^3$ whose linear dependencies can not be achieved on a set from $\mathcal{F}^3$. In other words, we want to construct a configuration of points on the projective plane $\mathbf{P}^2$ such that its "type" of linear dependencies can not be realized by the points with coordinates in $\mathcal{F}$. This is much, much easier. To do that, we choose an algebraic number $\xi \in \mathcal{A} \setminus \mathcal{F}$ and its primitive polynomial $p$ and "simulate" the identity $p(\xi) = 0$ by a configuration of points using simulation for addition and multiplication.

See N.E. Mnëv, On the realizability over fields of the combinatorial types of convex polytopes, *Zapiski Nauchnukh Seminarov LOMI* (Russian), **123**(1983), 203–207. See also J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics, **1643**, Springer-Verlag, 1996.

PROBLEMS.

1. Every $d$-dimensional polytope with at most $(d+3)$ vertices is combinatorially isomorphic to a rational polytope.

2. This is how Gale diagrams look in the coordinates. Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope. Let us consider a linear space $\Lambda$ of all affine dependencies $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$, $\lambda_1 + \ldots + \lambda_n = 0$. Then $\dim \Lambda = n - d - 1$. Let $u_1, \ldots, u_{n-d-1}$ be a basis of $\Lambda$. Let $\widehat{v_i} \in \mathbb{R}^{n-d-1}$ be the point whose $j$-th coordinate is the $i$-th coordinate of the point $u_j$. Then $\widehat{v_1}, \ldots, \widehat{v_n}$ is a Gale diagram of $P$.

3. There exist precisely $[d/2]$ pairwise combinatorially non-isomorphic $d$-dimensional simplicial polytopes with $d+2$ vertices.

4. There exist precisely $[d^2/4]$ pairwise combinatorially non-isomorphic $d$-dimensional polytopes with $d+2$ vertices.

5*. Let $n(d)$ be the number of pairwise combinatorially non-isomorphic $d$-dimensional simplicial polytopes with $d+3$ vertices. Then

$$n(d) = 2^{[d/2]} - [(d+4)/2] + \frac{1}{4(d+3)} \sum_h \phi(h) 2^{(d+3)/h},$$

where the sum is taken over all odd divisors of $d+3$ and $\phi$ is the Euler's totient function ($\phi(k)$ is the number of positive integers not bigger than $k$ and coprime with $k$).

6*. Find a formula for the number of pairwise combinatorially non-isomorphic simplicial $d$-dimensional polytopes with $d+4$ vertices.

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Wednesday, March 8

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## 34. Lawrence Construction

Let $v_1, \dots, v_n \in \mathbb{R}^{r-1}$ be some points such that the affine hull of any $n-1$ of them coincide with $\mathbb{R}^{r-1}$. We consider $\mathbb{R}^{r-1}$ as a hyperplane $x_r = 1$ in $\mathbb{R}^r$. Let us consider the set of points

$$W = \Big\{ w_i : i = 1, \dots, 2n, \quad \text{where} \quad w_i = v_i, w_{2n+1-i} = -v_i : i \leq n \Big\} \subset \mathbb{R}^r.$$

**(34.1) Lemma.** *The set $W$ is a Gale diagram of a $(2n - r - 1)$-dimensional polytope with $2n$ vertices.*

**Proof.** It is seen that $w_1 + \ldots + w_{2n} = 0$, so the whole set $W$ is a coface. So we can apply the Gale Transform and get some polytope, say $P = \mathrm{conv}\{\widehat{w_1}, \dots, \widehat{w_{2n}}\}$. Since $W$ generates the whole space $\mathbb{R}^r$, the dimension of $P$ is $2n - r - 1$. Now let us show that all the points $\widehat{w_1}, \dots, \widehat{w_{2n}}$ are the vertices of $P$. What we have to check is that every set $W \setminus \{w_i\}$ is a coface. It is enough to check this for $w_1 = v_1$, say. Since the affine hull of $v_2, \dots, v_n$ is the whole space $\mathbb{R}^{r-1}$, we can find a linear dependence

$$\lambda_1(-v_1) + \sum_{i \in I} \lambda_i v_i = 0,$$

where $I \subset \{2, \dots, n\}$, all $\lambda_s$ are non-zero and $\lambda_1 > 0$. Replacing $v_i$ by $-v_i$ for $\lambda_i < 0$ we get a coface $H \subset W$ which contains $w_{2n} = -v_1$ but does not contain $v_1$. Since every subset $\{w_i, w_{2n+1-i}\}: \ i = 2, \dots, n$ is a coface and the union of cofaces is a coface we get that $W \setminus \{w_1\}$ is a coface and hence $\widehat{w_1}$ is a vertex of $P$. $\square$

What we did is called sometimes *Lawrence construction.* The polytope $P$, obtained in such a way, we call *Lawrence polytope.*

PROBLEM

1°. Lawrence polytope contains $2n$ vertices, say, $v_1, \dots, v_{2n}$ such that for every $i$ the vertices $v_i$ and $v_{2n+i-i}$ are joined by an edge whereas the remaining $(2n-2)$ vertices belong to a facet of $P$.

2°. There exists a $d$-dimensional Lawrence polytope with $2d$ vertices.

3. Suppose that $Q = \mathrm{conv}\{u_1, \dots, u_n\} \subset \mathbb{R}^d$ is a $d$-dimensional polytope. Let us consider its Gale diagram $V = \{v_1, \dots, v_n\}$ in $\mathbb{R}^{r-1}$, where $r = n - d$ and apply the Lawrence construction to $V$. We get a polytope $P$ with $2n$ vertices of the dimension $2n - r - 1 = n + d - 1$. How to construct $P$ from $Q$ directly?

Lawrence construction produces unusually "rigid" polytopes.

**(34.2) Theorem "Combinatorial rigidity".** *Let $P$ be a Lawrence polytope with $2n$ vertices $a_1, \dots, a_{2n}$. Let $Q$ be a $d$-dimensional polytope with $2n$ vertices $b_1, \dots, b_{2n}$ which is combinatorially isomorphic to $P$ via the isomorphism $b_i \longmapsto a_i : i = 1, \dots, 2n$. Suppose that there exists a hyperplane $H_a$ which contains the vertices $\{a_i : i \in I_0\}$ of $P$ and separates the vertices $\{a_i : i \in I_+\}$ from the vertices $\{a_i : i \in I_-\}$ of $P$. Then there exists a hyperplane $H_b$ which contains the vertices $\{b_i : i \in I_0\}$ of $Q$ and separates the vertices $\{b_i : i \in I_+\}$ from the vertices $\{b_i : i \in I_-\}$ of $Q$.*

**Proof.** Suppose that $H_a = \{x \in \mathbb{R}^d : l(x) = \alpha\}$, so $I_0 = \{i : l(a_i) = 0\}$, $I_+ = \{i : l(a_i) > \alpha\}$, and $I_- = \{i : l(a_i) < \alpha\}$. Let $W = \{w_1, \dots, w_{2n}\}$ be the Gale diagram of $P$, so $w_i = v_i, w_{2n+1-i} = -v_i$ for $i < n$. Let $\lambda_i = l(a_i)$. By Lemma 32.2 we have $\lambda_1 w_1 + \ldots + \lambda_{2n} w_{2n} = 0$. Subtracting $0 = \alpha(w_1 + \ldots + w_{2n})$ we get

$$\sum_{i \in I_0} (\lambda_i - \alpha) w_i + \sum_{i \in I_+} (\lambda_i - \alpha) w_i + \sum_{i \in I_-} (\lambda_i - \alpha) w_i = 0.$$

Therefore, the set

$$\Big\{ w_i : i \in I_+ \text{ or } 2n + 1 - i \in I_- \Big\}$$

is a coface of $W$.

Now let $U = \{u_1, \dots, u_{2n}\}$ be the Gale Transform of $Q$. Since $Q$ is combinatorially isomorphic to $P$, the set $\{u_i, u_{2n+1-i}\}$ is a coface of $U$. Thus $u_{2n+1-i} = \mu_i u_i$ for some $\mu_i < 0$. Furthermore, the set

$$\Big\{ u_i : i \in I_+ \text{ or } 2n + 1 - i \in I_- \Big\}$$

is a coface. Hence there exist $\lambda_i : i = 1, \dots, 2n$ such that $\lambda_1 u_1 + \ldots + \lambda_{2n} u_{2n} = 0$, where $\lambda_i > 0$ for $i \in I_+$ or $(2n + 1 - i) \in I_-$ and $\lambda_i = 0$ elsewhere. If for some $i$ we have $i \in I_+$ and $(2n + 1 - i) \in I_-$ we split the coefficient $\lambda_i$ between $u_i$ and $u_{2n+1-i}$ equally. Now for $(2n + 1 - i) \in I_-$ we replace $\lambda_i u_i$ by $(\lambda_i / \mu_i) u_{2n+1-i}$. Hence we get a new linear dependence, say, $\delta_1 u_1 + \ldots + \delta_{2n} u_{2n} = 0$, where $\delta_i > 0$ for $i \in I_+$, $\delta_i < 0$ for $i \in I_-$ and $\delta_i = 0$ for $i \in I_0$. Subtracting $\alpha = (\delta_1 + \ldots + \delta_{2n})/2n$ we get an affine dependence and by Lemma 32.2 there exists a linear function $l$ such that $l(b_i) > \alpha$ for $i \in I_+$, $l(b_i) < \alpha$ for $i \in I_-$ and $l(b_i) = \alpha$ for $i \in I_0$. Then we may choose $H_b = \{x : l(x) = \alpha\}$. □

PROBLEM.

1. Check the rigidity of a Gale diagram of a Lawrence polytope: Let $P$ be a Lawrence polytope, $Q$ be a polytope, combinatorially isomorphic to $P$, $W$ be a Gale diagram of $P$ and $U$ be a Gale diagram of $Q$. If some set $A \subset W$ of points is linearly dependent, then the corresponding set of points $B \subset U$ is linearly dependent.

2. Triangular prism in $\mathbb{R}^3$ is a Lawrence polytope. In particular, it is combinatorially rigid.

3°. Neither cube, nor octahedron are combinatorially rigid.

## 35. Graphs of Polytopes

**(35.1) Definitions.** Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope with $n$ vertices $v_1, \ldots, v_n$. The *graph* $G(P)$ of $P$ is the (undirected, simple) graph with the vertex set $V = \{v_1, \ldots, v_n\}$ and the set of edges $E = \big\{(v_i, v_j) : \ [v_i, v_j] \text{ is an edge of } P\big\}$. There are many ways to introduce an orientation on $G(P)$, making every edge $(v_i, v_j)$ directed $\overrightarrow{v_i v_j}$ or $\overrightarrow{v_j v_i}$. An orientation is called *acyclic* iff there is no oriented cycles $v_{i_1} \longmapsto v_{i_2} \longmapsto \ldots \longmapsto v_{i_k} \longmapsto v_{i_1}$. A vertex $v \in V$ is called a *sink* iff there is no edge coming out of $v$. A subset $H$ is called an *initial set* iff there is no edge $\overrightarrow{uv}$, where $u \in V \setminus H$ and $v \in H$. An undirected graph $G = (V, E)$ is called *$d$-connected* iff for any set $S \subset V$ of not more than $(d-1)$ vertices and for any two distinct vertices $u, v \in V \setminus S$ there exists a path in $G$ starting in $u$, finishing in $v$ and not containing any vertex from $S$.

PROBLEMS.

1°. An acyclic orientation of a graph induces a poset structure on $V$: we let $x < y$ iff there exists a directed path beginning in $x$ and ending in $y$.

2°. Reversing every edge in an acyclic orientation we get yet another acyclic orientation.

3. This is an easy way to see if a given orientation is acyclic. Let $G = (V, E)$ be an oriented graph with $n$ vertices and $A$ be its $n \times n$ adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } \overrightarrow{v_i v_j} \text{ is a directed edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the orientation is acyclic iff $\operatorname{trace}(A + A^2 + \ldots + A^n) = 0$.

4. A graph is $d$ connected iff for any two distinct vertices $u, v$ there exist $d$ paths $P_1, \ldots, P_d$ starting in $u$, finishing in $v$ and such that $P_i \cap P_j = \{u, v\}$ for every two $i \neq j$.

**(35.2) Linear orientation.** Let $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function such that $l(v_i) \neq l(v_j)$ for any two different vertices of the polytope $P \subset \mathbb{R}^d$. The function $l$ induces a *linear orientation* of $G(P)$: we direct every edge $\overrightarrow{v_i v_j}$ whenever $l(v_j) > l(v_i)$.

PROBLEMS.

1°. A linear orientation is acyclic and possesses the unique sink, namely, the vertex where the maximal value of $l$ on $P$ is attained.

2°. For every face $F$ there exists a linear orientation of $G(P)$ where the set of vertices on $F$ is an initial set.

*Hint:* Consider a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $l$ attains its minimum precisely on the face $F$. Then perturb $l \longmapsto \tilde{l}$ slightly.

We will use the following simple result.

**(35.3) Lemma.** *Let $P \subset \mathbb{R}^d$ be a polytope and $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. For every vertex $v$ of $P$ either $l(v) = \max\{l(x) : x \in P\}$ or there exists a neighbor $u$ of $v$ such that $l(u) > l(v)$.*

**Proof.** Follows by Lemma 16.5. □

**(35.4) Theorem ("Balinski's Theorem").** *A graph of a $d$-dimensional polytope is $d$-connected.*

**Proof.** We prove this by induction on $d$. If $d = 1$ then the polytope is an interval and its graph is connected. Suppose that $d > 1$. Let us choose a set $S$ of $s < d$ vertices of a $d$-dimensional polytope $P$, say, $S = \{v_1, \ldots, v_s\}$. Let us choose some another vertex, say, $v$. Since $s+1 \leq d$ there exists a hyperplane $H$ passing through $v_1, \ldots, v_s, v$. Thus there exists a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ and a number $\alpha$ such that $l(v_1) = \ldots = l(v_s) = l(v) = \alpha$. There are two possibilities: the hyperplane $H$ is either a supporting hyperplane of a face of $P$ or is not a supporting hyperplane. Let us consider the first case. Without loss of generality we may assume that $\alpha = \min\{l(x) : \ x \in P\}$ and $F = \{x \in P : l(x) = \alpha\}$ is a face containing $v_1, \ldots, v_s$. Let $\beta = \max\{l(x) : \ x \in P\}$. So $\beta > \alpha$ and $G = \{x \in P : l(x) = \beta\}$ is a face. Let us choose any two vertices $u, w \in V \setminus S$. We have $l(u), l(w) \geq \alpha$. Using Lemma 35.3 we can construct two paths $P_u$ and $P_w$, starting in $u$ and $w$ correspondingly and ending somewhere on the face $G$ such that the value of $l$ strictly increases along the paths. Therefore they do not contain any vertex from $S$. Applying the induction conjecture to the face $G$ we can connect the ends of $P_u$ and $P_v$ by a path in $G$. Thus we have constructed a path connecting $u$ and $w$ in $V \setminus S$.

Let us consider the second case: $H$ is not a supporting hyperplane of a a face. Let $\alpha^+ = \max\{l(x) : x \in P\}$ and $\alpha^- = \min\{l(x) : x \in P\}$. Then $\alpha^- < \alpha < \alpha^+$ and $F^+ = \{x \in P : \ l(x) = \alpha^+\}$, $F^- = \{x \in P : l(x) = \alpha^-\}$ are faces of $P$. Let us choose a vertex $u \in V \setminus S$. If $l(u) \leq \alpha$ then using Lemma 35.3 we construct a path $P_u = P_u^-$ starting in $u$, finishing somewhere in $F^-$ and such that $l$ strictly decreases along $P_u^-$. Hence $P_u$ does not contain any vertex from $S$. If $l(u) \geq \alpha$ then using Lemma 35.3 we construct a path $P_u = P_u^+$ starting in $u$, finishing somewhere in $F^+$ and such that $l$ strictly increases along $P_u^+$. Again, $P_u$ does not contain any vertex from $S$. Now, for every two given vertices $u_1, u_2 \in V \setminus S$ we construct the paths $P_{u_1}$ and $P_{u_2}$. If their ends belong to the same face ($F^+$ or $F^-$) we can connect them there (using the induction conjecture) and thus construct a path from $u_1$ to $u_2$. Otherwise, we observe that for the special vertex $v$ we have chosen we can construct both path: the increasing path $P_v^+$ to $F^+$ and the decreasing path $P_v^-$ to $F^-$. So we can connect the vertex sets of $F^+$ and $F^-$ by a path in $V \setminus S$. Therefore, if the ends of $P_{u_1}$ and $P_{u_2}$ belong to different faces then we can still connect their ends in $V \setminus S$ □

## 36. Reconstructing a Simple Polytope from its Graph

Let $P$ be a simple $d$-dimensional polytope. Thus every vertex of $P$ has precisely $d$ neighbors and, therefore, the graph $G(P)$ is *$d$-regular.* Furthermore, if $F \subset P$ is a face of $P$ then $F$ itself is a simple polytope, so $G(F)$ is a $k$-regular subgraph of $G(P)$, where $k = \dim F$. We shall use the following fact: for any vertex $v$ of $P$ and any $k$ neighbors $u_1, \ldots, u_k$ of $v$ there exists the unique $k$-dimensional face containing $u_1, \ldots, u_k$ and $v$ (Theorem 15.4).

We show that the polytope $P$ can be reconstructed from its graph $G(P)$. First, we describe the algorithm and then prove its correctness. Abusing the language we identify a face of $P$ with its vertex set.

**Input:** a $d$-regular graph $G = (V, E)$ such that there exists a simple $d$-dimensional polytope $P$ with $G(P) = G$.

**Output:** a family of subsets $\{F \subset V\}$, where $F$ is a face of $P$.

**Algorithm:**

1. Consecutively generate all possible acyclic orientations $O$ of $G$. For an acyclic orientation $O$ compute the number $h_k^O$ of vertices with in-degree $k$. Then compute

$$f^O = h_0^O + 2h_1^O + \ldots + 2^k h_k^O + \ldots + 2^d h_d^O.$$

Find $f = \min\Big\{ f^O : O \text{ is acyclic orientation } \Big\}$.

2. Call an acyclic orientation *good* iff $f^O = f$. Consecutively generate all good acyclic orientations $O$. For every good acyclic orientation $O$ and every $k = 1, \ldots, d-1$ find all connected $k$-regular subgraphs $(F, E_1) \subset (V, E)$ such that $F$ is an initial set with respect to the orientation $O$. Output $F$.

Now we have to prove that the algorithm is correct.

**(36.1) Lemma.** *An acyclic orientation $O$ of $G(P)$ is good iff for every face $F$ of $P$ the subgraph $G(F)$ has the unique sink.*

**Proof.** Every subgraph $G(F)$ has at least one sink. For every vertex $v \in V$ with in-degree $k$ there exist $2^k$ faces with a sink in $v$ (any subset of incoming edges defines a face). Therefore, if for every face $F$ the subgraph $G(F)$ has the unique sink then $f^O$ is the number of non-empty faces of $P$. Otherwise, $f^O$ is greater than the number of non-empty faces of $P$ since the faces with more than one sink are counted several times. Finally, for any linear orientation $O$ every subgraph $G(F)$ has precisely one sink, so $f$ is the number of non-empty faces of $P$ and hence $f^O = f$ iff $G(F)$ has the unique sink for every face $F$ of $P$. □

**(36.2) Theorem.** *The algorithm is correct.*

**Proof.** Let $F$ be a $k$-dimensional face of $P$. Then $G(F)$ is a $k$-regular subgraph of $G(P)$ and we present a good acyclic orientation for which $F$ is an initial set: let $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function such that $l(v) = \min\{l(x) : x \in P\}$ whenever $v \in F$. Perturb $l \longmapsto \tilde{l}$ slightly and consider the linear orientation corresponding to $\tilde{l}$.

Suppose now, that for some good acyclic orientation $O$ and some $k$-regular subgraph $(F, E_1) \subset (V, E)$ the set $F$ is an initial set with respect to the orientation $O$. We have to prove that $F$ is a face of $P$. Let us consider a sink $v$ of $(F, E_1)$. Let $\tilde{F}$ be a face of $P$ generated by the $k$ edges coming in $v$. Then $G(\tilde{F})$ is a $k$-regular subgraph. Since the orientation $O$ is good, $v$ is the unique sink on $G(\tilde{F})$ , and, therefore, every vertex $u \in \tilde{F}$ can be connected with $v$ by a directed path in $G(\tilde{F})$. Therefore $u \in F$ since $F$ is an initial set with respect to $O$. Thus $(F, E_1), G(\tilde{F})$ are two connected $k$-regular graphs such that $\tilde{F} \subset F$. Hence we must have $\tilde{F} = F$. □

The algorithm and the proof belong to Gil Kalai: G. Kalai, A simple way to tell a simple polytope from its graph, *Journal of Combinatorial Theory, Ser. A*, **49**(1988), 381-383.

PROBLEMS.

1. What is the computational complexity of the algorithm?

2*. Is there an easier way to compute, say, $f$-vector of $P$ from the graph $G(P)$?

3. The graph $G(P)$ of a $d$-dimensional polytope contains the complete graph $K_{d+1}$ with $d+1$ vertices as a minor (that is, there exist $d+1$ different vertices in $G(P)$ pairwise joined by non-intersecting paths). In particular, if dim $G \geq 4$ then $G(P)$ is not planar.

4*. Let $P$ be a simple polytope. Is it true that every connected $(d-1)$-regular subgraph which does not separate $G(P)$ is the graph of a facet of $P$?

5*. There exists a function $f(k)$ such that for every $d$ there exist at most $f(k)$ pairwise non-isomorphic graphs of $d$-dimensional polytopes with $d+k$ vertices.

6. A simplicial $d$-dimensional polytope can be uniquely reconstructed from the complex of its $[d/2]$ dimensional faces.

7. A $d$-dimensional polytope can be uniquely reconstructed from the complex of its faces of dimension $\leq d-2$.

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Wednesday, March 15

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## 37. Empty Edges

**(37.1) Defect.** Let $P$ be a polytope. Let us denote by $V(P)$ the vertex set of $P$. Similarly, if $F$ is a face of $P$ then by $V(F)$ we denote vertex set of $F$. The *defect* of $P$ is the number

$$\nu(P) = \max_F \{|V(P)| - |V(F)|\},$$

where $F$ ranges over all facets of $P$.

PROBLEM.

1. $\nu(P) = 1$ iff $P$ is a simplex.

**(37.2) Theorem.** *If $F$ is a facet of $P$ then $\nu(P) \geq \nu(F)$.*

**Proof.** Let $F$ be a facet of $P$ and $G$ be a facet of $F$. Then there exists another facet $H$ of $P$ such that $F \cap H = G$ (see Corollary 12.1). Then $V(F) \setminus V(G) \subset V(P) \setminus V(H)$. Therefore $|V(F)| - |V(G)| \leq |V(P)| - |V(H)| \leq \nu(P)$. Hence $\nu(F) \leq \nu(P)$. □

**(37.3) Corollary.** *A $d$-dimensional polytope with $d + k$ vertices contains a $(d - k + 1)$-dimensional face which is a simplex.*

**Proof.** Let $P$ be a $d$-dimensional polytope with $d + k$ vertices. Let us choose a sequence

$$P = F_0 \supset F_1 \supset F_2 \supset \ldots \supset F_{k-1},$$

where $F_i$ is a facet of $F_{i-1}$ with the smallest possible number of vertices. In particular,

$$|V(F_{i-1})| = |V(F_i)| + \nu(F_{i-1}).$$

Suppose that $F_{k-1}$ is not a simplex. Then by Problem 1, Section 37.1, we have $\nu(F_{k-1}) \geq 2$. By Theorem 37.2 we must have $\nu(F_i) \geq 2$ for every $i \leq k - 1$. Since $\dim F_{k-1} = d - k + 1$, the face $F_{k-1}$ should contain at least $d - k + 3$ vertices (it is not a simplex). Therefore, $|V(P)| \geq d - k + 3 + 2(k - 1) = d + k + 1$ which is a contradiction. □

Let $P$ be a polytope. A pair of its vertices $\{u, v\}$ is called an *empty edge* iff $[u, v]$ is not an edge of $P$.

We are going to prove the following result.

**(37.1) Theorem.** *There exists a function $f(k) = 2k(k - 1)$ such that for every $d$-dimensional polytope $P$ with $d + k$ vertices the graph $G(P)$ has at most $f(k)$ empty edges.*

**Proof.** Let us consider the set $\mathcal{E}$ of empty edges. Let $A \subset \mathcal{E}$ be a maximal (under inclusion) subset of pairwise disjoint empty edges. We claim that the set $A$ can not contain more than $2k - 2$ empty edges. Indeed, let $\Delta \subset P$ be a simplicial face which exists by Corollary 37.3. Every empty edge should have at least one vertex outside of $\Delta$, so in the remaining $2k - 2$ vertices. Thus there is a possibility to place at most $2k - 2$ pairwise disjoint empty edges. Now, every empty edge from $\mathcal{E} \setminus A$ should have a common vertex with some empty edge from $A$. On the other hand, there can not be more than $k$ empty edges with a given common vertex, since every vertex should belong to at least $d$ real edges. Therefore, the total number of empty edges can not be greater than $2k(k-1)$. □

For more information see G. Kalai, Some aspects of the combinatorial theory of convex polytopes, in: *Polytopes: Abstract, Convex and Computational*, Kluwer Acad. Publ., Dordrecht, 205–229 (NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, **440**).

PROBLEM.

1. Do problem 5 from Section 36.2 now.

2. An empty face of a polytope $P$ is a subset $S \subset V(P)$ of vertices of $P$ such that $S$ is not the set of vertices of a face of $P$ but every proper subset of $S$ is the set of vertices of a face of $P$.

a°). A set of vertices of $P$ is the set of vertices of a face iff it does not contain any empty face.

b°). An empty edge is an empty face.

c). There exists a function $f(k)$ such that every $d$-dimensional polytope with at most $d+k$ vertices contains at most $f(k)$ empty faces.

## 38. Diameter

Let $G=(V,E)$ be an undirected graph. For any pair of vertices $u,v \in V$ we introduce a number $d(u,v)$ as the smallest possible number of edges in a path which joins $u$ and $v$ in $G$. The number $\delta(G)=\max\{d(u,v): u,v \in V\}$ is called the *diameter* of $G$.

**(38.1) Hirsh Conjecture.** If $G=G(P)$ is the graph of a $d$-dimensional polytope with $n$ facets then $\delta(G) \leq n-d$.

PROBLEMS.

1°. Check the Hirsh Conjecture for $d=2$.

2. Check the Hirsh Conjecture for $d=3$.

3. Check the Hirsh Conjecture for the $d$-dimensional cube.

4. Prove that the Hirsh Conjecture is correct iff it is correct for $d$-dimensional polytopes with $2d$ facets.

5. Check the Hirsh Conjecture for the polar of the Cyclic polytope.

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Friday, March 17

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## 39. A Pseudopolynomial Bound for the Diameter of a Polyhedron

**(39.1) Lemma.** *Let $P \subset \mathbb{R}^d$ be a polyhedron and let $v \in P$ be its vertex. Let $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a supporting function at $v$, that is, $l(v) = \alpha$ and $l(x) < \alpha$ for any $x \neq v$ in $P$. For any $\beta < \alpha$ let $P_\beta = \{x \in P : \ l(x) \geq \beta\}$. Then $P_\beta$ is a polytope.*

**Proof.** Obviously, $P_\beta$ is a polyhedron. So we have to prove that $P_\beta$ is bounded. Let $H = \{x \in \mathbb{R}^d : \ l(x) = \alpha\}$ be the supporting hyperplane of $v$. Suppose that for any $R > 0$ there is a point $x_R \in P_\beta$ whose distance from $v$ is at least $R$. Then for the point $y_R = v + (x_R - v)/\|x_R - v\|$ we have $|l(y_R) - \alpha| \leq (\alpha - \beta)/R$. Thus we get a sequence of points on the unit sphere centered at $v$ whose distances from $H$ tend to 0. A limit point $y$ of this sequence would belong to $H \cap P$ and would be different from $v$. This is a contradiction. $\square$

If $P$ is an unbounded polyhedron we still may consider its graph $G(P)$ as a graph whose vertices are the vertices of $P$ and whose edges are the edges of $P$, each of which contains precisely two vertices of $P$ (since $P$ is unbounded, it may have unbounded edges and they are not included in $G(P)$).

**(39.2) Theorem.** *Let $P$ be a polyhedron. Then $G(P)$ is connected. If $G(P)$ is not empty (that is, there is a vertex of $P$) then every face of $P$ contains a vertex.*

**Proof.** If $P$ does not contain any vertex, the theorem is clear. Since $P$ is a polyhedron it must have only finitely many vertices (for a $d$-dimensional polyhedron given by $n$ linear inequalities the number of vertices does not exceed $\binom{n}{d}$). Suppose that $v$ is a vertex of $P$ and let $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be the supporting function of $l$. Let us choose a sufficiently small $\beta$ so that the polytope $P_\beta$, constructed as in Lemma 39.1 contains all the vertices of $P$. In particular, $G(P_\beta)$ is a subgraph of $G(P)$. Let $u, w$ be two vertices of $P$ and hence vertices of $P_\beta$. Using Lemma 35.3 we may connect them in $P_\beta$ with $v$ by increasing (with respect to $l$) paths. Therefore, they are connected in $G(P)$. Now, let $F$ be a face of $P$. Then, for a sufficiently small $\beta$ the set $F_\beta = F \cap P_\beta$ is a face of $P_\beta$ and the maximal value of $l$ on $F_\beta$ is greater than $\beta$. Therefore there is a $u$ vertex of $P_\beta$ on $F_\beta$ such that $l(u) > \beta$. Then $u$ is a vertex of $P$ on $F$. $\square$

Here are some problems concerning general (unbounded) polyhedra.

PROBLEMS.

1. A non-empty polyhedron contains a vertex iff it does not contain a straight line.
2. A polyhedron is bounded iff it does not contain a ray.
3. A polyhedron is the Minkowski sum of a linear subspace, a cone without straight lines, and a polytope.

Let $\Delta(d, n)$ be the largest diameter of the graph $G(P)$, where $P$ is a $d$-dimensional polyhedron with at most $n$ facets.

The following remarkable theorem is proven in G. Kalai and D. Kleitman, A quasi-polynomial bound for the diameter of graphs of polyhedra, *Bulletin of the American Mathematical Society (New Series)*, **26**(1992), 315–316.

**(39.3) Theorem.** $\Delta(d,n) \leq n^{\log d+2}$.

**Proof.** Let $P$ be a $d$ dimensional polyhedron with $n$ facets. We may assume that $P$ is given by a system of linear inequalities $P = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i\}$, where every inequality defines a facet of $P$. If $P$ contains fewer than two vertices, we are done. Otherwise, let us take two vertices $u$ and $v$ of $P$. Let $k_v$ (resp. $k_u$) be the maximal number such that there are at most $n/2$ facets which can be visited by some path in $G(P)$ starting from $v$ (resp. from $u$) of length at most $k_v$ (resp. $k_u$). If $v$ (resp. $u$) belongs to more than $n/2$ facets we let $k_v = -1$ (resp. $k_u = -1$). Theorem 39.2 implies that both numbers are correctly defined ($< +\infty$). Then there is a facet $F$ of $P$ which is reachable by a path of length $k_v + 1$ from $v$ and by a path of length $k_u + 1$ from $u$. The facet $F$ is a $(d-1)$-dimensional polytope with at most $n-1$ facets. Therefore we get

$$d(u,v) \leq 2 + k_u + k_v + \Delta(d-1, n-1).$$

Now we claim that $k_u, k_v \leq \Delta(d, n/2)$. Let us prove the bound for $k_v$. Let $I_v = \Big\{i :$ there exists a path of length $\leq k_v$ from $v$ which visits the facet $l_i(x) = \alpha_i\Big\}$. So $|I_v| \leq n/2$. Consider the polyhedron $Q_v$ defined by the inequalities $Q_v = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i : i \in I_v\}$. Then $v$ is a vertex of $Q_v$ and $Q_v$ is a $d$-dimensional polyhedron with at most $n/2$ facets. Let us choose a vertex $w$ of $P$ with $d(v,w) = k_v$ in $P$. Then $w$ is a vertex of $Q_v$ and we claim that $d(v,w) = k_v$ also in $Q_v$. Suppose that there exists a shorter path from $v$ to $w$ in $G(Q_v)$. This path shouldn't exist in $G(P)$, therefore one of its edges must cross a hyperplane $H_i = \{x : l_i(x) = \alpha_i\}$ for $i \notin I_v$. Let us consider the first such edge. Its intersection with $H_i$ is a vertex, say, $\omega$ of $P$. Following along the path from $v$ to $\omega$ we get a path of length $\leq k_v$ in $P$ which meets a facet of $P$ whose index does not belong to $I_v$. This contradicts to the definition of $I_v$. Hence $k_u, k_v \leq \Delta(d, n/2)$. So we got a path from $u$ to $v$ whose length does not exceed $\Delta(d-1, n-1) + 2 + 2\Delta(d, n/2)$. Therefore

$$\Delta(d,n) \leq \Delta(d-1, n-1) + 2\Delta(d, n/2) + 2.$$

Iterating this, we get that $\Delta(d,n) \leq 2d\Delta(d, n/2) + 2d$. Iterating this we get $\Delta(d,n) \leq (2d)^{\log n} + 2d \log n \leq n^{\log d+2}$. □

PROBLEM

1. Let $P$ be a $d$-dimensional polyhedron with $n$ facets and $l$ be a linear function which is not constant on any edge (including unbounded edges). For any vertex $v$ there exists a path of length $\leq n^{\log d+2}$ which starts in $v$, increases with respect to $l$ and either stops at the vertex with the largest value of $l$ or at an unbounded edge with increasing value of $l$ along the edge.

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Monday, March 20

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## 40. Linear Programming, Average Case

The problem of finding the maximal value (and the point where this value is attained) of a given linear function $l_0 : \mathbb{R}^d \longrightarrow \mathbb{R}$ on a given polyhedron $P = \{x \in \mathbb{R}^d : \ l_i(x) \leq \alpha_i, i = 1, \ldots, n\}$, where $l_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ are linear functions, is called a *Linear Programming Problem.* The problem of finding whether a given polyhedron is empty is called a *feasibility problem.* If the polyhedron $P = \{x \in \mathbb{R}^d : \ l_i(x) \leq \alpha_i, i = 1, \ldots, n\}$ is empty we suppose that the maximal value of $l_0$ is equal to $-\infty$.

**(40.1) Random problems.** Suppose that there is a probability distribution on the set of $n$ linear inequalities in $d$ variables. Thus every feasibility problem is encoded by a $2n$-tuple $(l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n)$, where $l_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ are linear functions and $\alpha_i \in \mathbb{R}$ are numbers. We make the following two assumptions about this probability distribution:

(40.1.1) it has a continuous density function (so as to exclude degenerate cases);
(40.1.2) it is invariant with respect to any transformation:

$$(l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n) \longmapsto (\epsilon_1 l_1, \ldots, \epsilon_n l_n, \epsilon_1 \alpha_1, \ldots, \epsilon_n \alpha_n),$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$.

Thus replacing any inequality $l_i(x) \leq \alpha_i$ by the inequality $l_i(x) \geq \alpha_i$ does not change the distribution.

Similarly, we suppose a probability distribution on the set of linear programming problems $(l_0, l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n)$. We assume that this distribution is continuous and symmetric with respect to changing signs of $l_i$ and $\alpha_i$ for $i > 0$ (we do not assume symmetry in $l_0$).

Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a collection of hyperplanes $H_i = \{x \in \mathbb{R}^d : l_i(x) = \alpha_i\}$ in $\mathbb{R}^d$. This set $\mathcal{H}$ is called a *hyperplane arrangement.* A non-empty polyhedron $P_\epsilon = \{x \in \mathbb{R}^d : \epsilon_i l_i(x) \leq \epsilon_i \alpha_i\}$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ is called a *cell* of the arrangement $\mathcal{H}$. A set $F \subset \mathbb{R}^d$ is called a *face* of $\mathcal{H}$ iff $F$ is a face of some cell $P_\epsilon$. The number of $k$-dimensional faces of $\mathcal{H}$ is denoted by $f_k(\mathcal{H})$. An arrangement $\mathcal{H}$ is called *simple* every $d$ hyperplanes have a common point and no $(d+1)$ hyperplanes intersect in one point.

**(40.2) Theorem.** *The number $f_k(\mathcal{H})$ of $k$-dimensional faces of a simple $d$-dimensional arrangement $\mathcal{H}$ with $n$ hyperplanes depends on $k, d$ and $n$ only (it will be denoted by $f_k(d, n)$). Furthermore,*

$$f_k(d, n+1) = f_k(d, n) + f_{k-1}(d-1, n) + f_k(d-1, n).$$

*In particular,*

$$f_0(d, n) = \binom{n}{d} \quad \text{and} \quad f_d(d, n) = \sum_{i=0}^{d} \binom{n}{i}.$$

**Proof.** We use induction on $d$. If $d = 1$ then every simple arrangement consists of $n$ distinct points, so $f_0(1,n) = n$, $f_1(1,n) = n+1$ and $f_k(1,n) = 0$ for $k > 1$. Suppose that $d > 1$. Now we use induction on $n$. We have $f_{d-1}(d,1) = 1$, $f_d(d,1) = 2$ and $f_k(d,1) = 0$ for $k < d-1$. Let us add an $(n+1)$-st hyperplane to a simple arrangement $\mathcal{H}_n$ with $n$ hyperplanes so as to get a simple arrangement of $\mathcal{H}_{n+1}$ with $(n+1)$ hyperplanes. The $(n+1)$-st hyperplane either intersects a $k$-dimensional face $F$ of $\mathcal{H}_n$ or it does not. In the first case (we use that $\mathcal{H}_{n+1}$ is simple) it cuts $F$ into two $k$-dimensional faces and the intersection $F \cap H_{n+1}$ is a $(k-1)$-dimensional face of a simple $(d-1)$-dimensional arrangement in $H_{n+1} \cap H_i : i = 1, \ldots, n$. And we also have $f_k(d-1,n)$ of new $k$-dimensional faces on $H_{n+1}$. Thus we get $f_k(d,n+1) = f_k(d,n) + f_{k-1}(d-1,n) + f_k(d-1,n)$. The formula for $f_d(d,n)$ follows by this recursion and the initial conditions. Since the intersection of every $d$ hyperplanes is a vertex of $\mathcal{H}$ and every vertex is the intersection of some $d$ hyperplanes we get the formula for $f_0(d,n)$. $\square$

PROBLEMS.

1. The number $f_k^C(d,n)$ of bounded $k$-dimensional faces of a simple $d$-dimensional arrangement with $n$ hyperplane depends on $k, d$ and $n$ only. Furthermore, $f_k^C(d,n+1) = f_k^C(d,n) + f_{k-1}^C(d-1,n) + f_k^C(d-1,n)$. In particular, the number of polytopal cells in a simple arrangement of $n$ hyperplanes in $\mathbb{R}^d$ is equal to $\binom{n-1}{d}$.

2. Find general formulas for $f_k(d,n)$ and $f_k^C(d,n)$.

3. Let $\mathcal{H}$ be any arrangement of $n$ hyperplanes in $\mathbb{R}^d$. Then $f_k(\mathcal{H}) \le f_k(d,n)$.

4*. The characteristic function $\chi_S$ of a set $S \subset \mathbb{R}^d$ is the function $\chi_S : \mathbb{R}^d \longrightarrow \mathbb{R}$:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $\mathcal{H}$ be an arrangement of $n$ hyperplanes $H_1, \ldots, H_n$ in $\mathbb{R}^d$. For every $i = 1, \ldots, n$ consider the characteristic functions $\chi_i^+, \chi_i^-$ and $\chi_i^0$ of the two open halfspaces bounded by $H_i$ and the hyperplane $H_i$ itself.

a) The characteristic function of every cell is a polynomial of degree $\le d$ in $\chi_i^+, \chi_i^-$ and $\chi_i^0$;

b) The characteristic function of every face of $\mathcal{H}$ is a polynomial of degree $\le d$ in $\chi_i^+, \chi_i^-$ and $\chi_i^0$;

c) Any function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ that can be represented as a polynomial in $\chi_i^+, \chi_i^-$ and $\chi_i^0$ can be represented as a polynomial in $\chi_i^+, \chi_i^-$ and $\chi_i^0$ of degree $\le d$.

See A.N. Varchenko and I.M. Gel'fand, On Heaviside functions of a configuration of hyperplanes, *Functional Analysis and its Applications*, **21**(1987), N 4, 225–270.

5. A simple arrangement of more than $(d+1)$ hyperplanes in $\mathbb{R}^d$ contains a cell which is a simplex.

*Hint:* Consider a pair $\{v, H\}$, where $v$ is a vertex of the arrangement, $H$ is a hyperplane of the arrangement $v \notin H$ and $v, H$ has the smallest non-zero distance from $v$ to $H$.

6*. What is the smallest possible number of simplicial cells in a simple $d$-dimensional arrangements of $n$ hyperplanes?

7. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be an arrangement of $n$ hyperplanes in $\mathbb{R}^d$. Suppose that $0 \in H_i$ for all $i \in H_i$. There exists a cell in $\mathcal{H}$ which has not more than $d$ facets.

8. Let us consider the arrangement of $\binom{d}{2}$ hyperplanes in $\mathbb{R}^d$ given by the equations $x_i = x_j$ for $i \neq j$. There exist precisely $d!$ cells and every cell has $d$ facets.

**(40.3) Theorem.** *Let us consider a random system of $n$ linear inequalities in $\mathbb{R}^d$ assuming that the probability distribution satisfies (40.1.1)-(40.1.2). Then this system is feasible with the probability*

$$2^{-n} \sum_{i=0}^{d} \binom{n}{i}.$$

*The average number of vertices of a polyhedron given by a random system of $n$ linear inequalities in $\mathbb{R}^d$ is*

$$2^{d-n} \binom{n}{d}.$$

*In a random linear program $(l_0, l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n)$ the maximal value of $l_0$ is equal to $+\infty$ with the probability*

$$2^{-n} \sum_{i=0}^{d-1} \binom{n-1}{i}$$

*and is equal to $-\infty$ with the probability*

$$2^{-n} \sum_{i=d+1}^{n} \binom{n}{i}.$$

**Proof.** Because of (40.1.1), with probability 1 a random system of linear inequalities $l_i(x) \leq \alpha_i$ gives rise to a simple arrangements of hyperplanes. By Theorem 40.2 every simple arrangement gives rise to precisely $\sum_{i=0}^{d} \binom{n}{i}$ feasible systems $\epsilon_i l_i(x) \leq \epsilon_i \alpha_i$, $\epsilon_i \in \{-1, 1\}$. Furthermore, the total number of systems is $2^n$. By the assumption (40.1.2) all these $2^n$ instances have equal probability density. Hence the first estimate follows. Each of $\binom{n}{d}$ vertices of the arrangement is a vertex of precisely $2^d$ cells of the arrangement (we consider $d$ hyperplanes containing the vertex and $2^d$ regions bounded by them). Hence the second estimate follows. Let us prove the third estimate. Let us consider a random instance $(l_0, l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n)$ of a linear programming problem. Consider the hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_n\}$: $H_i = \{x \in \mathbb{R}^d : l_i(x) = \alpha_i\}$, $i = 1, \ldots, n$. Let $H_0(\beta) = \{x \in \mathbb{R}^d : l_0(x) = \beta\}$ be a hyperplane. If $H_0(\beta_1)$ and $H_0(\beta_2)$ for $\beta_1 < \beta_2$ intersect a cell of $\mathcal{H}$ then $H_0(\beta)$ intersects the cell for any $\beta_1 < \beta < \beta_2$. Therefore, there exists some $\beta_0$ such that $H_\beta$ intersects a cell of $\mathcal{H}$ for all sufficiently large $\beta$ iff it intersects the cell for $\beta = \beta_0$. Thus the number of cells of $\mathcal{H}$, where $l_0$ grows indefinitely is equal to the number of cells on the $(d-1)$-dimensional arrangement $H_i \cap H_0(\beta_0)$. Because of (40.1.1) this arrangement is simple with probability 1. Hence the number of cells of $\mathcal{H}$, where $l_0$ is unbounded is equal to $\sum_{i=0}^{d-1} \binom{n-1}{i}$. Then the second estimate follows. The last estimates follows by computing the probability of an infeasible region $l_i(x) \leq \alpha_i$ (thus follows by the first estimate). $\square$

PROBLEMS.

1. Fix $\alpha > 2$ and consider a random system of $n$ linear inequalities. If $n \geq \alpha d$ then the probability that this system in infeasible tends to 1 (exponentially fast) as $d$ tends to infinity.

2. Fix $\alpha < 2$ and consider a random system of $n$ linear inequalities in $\mathbb{R}^d$. If $n \leq \alpha d$ then the probability that this system is feasible tends to 1 (exponentially fast) as $d$ tends to infinity.

3. Prove that the average number of vertices of a polyhedron given by a random system of $n$ linear inequalities in $\mathbb{R}^d$ is bounded by a function of $d$ only.

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Wednesday, March 22

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## 41. Cooptimal Paths, Average Case

Let $P \subset \mathbb{R}^d$ be a polyhedron, $v \in P$ be a point, and $l : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. If $l(v) = \max\{l(x) : x \in P\}$, then we call $v$ *optimal* for $l$. Clearly, if a function has the unique optimal point $v$ then $v$ is a vertex of $P$.

**(41.1) Theorem.** *Let $P \subset \mathbb{R}^d$ be a polyhedron and $l_0, l_1 : \mathbb{R}^d \longrightarrow \mathbb{R}$ be linear functions. Suppose that $v_0$ is the unique optimal point for $l_0$ and $v_1 \neq v_0$ is the unique optimal point for $l_1$. For $\alpha \in [0,1]$ let $l_\alpha = (1-\alpha)l_0 + \alpha l_1$. We assume that the following conditions of "general position" are satisfied: for any $\alpha \in [0,1]$ there exist at most two optimal vertices of $l_\alpha$ on $P$ and $l_0$ or $l_1$ do not have a constant value on any face $F$ of $P$ of dimension greater than 0. Then there are finitely many $\alpha$-s for which there are exactly two optimal vertices of $P$ (let us call these $\alpha$ exceptional). Let us choose a point $\beta_i : i = 1, \ldots, m$ in every open interval between two adjacent exceptional values. Then for every $i = 1, \ldots, m$ there exists the unique optimal vertex $v_{\beta_i}$ of $l_{\beta_i}$ and $v_{\beta_i}$ is independent of the actual choice of $\beta_i$ inside the interval. Furthermore, the sequence $v_0, v_{\beta_1}, \ldots, v_{\beta_m}, v_1$ is a path in $G(P)$.*

**Proof.** Clearly, for any $\alpha \in [0,1]$ we have $l_\alpha(x) \leq \max\{l_0(x), l_1(x)\}$ and, therefore $\max\{l_\alpha(x) : x \in P\} < +\infty$. If $\alpha$ is exceptional and $c = \max\{l_\alpha(x) : x \in P\}$ then $F = \{x \in P : l_\alpha(x) = c\}$ is a face of dimension at least 1. If for two different $\alpha_1, \alpha_2$ the functions $l_{\alpha_1}$ and $l_{\alpha_2}$ have a constant value on $F$ then any function $l_\alpha$ has a constant value on $F$ (as a linear combination of $l_{\alpha_1}, l_{\alpha_2}$). But the functions $l_0, l_1$ do not have a constant value on $F$. The contradiction shows that for every face $F$ of $P$, dim $F \geq 1$ there exists at most one $\alpha$ such that $l_\alpha$ has a constant value on $F$. So there exist only finitely many exceptional $\alpha$. Now we observe that the set of $\alpha$ such that $l_\alpha$ has a given vertex $v$ of $P$ as its unique optimal point is open in $[0,1]$. Therefore the choice of $v_\beta$ is independent on the choice of $\beta$ between exceptional values.

Let us choose $\beta_1$ and $\beta_2$ separated by the unique exceptional value $\alpha$ and let $v_{\beta_1}$ and $v_{\beta_2}$ be the optimal vertices for $l_{\beta_1}, l_{\beta_2}$ correspondingly. We have to prove that $[v_{\beta_1}, v_{\beta_2}]$ is an edge of $P$. Let $c_\alpha = \max\{l_\alpha : x \in P\}$ and $F = \{x \in P : l_\alpha = c_\alpha\}$. Then $F$ is a face of $P$ and since $\alpha$ is exceptional we must have $\dim F = 1$, so $F$ is an edge. Now we have to prove that $v_{\beta_1}, v_{\beta_2} \in F$. To see this, we observe that $v_{\beta_1}$ is an optimal point for any $l_\beta$ from the interval $[\beta_1, \alpha)$ and by continuity arguments it should be optimal for $l_\alpha$. Similarly, $v_{\beta_2}$ should be optimal for $l_\alpha$. □

Such a path is called a *cooptimal path.* A vertex on the cooptimal path is called *cooptimal vertex.*

PROBLEMS.

1. Suppose that all $l_1$ grows indefinitely on $P$ and all other conditions of Theorem 41.1 are satisfied. Then our construction leads to a cooptimal path of vertices $v_0, v_1, \ldots, v_m$, where $v_m$ is the endpoint of an unbounded edge with infinitely increasing $l_1$ along this edge.

2. Show that the conditions of Theorem 41.1 are indeed of the "general position" type. That is, for every polyhedron $P \subset \mathbb{R}^d$ the set of pairs $(c_0, c_1) \in \mathbb{R}^d \times \mathbb{R}^d$ such that for every $\alpha \in \{0,1\}$ the linear function $l_\alpha(x) = \langle (1-\alpha)c_0 + \alpha c_1, x\rangle$ has at most two optimal vertices and $l_0$ and $l_1$ do not have constant values on any non-zero-dimensional face of $P$ contains an open and dense subset in $\mathbb{R}^d \times \mathbb{R}^d$.

Let us fix two linearly independent linear functions $f_0, f_1 : \mathbb{R}^d \longrightarrow \mathbb{R}$. Let us consider a random system of linear inequalities distributed according to (40.1.1)-(40.1.2) and the corresponding polyhedron $P = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i,\ i = 1, \ldots, n\}$. If $-\infty < \max\{f_0(x) : x \in P\} < +\infty$ (that is, $P$ is not empty and it is bounded from above with respect to $f_0$) then with probability 1 there exists a unique optimal vertex $v$ of $f_0$ and we can construct a cooptimal path either to an edge with indefinitely increasing $f_1$ or to the optimal vertex $v_1$ of $f_1$. We are interested in estimating the average length of this path, counting the average over those polyhedra only for which the cooptimal path exists.

**(41.2) Theorem.** *The average length of the cooptimal path taken over non-empty polyhedra bounded from above by $f_0$ and generated by a random system of $n$ linear inequalities in $d$ variables does not exceed*

$$(d+1)\binom{n}{d} \Big/ \sum_{i=d}^{n-1} \binom{n-1}{i}.$$

*The average taken over bounded polyhedra (polytopes) only does not exceed*

$$(d+1)\binom{n}{d} \Big/ \binom{n-1}{d} = \frac{n(d+1)}{n-d}.$$

**Proof.** Let us pick a system $(l_1, \dots, l_n, \alpha_1, \dots, \alpha_n)$ at random. Then with probability 1 the hyperplane arrangement $H_i = \{x \in \mathbb{R}^d : l_i(x) = \alpha_i\}$ is simple in $\mathbb{R}^d$ and for every choice of $\epsilon = (\epsilon_1, \dots, \epsilon_n), \epsilon_i \in \{-1, 1\}$ the system of linear inequalities $\epsilon_i l_i \leq \epsilon_i \alpha_i, i = 1, \dots, n$ appears with the same probability density. Let us choose a vertex $v$ of the arrangement. There exist $2^d$ cells that contain $v$. These cells $P_\epsilon$ are defined as follows. Let $I_v = \{i : l_i(v) = \alpha_i\}$, so $|I_v| = d$. Then $P_\epsilon = \{x \in \mathbb{R}^d : \epsilon_i l_i(x) \leq \epsilon_i \alpha_i\}$, where $\epsilon_i \in \{-1, 1\}$ are chosen arbitrarily for $i \in I_v$ and $\epsilon_i$ is uniquely determined by

$$\epsilon_i = \begin{cases} 1 & \text{if } l_i(v) < \alpha_i, \\ -1 & \text{if } l_i(v) > \alpha_i \end{cases}$$

for $i \notin I_v$. Let us count in how many of these $2^d$ cells $P_\epsilon$ $v$ may be a cooptimal vertex. If $v$ is a cooptimal vertex in $P_\epsilon$ then $v$ is optimal in $P_\epsilon$ for $f_\alpha = (1 - \alpha) f_0 + \alpha f_1$ for some $\alpha \in [0, 1]$. Then $f_\alpha$ belongs to the cone generated by $\epsilon_i l_i : i \in I_v$, that is,

$$f_\alpha = \sum_{i \in I_v} \lambda_i (\epsilon_i f_i), \ \ \lambda_i \geq 0.$$

Thus the number of cells $P_\epsilon \ni v$ where $v$ is cooptimal does not exceed the number of different cones among $2^d$ cones $K_\epsilon = \text{co}\{\epsilon_i l_i, i \in I_v\}$, $\epsilon_i \in \{-1, 1\}$ which have a non-empty intersection with the straight line $L = \{(1 - \alpha) f_0 + \alpha f_1 : \alpha \in \mathbb{R}\}$ passing through the two given linear functions. These $2^d$ cones $K_\epsilon$ are bounded in $\mathbb{R}^d$ by $d$ hyperplanes, say, $G_1, \dots, G_d$ and with probability 1 the line $L$ intersects them in $d$ different points. Therefore with probability 1 the line $L$ intersects exactly $(d + 1)$ of these cones and, therefore, the number of cells $P_\epsilon$ where $v$ is cooptimal does not exceed $(d + 1)$. Since the total number of vertices in the arrangement is $\binom{n}{d}$ we get that the total length of all cooptimal paths in all cells does not exceed $(d + 1)\binom{n}{d}$. Dividing this by the number of non-empty cells with $\max f_0 < +\infty$ (see Theorem 40.3) or by the number of polytopal cells (see Problem 1, Section 40.2) we get the desired estimates. □

PROBLEMS.

These problems concern with monotone paths on polyhedra, that is the paths where the value of a given linear function increases. It seems that they are somewhat worse than cooptimal paths but that may seem because of our inability to study them in a reasonable way.

1. Consider the system of linear inequalities in $\mathbb{R}^d$

$$0 \leq x_1 \leq 1, \quad \epsilon x_1 \leq x_2 \leq 1 - \epsilon x_1, \dots, \epsilon x_k \leq x_{k+1} \leq 1 - \epsilon x_k, \dots, \epsilon x_{d-1} \leq x_d \leq 1 - \epsilon x_{d-1}.$$

If $\epsilon > 0$ is sufficiently small then the set of solutions is a polytope $P$ (called the Klee - Minty Cube), combinatorially isomorphic to the $d$-dimensional cube. There exists a path in $G(P)$ which visits all the $2^d$ vertices of $P$ such that the value of the function $l(x) = x_d$ strictly increases along the way.

2. Fix a non-zero linear function $l_0 : \mathbb{R}^d \longrightarrow \mathbb{R}$. For a random system of $n$ linear inequalities $(l_1, \ldots, l_n, \alpha_1, \ldots, \alpha_n)$ in $d$ variables consider the average length of the longest monotone path starting at the vertex $v$ of the polyhedron $P = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i\}$ with the smallest value of $l_0$ and leading either to an unbounded edge where $l_0$ infinitely increases or to the vertex where $l_0$ has the maximal value (we take the average over those polyhedra only, where the construction makes sense). Show that the average can be bounded by a function which is linear in $n$.

3*. Can you bound the average in Problem 2 by a polynomial in $d$?

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Friday, March 24

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## 42. Overview of Linear Programming Algorithms

**(42.1) Combinatorial algorithms.** Combinatorial algorithms do not take into account numerical issues (size of the coefficients, etc.): they operate with data as with symbols and, in principle, applicable to problems over any ordered field.

**(42.1.1) Classical Simplex Method.** We start from a vertex of a polyhedron and follow a path with the objective function increasing on every step. The initial vertex can be constructed as indicated in the following problem.

PROBLEM.

1. Let $P = \{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \alpha_i\}$, $c_i \in \mathbb{R}^d$ be a (possibly empty) polyhedron. Let $I = \{i : \alpha_i < 0\}$. Consider a polyhedron $Q$ given by the system of linear inequalities: $\langle c_i, y - z\rangle + u_i = \alpha_i : i \notin I$, $\quad \langle c_i, y - z\rangle + u_i - w_i = \alpha_i : i \in I$, where $y = (y_1, \ldots, y_d)$, $z = (z_1, \ldots, z_d)$ and $y_j, z_j \geq 0, j = 1, \ldots, d$, $u_i \geq 0 : i = 1, \ldots, n$ and $w_i \geq 0 : i \in I$.

Then the point $y_j = z_j = 0 : j = 1, \ldots, d$, $u_i = 0 : i = 1, \ldots, n$ and $w_i = -\alpha_i$ for $i \in I$ is a vertex of $Q$ and the minimal value of the function $\sum_{i \in I} w_i$ on $Q$ is 0 iff $P$ is not empty.

**(42.1.2) Parametric Simplex Method.** Suppose we are given a linear programming problem:

Find $\max l_0(x)$ where $x \in \mathbb{R}^d$ and $l_i(x) \leq \alpha_i$, $i = 1, \ldots, n$. We assume that the configuration of hyperplanes $H_i = \{x \in \mathbb{R}^d : \ l_i(x) = \alpha_i\}$ is simple.

On Step 0 we find the unique point $v_0$ such that $l_i(v_0) = \alpha_i$, $i = 1, \ldots, d$.

On Step $k$ we start with a vertex $v_k$ of the polyhedron $P_k = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i : \ i = 1, \ldots, d + k\}$. We construct a linear function $L_k$ whose maximum on $P_k$ is attained in the unique point $v_k$. To construct $L_k$ we may proceed as follows: let $I_v$ be the set of indices of hyperplanes containing $v_k$, so $l_i(v_k) = \alpha_i : \ i \in I_v$. Set $L_k = \sum_{i \in I(v)} l_i(v)$. Then we check if $l_{k+1}(v_k) \leq \alpha_k$. If yes, we go to Step $k + 1$. Otherwise, we set $L_\alpha = (1 - \alpha)L_k - \alpha l_{k+1}$

and construct a cooptimal path to the first point $v_{k+1} \in P_{k+1}$ with $l_{k+1}(v_{k+1}) \leq \alpha_{k+1}$ and go to Step $k+1$. If such a vertex $v_{k+1}$ does not exists then the system is infeasible.

On Step $n+1$ we construct a cooptimal path on $P_{n-d} = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i, i = 1, \ldots, n\}$ leading to the optimal vertex of $l_0$ or to an unbounded edge with increasing $l_0$.

PROBLEMS.

1. Let us call a *pivot* every move along the cooptimal path which we perform in the course of the algorithm. Prove that the average number of pivots can be bounded by a function in $d$ alone.

2*. On the $k$-th step of the algorithm let us construct the function $L_k$ as follows. Let us list the indices of hyperplanes containing $v_k$ in the increasing order: $i_1 < i_2 < \ldots < i_d$, so $l_{i_j}(v_k) = \alpha_{i_j}$ for $j = 1, \ldots, d$. Let $L_k = \epsilon l_{i_1} + \epsilon^2 l_{i_2} + \ldots + \epsilon^d l_{i_d}$, where $\epsilon$ is very small but positive. Then the average number of pivots is bounded by a polynomial in $d$.

3*. Let us consider a system of linear inequalities $\{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i : i = 1, \ldots, n\}$. Let us permute the indices $i$ of inequalities at random and apply the algorithm. What is the average number of pivots? (taken over all $n!$ orderings of the same set of inequalities).

**(42.1.3) Randomized simplex algorithms.**

Let $P = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i, i = 1, \ldots, n\}$ be a polyhedron and $l_0 : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a linear function. As usual, we assume that the hyperplane arrangement $H_i = \{x \in \mathbb{R}^d : l_i(x) = \alpha_i\}$, $i = 0, \ldots, n$ is simple. Consider, for example, the following algorithm suggested by G. Kalai.

We start with a vertex $v$ of $P$. If there are no edges coming out of $v$ with the increasing value of $l_0$ then $v$ is optimal and we are done. If there is only one edge with increasing value of $l_0$ we move along this edge. Otherwise, we take a random facet $l_j(x) = \alpha_j$ among $d$ facets $l_i(x) = \alpha_i$ containing $v$ and solve the $(d-1)$-dimensional problem with $(n-1)$ inequalities on that facet applying the same procedure recursively. If we get a vertex $w$ we let $v := w$ and iterate the process again.

PROBLEM.

1*. Let us call a *pivot* every move along the edge in the algorithm. Show that there is an upper bound $E(d,n)$ for the average number of pivots (average is taken over all possible ways to run the algorithm with the same polyhedron $P$ and function $l_0$) which satisfies the following recursion $E(d,n) \leq \max\Big\{1 + E(d, n-1), E(d-1, n-1) + \frac{1}{d} \sum_{i=1}^{\max\{d, n-d\}} E(d, n-i)\Big\}$. Show that this implies that $E(d,n) \leq n^{O(\sqrt{d})}$.

See G. Kalai, A subexponential randomized simplex algorithm, in: *Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC'92)*, ACM Press, 475–482.

**(42.2) Continuous algorithms.** These algorithms heavily rely on the numerical aspect of the problem. The stopping rule of the algorithm depends on some a priori information on the numerical size of the coefficients, etc. The Ellipsoid Method is described in Section 14. Another approach, known under a unifying name *Interior point method* uses the following construction. Let $P_\alpha$ be a (possibly empty) polytope given by a system of linear inequalities

$P_\alpha = \{x \in \mathbb{R}^d : l_i(x) \leq \alpha_i\}$. We want to check whether the polytope is empty. We choose sufficiently large $\beta_i$ so that the polytope $P_\beta = \{x \in \mathbb{R}^d : l_i(x) \leq \beta_i\}$ is not empty and we can easily find a point, say, $x_0$ in its interior. For a point $x$ in the interior of $P_\beta$ we can define the value $f_\beta(x) = -\sum_{i=1}^n \ln\bigl(\beta_i - l_i(x)\bigr)$. Then $f(x)$ is convex (prove!) and has the unique minimum point $x_\beta$ on $P_\beta$, called *center*. To construct $x_\beta$ we use Newton's method, say. Now we deform $P_\beta$ into $P_\alpha$ and follow the path of its center $x_\beta$.

PROBLEM.

1*. Let $V = \{v_1, \ldots, v_n\}$ be a finite set of points in $\mathbb{R}^d$. The function $h_V(c) = \ln \sum_{i=1}^n \exp\{\langle c, v_i\rangle\}$ is convex and

$$\int_{\mathbb{R}^d} \det\Bigl(\frac{\partial^2}{\partial c_i \partial c_j} h_V(c)\Bigr)_{i,j=1,\ldots,d} = \text{vol } P,$$

where $P$ is the convex hull of $V$. Furthermore, $h_{V_1+V_2}(c) = h_{V_1}(c) + h_{V_2}(c)$, where "+" denotes the Minkowski sum.

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Monday, March 27

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## 43. The Löwner - John Ellipsoid

**(43.1) Definitions.** A set $K \subset \mathbb{R}^d$ is called *centrally symmetric* iff $x \in K \Longleftrightarrow -x \in \mathbb{R}^d$. We consider some general metric properties of centrally symmetric convex bodies. An *ellipsoid* $E \subset \mathbb{R}^d$ is the image of the ball $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ under a linear non-degenerate transform. So we consider only centrally symmetric ellipsoids. If $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a non-degenerate linear transform and $E = T(B)$ is an ellipsoid then vol $E = (\det T)\cdot$ vol $B$. The volume of the $d$-dimensional unit ball $B_d$ we denote by $\kappa_d$.

PROBLEMS.

1°. An ellipsoid $E \subset \mathbb{R}^d$ can be defined as $E = \{x \in \mathbb{R}^d : \langle x, Qx\rangle \leq 1\}$, where $Q$ is a $d \times d$ positive definite matrix. Furthermore, vol $E = (\det^{-1/2} Q)\kappa_d$.

2°. For every ellipsoid $E \subset \mathbb{R}^d$ there is an orthogonal transformation of coordinates in $\mathbb{R}^d$ such that $E$ can be written in the form

$$\lambda_1 x_1^2 + \ldots + \lambda_d x_d^2 \leq \rho^2,$$

where $\rho, \lambda_i \geq 0$. Furthermore, vol $E = \kappa_d(\lambda_1 \ldots \lambda_d)^{-1/2}\rho^d$. The intersection of the coordinate axe $x_i$ and $E$ is called an *axe* of the ellipsoid $E$. If $l_1, \ldots, l_d$ are the lengths of axes of $E$ then vol $E = 2^{-d}\kappa_d l_1 \ldots l_d$.

3. If $E = \{x \in \mathbb{R}^d : \langle x, Qx\rangle \leq 1\}$ is an ellipsoid than $E^\circ = \{x \in \mathbb{R}^d : \langle x, Q^{-1}x\rangle \leq 1\}$ is an ellipsoid. Furthermore, (vol $E$) (vol $E^\circ$)$= \kappa_d^2$.

4*. If $K \subset \mathbb{R}^d$ is a centrally symmetric convex compact set then (vol $K$) (vol $K^\circ$)$\leq \kappa_d^2$.

5*. Is it true that the product (vol $K$) (vol $K^\circ$) attains its minimum over all centrally symmetric convex compact sets $K \subset \mathbb{R}^d$ when $K, K^\circ$ is a pair consisting of the standard octahedron and the standard cube?

**(43.2) Theorem.** *Let $K \subset \mathbb{R}^d$ be a d-dimensional convex compact set which is centrally symmetric about the origin. Then there exists an ellipsoid $E$ such that $E \subset K \subset \sqrt{d}E$.*

Such an ellipsoid is called a *Löwner - John Ellipsoid.*

**Proof.** Let us choose an ellipsoid $E \subset K$ of the largest volume among all the ellipsoids inscribed in $K$. To see that $E$ exists we observe that the set $A$ of possible axes $(a_1, \ldots, a_d)$ of an ellipsoid $E \subset K$ is compact and therefore the continuous function vol $E = 2^{-d}\kappa_d \|a\|_1 \ldots \|a_d\|$ attains its maximum on $A$.

Applying a suitable linear transform to $\mathbb{R}^d$ we may assume that $E = B$ is the unit ball, $E = \{x \in \mathbb{R}^d : x_1^2 + \ldots + x_d^2 \leq 1\}$. Thus we have to prove that there is no point $x \in K$ such that $\|x\| > \sqrt{d}$. Suppose that there is such a point. Without loss of generality we assume that $x$ has the form $(a, 0, \ldots, 0)$, where $|a| > \sqrt{d}$. We are going to obtain a contradiction. For $0 < \theta \leq 1$ we consider the ellipsoid

$$E_\theta = \Big\{x \in \mathbb{R}^d : (\theta x_1)^2 + \ldots + x_d^2 \leq \frac{(\theta a)^2}{1 + \theta^2(a^2 - 1)}\Big\}.$$

Thus $E_1 = E$ and

$$\text{vol } E_\theta = \kappa_d \Big(\frac{\theta^2 a^2}{1 + \theta^2(a^2 - 1)}\Big)^{d/2} \theta^{-1}.$$

Furthermore,

$$\frac{d}{d\theta} \text{vol}(E_\theta)\bigg|_{\theta=1} = \kappa_d \Big(-1 + \frac{d}{a^2}\Big) < 0$$

since we assumed that $a^2 > d$. Thus if we prove that $E_\theta \subset K$ for any $\theta < 1$ we are going to obtain a contradiction (since we can slightly decrease $\theta$ and get a larger volume of an inscribed ellipsoid).

To prove that $E_\theta \subset K$ it is enough to prove that any section of $E_\theta$ by a two-dimensional plane $\Pi$ passing through the first coordinate axe is contained in $K$. Let us call the second coordinate of $\Pi$, say, $y$. Thus we have to prove that the ellipse

$$e_\theta = \Big\{(x_1, y) : (\theta x_1)^2 + y^2 \leq \frac{(\theta a)^2}{1 + \theta^2(a^2 - 1)}\Big\}$$

belongs to the section $C = \Pi \cap K$. The set $C$ is a convex compact set on the plane $\Pi = \{(x_1, y)\}$ which contains the disc $x_1^2 + y_2 \leq 1$ and two points $(a, 0)$ and $(-a, 0)$. Let us consider the rhomb $R$ with the vertices $(a, 0), (-a, 0), (0, \dfrac{a}{\sqrt{a^2 - 1}}), (0, -\dfrac{a}{\sqrt{a^2 - 1}})$. The disc $x_1^2 + y^2 \leq 1$ is inscribed in the rhomb and touches its sides in the points whose distance to the closest vertex $A^+ = (a, 0)$ or $A^- = (-a, 0)$ is equal to $\sqrt{a^2 - 1}$. Let $B^+, C^+, B^-, C^-$ be these touching points on the sides of the rhomb. It is enough to prove that the ellipse $e_\theta$ is contained in the union of the two triangles $C^+A^+B^+$, $C^-B^-A^-$ and the disc $x_1^2 + y_2 \leq 1$. Next, we observe that the ellipse $e_\theta$ is inscribed in the rhomb $R$. To see this, one may consider a linear transform of the plane $x_1 \longmapsto \theta x_1, y \longmapsto y$ which maps the rhomb $R$ onto the rhomb with the vertices $(\theta a, 0), (-\theta a, 0), (0, \frac{a}{\sqrt{a^2-1}}), (0, -\frac{a}{\sqrt{a^2-1}})$ and the ellipse $e_\theta$ into the disc $x_1^2 + y^2 \leq \frac{(\theta a)^2}{1+\theta^2(a^2-1)}$ which touches all four sides of the rhomb. Since $\theta < 1$ the touching points $C_\theta^+$ and $B_\theta^+$ of $e_\theta \subset R$ are closer to $A^+$ than $C^+$ and $B^+$ whereas the touching points $B_\theta^-$ and $C_\theta^-$ are closer to $A^-$ than the points $B^-$ and $C^-$. Now the picture is a sort of clear. $\square$

PROBLEMS.

1. If $E \subset K$ is an ellipsoid of the largest volume inscribed in $K$ then the ellipsoid $E^\circ \supset K^\circ$ is an ellipsoid of the smallest volume which contains $K^\circ$ ($K$ is convex compact and $E, K$ are centrally symmetric).

2. There exists the unique ellipsoid of the largest volume contained in a centrally symmetric convex compact set $K$ (cf. Problems from Section 14.3) This ellipsoid is called *the John* ellipsoid. The ellipsoid of the smallest volume containing $K$ is called the *Löwner Ellipsoid*.

3°. Let $I_d$ be the standard $d$-dimensional cube, $I_d = \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1 : i = 1, \ldots, d\}$. Let $r$ be the radius of the ball inscribed in $I_d$ and $R$ be the radius of the ball circumscribed about $I_d$. Then $R/r = \sqrt{d}$.

4°. Let $O_d$ be the standard octahedron in $\mathbb{R}^d$, that is

$O_d = \text{conv}\{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. Let $r$ be the radius of the ball inscribed in $O_d$ and $R$ be the radius of the ball circumscribed about $O_d$. Then $R/r = \sqrt{d}$.

5. The ellipsoid of the largest volume inscribed in the standard cube (standard octahedron) is a ball. The ellipsoid of the smallest volume containing the standard octahedron (standard cube) is a ball.

*Hint:* Use Problem 2. Let $E \subset \mathbb{R}^d$ be the ellipsoid of the largest (resp. smallest) volume contained in (resp. containing) the standard cube $I_d$ or the octahedron $I_d$. Then every symmetry of $I_d$ or $O_d$ must map $E$ onto itself. There are no proper subspaces in $\mathbb{R}^d$ which are invariant under the symmetries of the cube (resp. octahedron).

6. Let $p$ be a norm in $\mathbb{R}^d$, that is a function $p : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$p(x) \geq 0$ and $p(x) = 0 \Longrightarrow x = 0$;

$p(x+y) \leq p(x) + p(y)$;

$p(-x) = -p(x)$. Then the set $K_p = \{x \in \mathbb{R}^d : p(x) \leq 1\}$ is a convex compact centrally symmetric set containing the origin in its interior.

Furthermore, if $K \subset \mathbb{R}^d$ is a centrally symmetric convex compact set containing the origin in its interior then the function $p_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$ is a norm on $\mathbb{R}^d$ such that $K_{p_K} = K$.

7. The *Banach - Mazur* distance $d(K_1, K_2)$ between two convex compact centrally symmetric sets $K_1, K_2 \subset \mathbb{R}^d$ containing the origin in their interiors is defined as

$d(K_1, K_2) = \inf\{\log \|T\| + \log \|T^{-1}\|, T : \mathbb{R}^d \longrightarrow \mathbb{R}^d\}$, where a linear operator $T$ is considered as the operator from $\mathbb{R}^d$ with $p_{K_1}$ norm to $\mathbb{R}^d$ with $p_{K_2}$ norm. Prove that $d(K_1, K_2) = d(K_2, K_1)$, that $d(K_1, K_2) \leq d(K_1, K) + d(K, K_2)$, that $d(K_1, K_2) \geq 0$ and that $d(K_1, K_2) = 0$ iff there exists a non-degenerate linear transform $T$ such that $T(K_1) = K_2$.

8. From Theorem 43.2 deduce that $d(K_1, K_2) \leq \log d$, for any two $K_1, K_2 \subset \mathbb{R}^d$ (again compact centrally symmetric and containing the origin in the interiors).

9*. Compute $d(I_d, O_d)$, where $I_d$ is the standard cube whereas $O_d$ is the standard octahedron.

10*. There exists a constant $c$ such that for all sufficiently large $d$ there exist a pair of centrally symmetric convex compact sets $K_1, K_2 \subset \mathbb{R}^d$ containing the origin in their interiors such that $d(K_1, K_2) \geq \log d - c$.

See E.D. Gluskin, Probability in the geometry of Banach spaces, in: *Proceedings of the International Congress of Mathematicians (Berkeley, CA 1986)*, Amer. Math. Soc., Providence, RI, 1987, 924–938.

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Wednesday, March 29

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## 44. The Lower Bound Theorem

**(44.1) A bit of spherical geometry.** Let $S^{d-1} = \{x \in \mathbb{R}^d : \ \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$. We introduce a metric on $S^{d-1}$, namely $\operatorname{dist}(x, y) = \arccos\big(\langle x, y\rangle / \|x\|\|y\|\big)$ is the angle between $x$ and $y$. We also introduce a probabilistic rotation invariant measure $\nu$ on the sphere, that is $\nu(S^{d-1}) = 1$ and $\nu(g(X)) = \nu(X)$ for every Borel set $X \subset S^{d-1}$ and any orthogonal linear transform $g$.

If $X \subset S^{d-1}$ is a finite set of points on the sphere and $y \in S^{d-1}$ is a point then we let $\operatorname{dist}(y, X) = \min\{\operatorname{dist}(x, y) : x \in X\}$. Let $y \in S^{d-1}$. The *spherical cap* of radius $r$ is the set of points $C(y, r) = \{x \in S^{d-1} : \operatorname{dist}\ (y, x) \leq r\}$. Thus $C(y, 0) = \{y\}$, $C(y, \pi) = S^{d-1}$ and $C(y, \pi/2)$ is a hemisphere.

PROBLEMS

1. Check that dist is indeed a metric on $S^{d-1}$.

2°. Compute $\nu\big(C(y, \pi/2)\big)$.

3°. Prove that $\nu\big(C(y, \pi/2 - \epsilon)\big) + \nu\Big(C(y, \pi/2 + \epsilon)\Big) = 1$ for any $0 \leq \epsilon \leq \pi/2$.

**(44.2) Theorem.** *For $1 > \epsilon > 0$ for spherical caps on $S^{d-1}$ we have*

$$\nu\Big(C(y, \pi/2 - \epsilon)\Big) \leq 2 \exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}, \quad \nu\Big(C(y, \pi/2 + \epsilon)\Big) \geq 1 - 2 \exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}.$$

**Proof.** It is enough to prove the first inequality only (see Problem 3° above). We have

$$\nu\Big(C(y,\pi/2-\epsilon)\Big)=\left(\int_0^{\pi/2-\epsilon}\sin^{d-2}\phi\ d\phi\right)\Big/\left(\int_0^{\pi}\sin^{d-2}\phi\ d\phi\right).$$

Let us estimate the integrals. The first integral

$$\int_0^{\pi/2-\epsilon}\sin^{d-2}\phi\ d\phi=\int_\epsilon^{\pi/2}\cos^{d-2}\phi\ d\phi\le\int_\epsilon^{\pi/2}\exp\Big\{\frac{-(d-2)\phi^2}{2}\Big\}\ d\phi$$

$$\le\exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}\int_0^{+\infty}\exp\Big\{-\frac{(d-2)\phi^2}{2}\Big\}d\phi=\Big(\frac{\pi}{2(d-2)}\Big)^{\frac{1}{2}}\exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}.$$

We used that $\cos\phi\le\exp\{-\phi^2/2\}$ for $0\le\phi\le\pi/2$ and that $\phi^2-\epsilon^2\ge(\phi-\epsilon)^2$ for $\phi\ge\epsilon$.

Let us estimate the second integral. We use the recursion (integrating by parts)

$$\int_0^{\pi}\sin^{d-2}\phi\ d\phi=\frac{d-3}{d-2}\int_0^{\pi}\sin^{d-4}\phi\ d\phi$$

together with the initial condition

$$\int_0^{\pi}\sin^{k}\phi\ d\phi=\begin{cases}\pi & \text{for } k=0,\\ 2 & \text{for } k=1.\end{cases}$$

to get that

$$\int_0^{\pi}\sin^{d-2}\phi\ d\phi\ge\frac{(d-3)(d-5)\ldots}{(d-2)(d-4)\ldots}\ge\left(\frac{(d-3)(d-4)\ldots}{(d-2)(d-3)\ldots}\right)^{1/2}\ge(d-2)^{-\frac{1}{2}},$$

from which the proof follows. □

The following “Lower Bound Theorem for Centrally Symmetric Polytopes” was proven in the paper T. Figiel, J. Lindenstrauss and V.D. Milman, The dimension of almost spherical sections of convex bodies, *Acta Mathematica*, **139**(1977), 53–94.

**(44.3) Theorem.** *There exists a constant $\gamma>0$ such that for any centrally symmetric $d$-dimensional polytope $P$ with $|V|$ vertices and $|F|$ facets one has*

$$\ln|V|\cdot\ln|F|\ge\gamma d.$$

**Proof.** Let $P$ be a $d$-dimensional centrally symmetric polytope. Applying a suitable linear transform we may assume that $P$ contains the unit ball $B=\{x\in\mathbb{R}^d:\|x\|\le1\}$ and is contained in the ball $\sqrt{d}B$ (see Theorem 43.2). For $x\ne0$ let us denote by $\overline{x}=x/\|x\|$ the projection of $x$ onto the unit sphere $S^{d-1}$. Let $\mathcal{V}$ be the projection of the set of vertices

$V$ of $P$ onto $S^{d-1}$. Let us choose some $\epsilon > 0$ and consider the set $A_\epsilon \subset S^{d-1}$ defined as follows:

$$A_\epsilon = \Big\{ x \in S^{d-1} : \operatorname{dist}(x, \mathcal{V}) \leq \pi/2 - \epsilon \Big\}.$$

Then $A_\epsilon$ is the union on $|V|$ spherical caps $C\big(v, \pi/2 - \epsilon\big)$ centered at the projections of the vertices $v$ of $P$. Using Theorem 44.2 we get that

$$\nu(A_\epsilon) \leq 2|V| \exp\Big\{ -\frac{(d-2)\epsilon^2}{2} \Big\}.$$

For every $c \in S^{d-1} \setminus A_\epsilon$ the distance between the projection of every vertex $v$ of $P$ and $c$ is greater than $\pi/2 - \epsilon$. Since $P \subset \sqrt{d}B$ and hence $\|v\| \leq \sqrt{d}$ we get that

$$\max\{\langle c, x\rangle : x \in P\} = \max\{\langle c, v\rangle : \ v \in V\} \leq \sqrt{d}\cos(\pi/2 - \epsilon) = \sqrt{d}\sin\epsilon \leq \sqrt{d}\epsilon.$$

We want to choose an $\epsilon > 0$ so that $\nu(A_\epsilon) \leq 1/3$. We can take, say $\epsilon = \gamma_1 d^{-1/2} \ln^{1/2}|V|$, where $\gamma_1 > 0$ is a sufficiently large number. Then the set $C_1 = S^{d-1} \setminus A_\epsilon$ has measure at least $2/3$ and for every $c \in C_1$ we have that

$$\max\{\langle c, x\rangle : x \in P\} \leq \gamma_1 \ln^{\frac{1}{2}}|V|. \tag{44.3.1}$$

Suppose that $P$ is given by a system of linear inequalities $P = \Big\{ x \in \mathbb{R}^d : \langle l_i, x\rangle \leq \alpha_i, i = 1, \ldots, |F| \Big\}$ for some $l_i \in \mathbb{R}^d$. Since $P$ contains the unit ball we must have $\alpha_i > 0$ and, therefore, using a proper scaling we can assume that $\alpha_i = 1$ for all $i$. Since the point $x = l_i/\|l_i\| = \overline{l_i}$ is in $P$ we must have $\|l_i\| \leq 1$ for $i = 1, \ldots, |F|$. Thus

$$P = \Big\{ x \in \mathbb{R}^d : \langle l_i, x\rangle \leq 1, i = 1, \ldots, |F| \Big\} \quad \text{where} \quad \|l_i\| \leq 1, i = 1, \ldots, |F|.$$

Let $\mathcal{F} \subset S^{d-1}$ be set of projections $\overline{l_i}$. Let us choose some $\delta > 0$ and consider the set $B_\delta \subset S^{d-1}$ defined as follows:

$$B_\delta = \Big\{ x \in S^{d-1} : \operatorname{dist}(x, \mathcal{F}) \leq \pi/2 - \delta \Big\}.$$

By Theorem 44.2 we get

$$\nu(B_\delta) \leq 2|F| \exp\Big\{ -\frac{(d-2)\delta^2}{2} \Big\}.$$

Let us choose $c \in S^{d-1} \setminus B_\delta$. Since $\operatorname{dist}(c, \overline{l_i}) \geq \pi/2 - \delta$ and $\|l_i\| \leq 1$ we have that $\langle c, l_i\rangle \leq \sin\delta \leq \delta$. Therefore, the vector $x = \delta^{-1}c$ satisfies the system of linear inequalities $\langle l_i, x\rangle \leq 1$ and hence belongs to the polytope $P$. Hence

$$\max\{\langle c, x\rangle : x \in P\} \geq \delta^{-1}.$$

Now we want to choose $\delta > 0$ so that $\nu(B_\delta) \leq 1/3$. We can take, say, $\delta = \gamma_2 d^{-1/2} \ln^{1/2} |F|$, where $\gamma_2 > 0$ is a sufficiently large number. Then the measure of the set $C_2 = S^{d-1} \setminus B_\delta$ is at least $2/3$ and for every $c \in C_2$ we have that

$$\max\{\langle c, x \rangle : x \in P\} \geq \gamma_2^{-1} d^{\frac{1}{2}} \ln^{-\frac{1}{2}} |F|. \tag{44.3.2}$$

Since the sets $C_1 \subset S^{d-1}$ and $C_2 \subset S^{d-1}$ both have measure at least $2/3$ there should be a point $c \in C_1 \cap C_2$, so (44.3.1) and (44.3.2) both will be satisfied. Thus we have

$$\gamma_2^{-1} d^{\frac{1}{2}} \ln^{-\frac{1}{2}} |F| \leq \gamma_1 \ln^{\frac{1}{2}} |V|, \quad \text{or, in other words,}$$

$$\ln |V| \cdot \ln |F| \geq \gamma d,$$

where $\gamma = (\gamma_1 \gamma_2)^{-2}$. □

PROBLEMS.

1°. Let $P$ be the $d$-dimensional cube or the $d$-dimensional octahedron. Then for all sufficiently large $d$ the inequality of Theorem 44.3 is satisfied for any $\gamma < \ln 2$.

2*. Is it true that for any $\gamma < \ln 2$ the inequality of Theorem 44.3 is satisfied for all sufficiently large $d$?.

3. There exists $\gamma > 0$ such that for any $k \leq d$ and for any centrally symmetric $d$-dimensional polytope $P$ we have $\ln f_k \cdot \ln f_{d-1} \geq \gamma(d-k)$. (Here $f_i$ is the $f$-vector of $P$).

4. There exists $\gamma > 0$ such that for any $k \leq d$ and for any centrally symmetric $d$-dimensional polytope $P$ we have $\ln f_k \cdot \ln f_0 \geq \gamma k$.

5. There exists $\gamma > 0$ such that for any $k \leq l \leq d$ and for any centrally symmetric $d$-dimensional polytope $P$ we have $\ln f_l \cdot \ln f_k \geq \gamma(l-k)$.

6* A simplicial $d$-dimensional centrally symmetric polytope has at least $2^d$ facets.

7*. Is it true that any $d$-dimensional centrally symmetric polytope has at least $3^d$ faces?

8*. What is the maximal number of edges that a centrally symmetric 4-dimensional polytope can have?

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Friday, March 31

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## Solutions to Some Problems from the Second Set

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Monday, April 3

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## 45. An Average Linear Function on a Polytope. Examples

Let $P \subset \mathbb{R}^d$ be a polytope and $l(x) = \langle c, x \rangle$ be a linear function $l : \mathbb{R}^d \longrightarrow \mathbb{R}$. We are interested to estimate $\max\{l(x) : x \in P\}$ for a typical linear function $l$. To do that, we must choose a certain distribution on the set of linear functions $l$. We assume that the function $l$ is distributed with the spherical symmetry. Since a linear function is homogeneous, we may assume that we choose a vector $c$ at random from the unit sphere $S^{d-1}$.

**(45.1) Cube.** Let $I_d \subset \mathbb{R}^d$ be the standard cube defined by the system of linear inequalities $|x_i| \leq 1$, $i = 1, \ldots, d$.

Then for some $\gamma > 0$ the probability that

$$\gamma \frac{\sqrt{d}}{(\ln d)^{\frac{1}{2}}} \|c\| \leq \max\{\langle c, x \rangle : x \in I_d\} \leq \sqrt{d}\|c\|$$

tends to 1 as $d \longrightarrow +\infty$.

**Proof.** Let $V$ be the set of vertices of $I_d$. Then

$$\max\{\langle c, x \rangle : x \in I_d\} = \max\{\langle c, v \rangle : v \in V\} \leq \|c\|\|v\| \leq \sqrt{d}\|c\|,$$

so the upper bound is clear. We may write $I_d = \big\{x \in \mathbb{R}^d : \langle u_i, x \rangle \leq 1 : i = 1, \ldots, 2d\big\}$, where $u_i = e_i$ or $u_i = -e_i$ for the standard orthonormal basis of $\mathbb{R}^d$. Let $U = \{u_i : \ i = 1, \ldots, 2d\} \subset S^{d-1}$. For $\epsilon > 0$ let

$$U_\epsilon = \big\{x \in S^{d-1} : \operatorname{dist}(x, U) \leq \frac{\pi}{2} - \epsilon\big\}.$$

Then by Theorem 44.2 we have $\nu(U_\epsilon) \leq 4d \exp\big\{-\frac{(d-2)\epsilon^2}{2}\big\}$. Let us choose

$\epsilon = 2d^{-\frac{1}{2}}(\ln d)^{\frac{1}{2}}$. Then $\nu(U_\epsilon) \longrightarrow 0$ and $c \notin U_\epsilon$ with probability which tends to 1 as $d \longrightarrow +\infty$. If $c \notin U_\epsilon$ then $\operatorname{dist}(c, U) \geq \pi/2 - \epsilon$. If $c \notin U_\epsilon$ then $\langle c, u_i \rangle \leq \cos(\pi/2 - \epsilon) = \sin \epsilon \leq \epsilon$ for any $i$. Thus $\epsilon^{-1} c \in I_d$. Therefore $\max\{\langle c, x \rangle : x \in I_d\} \geq \langle c, \epsilon^{-1} c \rangle = \gamma \frac{\sqrt{d}}{\ln^{\frac{1}{2}} d}$. □

**(45.2) Octahedron.** Let $O_d \subset \mathbb{R}^d$ be the standard octahedron defined by the inequality $|x_1| + \ldots + |x_d| \leq 1$. Then for some $\gamma > 0$ the probability that

$$\gamma \frac{(\ln d)^{\frac{1}{2}}}{\sqrt{d}} \geq \max\{\langle c, x \rangle : x \in O_d\} \geq \frac{1}{\sqrt{d}}$$

tends to 1 as $d \longrightarrow +\infty$.

**Proof.** It is easy to see that the octahedron $O_d$ contains the inscribed ball $\|x\| \leq d^{-1/2}$. Therefore the lower estimate holds trivially. Let $V = \{v_i : i = 1, \ldots, 2d\}$ be the set of vertices of $O_d$. Then $V \subset S^{d-1}$ and for any $\epsilon > 0$ we may consider the set

$$V_\epsilon = \left\{x \in S^{d-1} : \operatorname{dist}(x, U) \leq \frac{\pi}{2} - \epsilon\right\}.$$

Then by Theorem 44.2 we have $\nu(V_\epsilon) \leq 4d \exp\left\{-\frac{(d-2)\epsilon^2}{2}\right\}$. Let us choose $\epsilon = 2d^{-\frac{1}{2}}(\ln d)^{\frac{1}{2}}$. Then $\nu(V_\epsilon) \longrightarrow 0$ and $c \notin V_\epsilon$ with probability which tends to 1 as $d \longrightarrow +\infty$. If $c \notin V_\epsilon$ then $\langle c, v_i\rangle \leq \cos(\pi/2 - \epsilon) = \sin \epsilon \leq \epsilon$ for any $i = 1, \ldots, 2d$. □

**(45.3) Simplex.** Let $\Delta_d \subset \mathbb{R}^d$ be a $d$-dimensional simplex. We assume that $\Delta$ is regular, that is, inscribed in the ball of radius 1 and circumscribed about the ball of radius $1/d$, both centered in the origin. Then for some $\gamma_1, \gamma_2 > 0$ the probability that

$$\gamma_1 \frac{(\ln d)^{-\frac{1}{2}}}{\sqrt{d}}\|c\| \leq \max\{\langle c, x\rangle : x \in \Delta_d\} \leq \gamma_2 \frac{(\ln d)^{\frac{1}{2}}}{\sqrt{d}}\|c\|$$

tends to 1 as $d \longrightarrow +\infty$.

**Proof.** Let $V = \{v_1, \ldots, v_d\}$ be the vertex set of $\Delta_d$. For $\epsilon > 0$ let

$$V_\epsilon = \left\{x \in S^{d-1} : \operatorname{dist}(x, V) \leq \frac{\pi}{2} - \epsilon\right\}.$$

Then by Theorem 44.2 we have $\nu(V_\epsilon) \leq 2d \exp\left\{-\dfrac{(d-2)\epsilon^2}{2}\right\}$. Let us choose $\epsilon = 2d^{-\frac{1}{2}}(\ln d)^{\frac{1}{2}}$. Then $\nu(V_\epsilon) \longrightarrow 0$ and $c \notin V_\epsilon$ with probability which tends to 1 as $d \longrightarrow +\infty$. If $c \notin V_\epsilon$ then for $i = 1, \ldots, d$ we have $\langle c, v_i\rangle \leq \cos(\pi/2 - \epsilon) = \sin \epsilon \leq \epsilon$, so

$$\max\{\langle c, x\rangle : x \in \Delta_d\} \leq \gamma_2 \frac{(\ln d)^{\frac{1}{2}}}{\sqrt{d}}\|c\|$$

with probability which tends to 1 as $d \longrightarrow +\infty$.

Let $\Delta_d = \left\{x \in \mathbb{R}^d : \langle u_i, x\rangle \leq \alpha_i : i = 1, \ldots, d\right\}$ be a representation of the simplex by a system of linear inequalities. We can assume that $\|u_i\| = 1$ for $i = 1, \ldots, d$. Since $\Delta_d$ contains a ball of radius $1/d$ we get that $\alpha_i \geq 1/d$ for $i = 1, \ldots, d$. For $\epsilon > 0$ let

$$U_\epsilon = \left\{x \in S^{d-1} : \operatorname{dist}(x, U) \leq \frac{\pi}{2} - \epsilon\right\}.$$

Again, if we choose $\epsilon = 2d^{-\frac{1}{2}}(\ln d)^{\frac{1}{2}}$. Then $\nu(U_\epsilon) \longrightarrow 0$ and $c \notin U_\epsilon$ with probability which tends to 1 as $d \longrightarrow +\infty$. If $c \notin U_\epsilon$ then $\langle c, u_i\rangle \leq \epsilon$ for $i = 1, \ldots, d$. Since $\alpha_i \geq 1/d$ then $\epsilon^{-1}d^{-1}c \in \Delta_d$ and

$$\max\{\langle c, x\rangle : x \in \Delta_d\} \geq d^{-1}\epsilon\|c\| = \gamma_1 \frac{(\ln d)^{-\frac{1}{2}}}{\sqrt{d}}\|c\|.$$

$\square$

**(45.4) Birkhoff Polytope.** Let $B_d \subset \mathbb{R}^{d^2}$ be the Birkhoff Polytope (see Section 10). Thus $B_d$ is the polytope defined in the space $\mathbb{R}^{d^2}$ of square $d \times d$ matrices $x = (x_{ij})$ by the equations

$$\sum_{i=1}^{d} x_{ij} = 1 : j = 1, \ldots d \quad \sum_{j=1}^{d} x_{ij} = 1 : i = 1, \ldots, d$$

and inequalities $x_{ij} \geq 0$. The vertex set of $B_d$ is the set of $d!$ permutation matrices. Let $q = (q_{ij}) : q_{ij} = 1/d$ be the "center" of the Birkhoff Polytope. The polytope $B_d$ is contained in the affine space $L_d$ of dimension $(d-1)^2$. We assume for simplicity that $c$ is distributed with some spherical symmetry in $L_d$ around the center $q$. For a matrix $c = (c_{ij}) : \ 1 \leq i, j \leq d$ let $\overline{c} = \dfrac{1}{d} \sum\limits_{ij} c_{ij} = \langle c, q \rangle$. Then there exist positive constants $\gamma_1, \gamma_2$ such that the probability that

$$\overline{c} + \gamma_1 (\ln d)^{-\frac{1}{2}} \|c - q\| \leq \max\{\langle c, x \rangle : x \in B_d\} \leq \overline{c} + \gamma_2 (\ln d)^{\frac{1}{2}} \|c - q\|$$

tends to 1 as $d \longrightarrow +\infty$.

**Proof.** Let $m = (d-1)^2$ and $S^{m-1}$ be the unit sphere in $L_d$ centered at $q$. If $v$ is a vertex of $B_d$ then the distance $\|v - q\|$ from $v$ to the center $q$ is equal to $\sqrt{d-1}$. Let $\mathcal{V} \subset S^{m-1}$ be the projection of the vertex set $V$ onto the sphere $S^{m-1}$. For $\epsilon > 0$ let us define

$$\mathcal{V}_\epsilon = \left\{x \in S^{m-1} : \operatorname{dist}(x, \mathcal{V}) \leq \frac{\pi}{2} - \epsilon\right\}.$$

Since the Birkhoff Polytope has $d!$ vertices, from Theorem 44.2 we get

$$\nu(\mathcal{V}_\epsilon) \leq 2d! \exp\left\{-\frac{(m-2)\epsilon^2}{2}\right\}.$$

Let us choose $\epsilon = 2d^{-1/2}(\ln d)^{1/2}$. Since $d! \leq d^d$ we get that $\nu(\mathcal{V}_\epsilon) \longrightarrow 0$ as $d \longrightarrow +\infty$. Therefore, the probability that a point $c \in S^{m-1}$ does not belong to $\nu(\mathcal{V}_\epsilon)$ tends to 1 as $d \longrightarrow +\infty$. If $c \notin \mathcal{V}_\epsilon$ then $\langle c, v \rangle = \overline{c} + \langle c - q, v - q \rangle \leq \|v - q\| \epsilon \leq \overline{c} + \gamma_2 (\ln d)^{1/2} \|c - q\|$ so the upper bound is proven.

To prove the upper bound, we observe that $B_d$ contains a ball centered in $q$ of radius $(d-1)^{-1}$. Therefore, in its affine span $L_d$ the polytope $B_d$ can be given by a system of $d^2$ inequalities

$$B_d = \left\{x \in L_d : \langle u_i, x - q \rangle \leq \alpha_i : i = 1, \ldots, d^2\right\},$$

where $\|u_i\| \leq 1$. Since $B_d$ contains a ball of radius $(d-1)^{-1}$ we must have $\alpha_i \geq (d-1)^{-1}$. Now, let $U = \{u_1, \ldots, u_{d^2}\}$ and

$$U_\epsilon = \left\{x \in S^{m-1} : \operatorname{dist}(x, U) \leq \frac{\pi}{2} - \epsilon\right\}.$$

By Theorem 44.2 we have $\nu(U_\epsilon) \leq 2d^2 \exp\{-\frac{(m-2)\epsilon^2}{2}\}$. Choosing $\epsilon = 3d^{-1}(\ln d)^{\frac{1}{2}}$ we get that $\nu(U_\epsilon) \longrightarrow 0$ as $d \longrightarrow +\infty$ and finish the proof as in the examples above. $\square$

PROBLEMS

1. Let $\mathbb{R}^d$ be a sequence of spaces, $H_d \subset \mathbb{R}^d$ be a sequence of linear hyperplanes and $S^{d-1} \subset \mathbb{R}^d$ be a sequence of unit spheres in $\mathbb{R}^d$. For a vector $c \in S^{d-1}$ we denote by $c_H$ its orthogonal projection onto $H$. Prove that for any $\delta > 0$ the probability that $\|c_H\|^2 \geq (1 - d^{-1+\delta})$ tends to 1 as $d$ tends to $+\infty$.

2. Prove that the formula of example 45.4 is still true if we assume that $c$ is distributed with a spherical symmetry in the whole space $\mathbb{R}^{d^2}$ of $d \times d$ matrices.

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Wednesday, April 5

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## 46. Further Insight into Average Linear Function

Let $X \subset S^{d-1}$ be a closed set on the sphere. For any $\epsilon > 0$ we consider the $\epsilon$-neighborhood $X_\epsilon$ of $X$ defined as $X_\epsilon = \{x \in S^{d-1} : \operatorname{dist}(x, X) \leq \epsilon\}$.

**(46.1) Theorem ("Isoperimetric Inequality")** . *Let $X \subset S^{d-1}$ be a closed subset and $C(y,r) \subset S^{d-1}$ be a spherical cap such that $\nu\big(C(y,r)\big) = \nu(X)$. Then for any $\epsilon > 0$ one has $\nu(X_\epsilon) \geq \nu\big(C(y, r+\epsilon)\big)$.*

We don't prove this theorem. The proof can be found in T. Figiel, J. Lindenstrauss and V.D. Milman, The dimension of almost spherical sections of convex bodies, *Acta Mathematica*, **139**(1977), 53–94.

Let $P \subset \mathbb{R}^d$ be a convex compact set. The function $F_P(c) = \max\{\langle c, x\rangle : x \in P\}$, where $c \in \mathbb{R}^d$ is called the *support function* of $P$.

PROBLEM.

1°. Prove that $X_{\epsilon+\delta} = (X_\epsilon)_\delta$.

2°. The function $F_P(c)$ is convex and $F_P(\lambda c) = \lambda F_P(c)$ for $\lambda \geq 0$.

3°. If $P = P_1 + P_2$ is the Minkowski sum of two polytopes then $F_P(c) = F_{P_1}(c) + F_{P_2}(c)$.

4° Prove that $\max\{F_P(c) : c \in S^{d-1}\} = \max\{\|x\| : x \in P\}$.

5. If $F_P(c) = R > 0$ then $P$ is the ball $\{x : \|x\| = R\}$ of radius $R$.

**(46.2) Theorem.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope, $F_P(c)$ be its support function and $R_P = \max\{F_P(c) : c \in S^{d-1}\}$. Let $g : \mathbb{N} \longrightarrow \mathbb{R}$ be a function such that $g(d) \longrightarrow +\infty$ as $d \longrightarrow +\infty$. There exists a number $\rho(P)$ such that for a random $c \in S^{d-1}$*

$$\frac{|F_P(c) - \rho(P)|}{R_P} \leq d^{-\frac{1}{2}} g(d)$$

*with probability which tends to 1 as $d \longrightarrow +\infty$.*

**Proof.** For $\alpha \in \mathbb{R}$ let us define two closed subsets of the sphere $X^+(\alpha) = \{c \in S^{d-1} : F_P(c) \geq \alpha\}$ and $X^-(\alpha) = \{c \in S^{d-1} : F_P(c) \leq \alpha\}$. Thus $X^+(\alpha) = \emptyset$, $X^-(\alpha) = S^{d-1}$ for a sufficiently big $\alpha$ and $X^+(\alpha) = S^{d-1}$ and $X^-(\alpha) = \emptyset$ for a sufficiently small $\alpha$. Let $\rho(P) = \inf\{\alpha : \nu(X^-(\alpha)) \geq 1/2\}$. Since the function $\alpha \longmapsto \nu\big(X^-(\alpha)\big)$ is lower semicontinuous, we must have $\nu\big(X^-(\rho)\big) \geq 1/2$. On the other hand, for any $\alpha < \rho$ we have $\nu\big(X^-(\alpha)\big) < 1/2$, therefore $\nu\big(X^+(\alpha)\big) \geq 1/2$. Since the function $\alpha \longmapsto \nu\big(X^+(\alpha)\big)$ is upper semicontinuous, we get that $\nu\big(X^+(\rho)\big) \geq 1/2$. In other words, $\rho(P)$ is the median of $F_P(c)$.

Let us choose an $\epsilon > 0$. Then by Theorem 46.1 and Theorem 44.2 we get that

$$\nu\big(X^+(\rho)_\epsilon\big) \geq \nu\big(C(y, \pi/2 + \epsilon)\big) \geq 1 - 2\exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}$$

and

$$\nu\big(X^-(\rho)_\epsilon\big) \geq \nu\big(C(y, \pi/2 + \epsilon)\big) \geq 1 - 2\exp\Big\{-\frac{(d-2)\epsilon^2}{2}\Big\}.$$

Thus for $\epsilon = d^{-\frac{1}{2}} g(d)$ we have that $\nu\big(X^+(\rho)_\epsilon\big), \nu\big(X^-(\rho)_\epsilon\big) \longrightarrow 1$ as $d \longrightarrow +\infty$. Therefore

$$c \in X^+(\rho)_\epsilon \cap X^-(\rho)_\epsilon$$

with probability which tends to 1 as $d \longrightarrow +\infty$. Let us consider a $c \in X^+(\rho)_\epsilon \cap X^-(\rho)_\epsilon$. There exists $b \in X^+(\rho)$ such that $\|c - b\| \leq \epsilon$ and therefore $F_P(c) \geq F_P(b) - \epsilon \cdot R_P \geq \rho - \epsilon \cdot R_P$. There exists $a \in X^-(\rho)$ such that $\|c - a\| \leq \epsilon$ and therefore $F_P(c) \leq F_P(a) + \epsilon \cdot R_P \leq \rho + \epsilon \cdot R_P$. □

**(46.3) Examples.**

**(46.3.1) Cube.** Let us consider the standard cube $I_d$ (see (45.1)). Then for $\rho_d = \rho(I_d)$, $R_d = \max\{\|x\| : x \in I_d\} = \sqrt{d}$, $F_d(c) = \max\{\langle c, x\rangle : x \in I_d\}$ we have $d^{-\frac{1}{2}}|F_d(c) - \rho_d| \leq d^{-\frac{1}{2}} g(d)$ with probability which tends to 1 as $d \longrightarrow +\infty$. Hence we have that for any function $g(d) \longrightarrow +\infty$

$$|F_d(c) - \rho_d| \leq g(d)$$

with probability which tends to 1 as $d \longrightarrow +\infty$. By (45.1) it follows that $\rho_d \geq \gamma\sqrt{d}/(\ln d)^{\frac{1}{2}}$ for some $\gamma > 0$ and all sufficiently large $d$. Therefore, the relative error of estimating $F_d(c)$ by $\rho_d$ tends to 0 on probability.

**(46.3.2) Birkhoff Polytope.** Let $B_d$ be the Birkhoff Polytope (see Example (45.4)). We place the origin into the center $x_{ij} = 1/d$ of the polytope and consider $B_d$ in its $(d-1)^2$-dimensional affine hull. Let $F_d(c) = F_{B_d}(c)$, $R_d = R_{B_d} = \sqrt{d-1}$ and $\rho_d = \rho(B_d)$. Then we have $(d-1)^{-\frac{1}{2}}|F_d(c) - \rho_d| \leq (d-1)^{-1} g(d)$ with probability which tends to 1 as $d \longrightarrow +\infty$. Therefore

$$|F_d(c) - \rho_d| \leq (d-1)^{-\frac{1}{2}} g(d)$$

with probability which tends to 1 as $d \longrightarrow +\infty$.

From (45.4) it follows that $\rho_d \geq \gamma(\ln d)^{-\frac{1}{2}}$ for some $\gamma > 0$ and sufficiently large $d$. Hence the relative error of estimating $F_d(c)$ by $\rho_d$ tends to 0 on probability.

PROBLEMS.

For a polytope $P \subset \mathbb{R}^n$ let $\rho(P)$ denote the median of the support function $F_P : S^{n-1} \longrightarrow \mathbb{R}$.

1. Let $I_n = \big\{(x_1, \ldots, x_n) : |x_i| \leq 1 \text{ for } i = 1, \ldots, n\big\}$ be the cube. Then $\rho(I_n) = \sqrt{2n/\pi}\big(1 + o(1)\big)$.

*Hint:* Show that $\rho(I_n) \approx \displaystyle\int_{S^{n-1}} F_P(u)\ du$, where $du$ is the rotation invariant probability measure. To compute the integral, represent the cube $I_n$ as the Minkowski sum of intervals.

2*. Let $\Delta_n = \big\{(x_1, \ldots, x_n) : x_1 + \ldots + x_n = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, \ldots, n\big\}$ be the simplex. Then $\rho(\Delta_n) = \sqrt{2 \ln n/\pi}\big(1 + o(1)\big)$.

*Hint:* Use Hint to Problem 1. To estimate the integral, reduce it to the integral $\displaystyle\int_{\mathbb{R}^n} f(x) e^{-\pi\|x\|^2}\ dx$ with $f(x) = \max\{x_1, \ldots, x_n\}$ and consider the distribution function of $f(x)$ with respect to the probability measure $e^{-\pi\|x\|^2}\ dx$.

3*. Let $B_n \subset \mathbb{R}^{n^2}$ be the Birkhoff polytope. Then $\rho(B_n) = \sqrt{2 \ln n}\big(1 + o(1)\big)$.

More about "the measure concentration phenomenon" see in V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics, **1200**, Springer-Verlag, Berlin, New York, 1986.

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Friday, April 7

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## 47. Shelling

**(47.1) Polytopal Complexes** (see also Section 26). A finite set of polytopes $\mathcal{P} = \{P_1, \ldots, P_m\}, P_i \subset \mathbb{R}^n$ is called a *polytopal complex* iff the following two conditions are satisfied:

(47.1.1) If $P \in \mathcal{P}$ and $F$ is a face of $P$ then $F \in \mathcal{P}$;

(47.1.2) For any two polytopes $P_i, P_j \in \mathcal{P}$ the intersection $P_i \cap P_j$ is a face of both $P_i$ and $P_j$.

The set $|\mathcal{P}| = \displaystyle\bigcup_{i=1}^{m} P_i \subset \mathbb{R}^n$ is called the *realization* of $\mathcal{P}$.

A polytope $F \in \mathcal{P}$ is called a *face* of $\mathcal{P}$. A *facet* of $\mathcal{P}$ is a non-empty polytope $P \in \mathcal{P}$ which is not a proper face of any polytope from $\mathcal{P}$. The largest dimension of a polytope from $\mathcal{P}$ is called the *dimension* of $\mathcal{P}$. A $d$-dimensional polytopal complex $\mathcal{P}$ is called *pure* iff every facet $P_i \in \mathcal{P}$ has dimension $d$.

PROBLEMS.

1°. Every 0-dimensional polytopal complex is pure.

2°. Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope. The set of all faces $F \neq P$ of $P$ is a $(d-1)$-dimensional pure polytopal complex.

This complex is called the *boundary complex* of the polytope $P$ and denoted $\partial P$.

**(47.2) Shelling and Shellability.** Let $\mathcal{P}$ be a 0-dimensional polytopal complex. Any linear ordering of the facets (that is, points) from $\mathcal{P}$ is called a *shelling* of $\mathcal{P}$. Let $\mathcal{P}$ be a pure $d$-dimensional polytopal complex. A linear ordering $P_1, \ldots, P_m$ of the facets of $\mathcal{P}$ is called *shelling* iff

(47.2.1) The boundary complex $\partial P_1$ has a shelling,

(47.2.2) For every $m \geq j > 1$ the polytopal complex $\big\{P_j \cap F : F$ is a face of $P_i$ for $i < j\big\}$ is pure, $(d-1)$-dimensional and has a shelling which can be appended to a shelling of the boundary complex of $P_j$.

A polytopal complex is called *shellable* if it has a shelling.

PROBLEMS.

1°. A 1-dimensional complex (graph) is shellable if it is connected.

2. The boundary complex of a simplex is shellable.

3. The boundary complex of a cube is shellable.

4*. A pure 2-dimensional complex $\mathcal{P}$, where $|\mathcal{P}|$ is a topological ball or a topological sphere is shellable.

5*. There exists a pure 3-dimensional polytopal complex $\mathcal{P}$, where $|\mathcal{P}|$ is a topological ball but $\mathcal{P}$ is not shellable.

6* (for topologists). Let $\mathcal{P}$ be a pure $d$-dimensional polytopal complex such that $|\mathcal{P}|$ is a topological sphere. Is it true that one can apply sufficiently many barycentric subdivisions to $\mathcal{P}$ so that the resulting complex will be shellable? Is it true that the number of such subdivisions can be bounded by a recursive function in the number of polytopes of $\mathcal{P}$?

**(47.3) Lemma.** *Let $F_1, \ldots, F_m$ be a shelling of the boundary complex $\partial P$ of a $d$-dimensional polytope $P$. Then the reverse order is also a shelling of $\partial P$.*

**Proof.** We use induction by $d$. If $d = 1$ the statement is clear. Suppose that $d > 1$. First, we must establish that $\partial F_m$ is shellable. Since $F_1, \ldots, F_m$ is a shelling we must have by (47.2.2) that $\{F_m \cap F : F$ is a face of $F_i : i < m\}$ is shellable. But this is just the boundary complex $\partial F_m$ of $F_m$. Let us check (47.2.2). Consider a facet $F_j$. The complex $\Delta_j = \{F_j \cap F : F$ is a face of $F_i$ for $i < j\}$ is shellable and its shelling can be appended to a shelling of $\partial F_j$. So the facets of $F_j$ can be arranged in a shelling order $G_1, \ldots, G_n$ such that $G_1, \ldots, G_s$ is a shelling of $\Delta_j$. By the induction conjecture we can reverse this order and still get a shelling order of $\partial F_j$. But the first $n - s + 1$ facets $G_n, G_{n-1}, \ldots, G_{s+1}$ of this reverse order are the facets of the complex $\overline{\Delta_j} = \{F_j \cap F : F \text{ is a face of } F_i \text{ for } i > j\}$. Therefore the complex $\overline{\Delta_j}$ is shellable and its shelling can be extended to a shelling of $\partial F_j$. Hence the reverse order is a shelling of $\partial P$. □

PROBLEM.

1. Check that Lemma 47.3 remains valid for any polytopal complex $\mathcal{P}$ such that $|\mathcal{P}|$ is a topological sphere.

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Monday, April 10

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## 48. Polytopes are Shellable

**(48.1) Visibility.** Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope and $p \in \mathbb{R}^d$ be a point. Let $F$ be a facet of $P$ and $H_F$ be the supporting hyperplane of this facet. Suppose that $v \notin H_F$. Then we say that a facet $F$ of $P$ is *visible* from $p$ iff the supporting hyperplane $H_F$ separates $p$ from $P$ and we say that the facet $F$ is *invisible* iff $H_F$ does not separate the polytope $P$ from $p$. For a $d$-dimensional polytope $P \subset \mathbb{R}^d$ and a point $p \in \mathbb{R}^d$ such that $p$ is not contained in the supporting hyperplane $H_F$ of any facet $F$ of $P$ we define the *visible complex* $V(P,p)$ as the complex consisting of all facets of $P$ that are visible from $p$ and all their faces and the *invisible complex* $I(P,p)$ as the complex consisting of all facets of $P$ that are invisible from $p$ and all their faces.

PROBLEMS.

1°. A visible complex of a $d$-dimensional polytope is pure and $(d-1)$-dimensional iff $p \notin P$.

2°. An invisible complex of a $d$-dimensional polytope is pure and $(d-1)$-dimensional.

3°. If $p$ is in the interior of $P$ then $V(P,p) = \emptyset$ and $I(P,p) = \partial P$.

4°. If $p \notin P$ then $|V(P,p)|$ and $I(P,p)|$ are the topological $(d-1)$-dimensional balls.

**(48.2) Theorem.** *Let $P \subset \mathbb{R}^d$ be a $d$-dimensional polytope and $p \notin P$ be a point which is not contained in the supporting hyperplane of any facet of $P$. Then $V(P,p)$ is shellable and a shelling or $V(P,p)$ can be extended to a shelling of $\partial P$.*

**Proof.** We use the induction on $d$. If $d = 1$ then the result is clear. Suppose that $d > 1$. Let us choose a point $a \in P$ and draw the straight line $l$ through $a$ and $p$. Without loss of generality we may assume that $l$ intersects the supporting hyperplane $H_F$ of every facet $F$ of $P$ and that the intersection points $l \cap H_F$ and $l \cap H_G$ are different for any two different facets $F$ and $G$ (otherwise we slightly perturb $a$ or $p$). Let us parameterize the line $l: \ l(t) = a + tu$, where $t \in \mathbb{R}$ and $u = p - a$. For every facet $F$ of $P$ we let $t_F : l(t_F) \in H_F$. Let us list the numbers $\{t_F\}$ in the following order. First, we list positive $t_F$ in increasing order and then we list negative $t_F$ in increasing order. We claim that this order is a shelling of $\partial P$ and that the first segment of this order with $t_F < 1$ is a shelling of $V(P,p)$.

It is clear that the facets of $V(P,p)$ are precisely the facets $F$ with $t_F < 1$. So it is enough to prove that the ordering is a shelling of $\partial P$. Let us consider the first facet $F_1$. It is a $(d-1)$-dimensional polytope and by the induction conjecture $\partial F_1$ has a shelling. Let us consider a facet $F = F_j$ with $t_F > 0$. Then the point $p_F = l(t_F)$ belongs to the supporting hyperplane $H_F$ of the facet $F$ and the supporting hyperplanes of the facets with smaller indices separate $p_F$ from $P$ whereas the supporting hyperplanes of the facets with bigger indices does not separate $p_F$ from $P$. Therefore, the polytopal complex

$$\{G \cap F_j : \ G \text{ is a face of } F_i \text{ for } i < j\}$$

is just the visible complex $V(F, p_F)$ of the $(d-1)$-dimensional polytope $F \subset H_F$. Applying the induction conjecture we conclude that this complex has a shelling that can be extended to a shelling of $\partial F$.

Let us consider now a facet $F = F_j$ with $t_F < 0$. Then the point $p_F = l(t_F)$ belongs to the supporting hyperplane $H_F$ of the facet $F$ and the supporting hyperplanes of the facets with bigger indices separate $p_F$ from $P$ whereas the supporting hyperplanes of the facets with smaller indices do not. Therefore, the polytopal complex

$$\{G \cap F_j : \ G \text{ is a face of } F_i \text{ for } i < j\}$$

is just the invisible complex $I(F, p_F)$ of the $(d-1)$-dimensional polytope $F \subset H_F$. By the induction conjecture there exists a shelling of $\partial F$ whose initial segment is a shelling of $V(F, p_F)$. Then by Lemma 47.3 we can reverse this shelling of $\partial F$. In this reversed shelling the facets from $I(F, p_F)$ will come first. Thus there exists the required shelling of $\partial F$ and the theorem is proven. □

PROBLEMS.

1. The construction of Theorem 48.2 provides the *topological shelling:* we list the facets $F_1, \ldots, F_m$ of $P$ in such an order that the intersection $F_j \bigcap (\bigcup_{i=1}^{j-1} F_i)$ is a topological $(d-1)$-dimensional ball for $j < m$ and a topological $(d-1)$-dimensional sphere for $j = m$.

2. Let $P$ be a $d$-dimensional simplicial polytope and $F_1, \ldots, F_m$ be a shelling of $\partial P$. For a facet $F_j$ let us define its index as the number of facets of $F_j$ that are facets of some $F_i$ with $i < j$. Then the number of facets of a given index $i$ is independent of the shelling and equal to $h_i(P)$.

*Hint:* Use that the equations

$$f_{k-1}(P) = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i(P)$$

determine $h_i(P)$ uniquely.

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Wednesday, April 12

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## 49. Further Insight into Shellability

A polytopal complex $\mathcal{P}$, such that each $P_i \in \mathcal{P}$ is a simplex, is called *simplicial complex.* As usual, by $f_k(\mathcal{P})$ we denote the number of $k$-dimensional faces of $\mathcal{P}$. We let $f_{-1}(\mathcal{P}) = 1$.

**(49.1) Theorem.** *Let $\mathcal{P}$ be a pure $(d-1)$-dimensional simplicial complex and $P_1, \ldots, P_m$ be a shelling of $\mathcal{P}$. For every facet $P_j$ (which is a $(d-1)$-dimensional simplex) let us define its index as the number of facets of $P_j$ (that is, ridges of $\mathcal{P}$) which are faces of some $P_i$ with $i < j$. Then the number of facets of index $i$ is independent of the shelling. We denote this number by $h_i(\mathcal{P})$. Furthermore,*

$$h_k(\mathcal{P}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathcal{P}), \quad k = 0, \ldots, d \quad \textit{and}$$

$$f_{k-1}(\mathcal{P}) = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i(\mathcal{P}), \quad k = 0, \ldots, d.$$

**Proof.** Let $s = (P_1, \ldots, P_m)$ be a shelling of $\mathcal{P}$. Let us denote for a moment by $h_i(P, s)$ the number of facets which have index $i$ in this shelling. For every face $F$ of $\mathcal{P}$ let us consider the smallest $n$ such that $F$ is a face of $P_n$. Let us count how many $(k-1)$-dimensional faces correspond to a given $n$. Suppose that index of $P_n$ is $i$. Then $i$ facets of $P_n$ are the faces of $\bigcup_{j=1}^{n-1} P_j$ and the remaining $(d-i)$ facets, say, $G_1, \ldots, G_{d-i}$ of $P_j$ are not faces of $\bigcup_{j=1}^{n-1} P_j$. The intersection of any $d-k$ of $G_1, \ldots, G_{d-i}$ is a $(k-1)$-dimensional face of $\mathcal{P}$ which correspond to the given $n$. Thus the total number of $(k-1)$-dimensional faces is

$$f_{k-1}(\mathcal{P}) = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i(\mathcal{P}, s).$$

On the other hand, these equations uniquely determine $h_i(\mathcal{P}, s)$ (see Theorem 16.2) as

$$h_k(\mathcal{P}, s) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathcal{P})$$

(consider $\mathbf{f}(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i}$ and $\mathbf{h}(x) = \sum_{i=0}^{d} h_i x^{d-i}$. The first system is equivalent to $\mathbf{f}(x) = \mathbf{h}(x+1)$, whereas the second one is equivalent to $\mathbf{h}(x) = \mathbf{f}(x-1)$). $\square$

**(49.2) Corollary.** *For a shellable simplicial $(d-1)$-dimensional complex $\mathcal{P}$ we have that*

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k}f_{i-1}(\mathcal{P}) \geq 0.$$

$\square$

**(49.3) Theorem.** *Let $\mathcal{P}$ be a shellable simplicial $(d-1)$-dimensional complex such that $|\mathcal{P}|$ is a topological sphere (what we really need is that every ridge of $\mathcal{P}$ belongs to exactly two facets of $\mathcal{P}$). Then $h_k(\mathcal{P}) = h_{d-k}(\mathcal{P})$ for $k = 0, \ldots, d$.*

**Proof.** Let $P_1, \ldots, P_m$ be a shelling of $\mathcal{P}$. Then the reverse order is also a shelling (see Problem 1, Section 47.3). Therefore $P_n$ has index $i$ in the first shelling, iff it has index $d-i$ in the second shelling. $\square$

**(49.3) General Questions.** Let $\mathcal{P}$ be a simplicial $(d-1)$-dimensional complex. Let us define

$$h_k(\mathcal{P}) = \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k}f_{i-1}(\mathcal{P}).$$

(49.3.1) When $h_k(\mathcal{P}) \geq 0$ for $k = 0, \ldots, d$ (non-negativity)?

(49.3.2) When $h_{d-k}(\mathcal{P}) = h_k(\mathcal{P})$ for $k = 0, \ldots, d$ (Dehn - Sommerville Equations)?

(49.3.3) When $h_{k+1}(\mathcal{P}) \geq h_k(\mathcal{P})$ for $k < [d/2]$ (unimodality)?

**Cohen - Macaulay Complexes.** Let $\mathcal{P}$ be a pure $(d-1)$-dimensional simplicial complex. Let $F \in \mathcal{P}$ be a face. The *link* $\mathrm{lk}(F, \mathcal{P})$ of the face $F$ is a simplicial complex consisting of the faces $G$ such that $G \cap F = \emptyset$ and there exists a facet, say, $H$ containing both $G$ and $F$. Besides, we agree that $\mathrm{lk}(\emptyset, \mathcal{P}) = \mathcal{P}$. The complex $\mathcal{P}$ is called *Cohen - Macaulay* (over a field $K$ ) iff $H_i\big(|\mathrm{lk}(F, \mathcal{P})|; K\big) = 0$ for $i < \dim \mathrm{lk}(F, \mathcal{P})$ (we consider reduced homological groups). For Cohen - Macaulay complexes we have non-negativity (49.3.1). If $|\mathcal{P}|$ is a topological sphere then $\mathcal{P}$ is Cohen - Macaulay.

**Eulerian Complexes.** Let $\mathcal{P}$ be a pure $(d-1)$-dimensional simplicial complex. We say that $\mathcal{P}$ is *Eulerian* iff for every face $H \in \mathcal{P}$ we have

$$\sum_{F \in \mathcal{P}: H \subset F}(-1)^{\dim F} = (-1)^{d-1}.$$

If $\mathcal{P}$ is Eulerian then we have (49.3.2). If $|\mathcal{P}|$ is a topological sphere then $\mathcal{P}$ is Eulerian.

Those complexes which are Cohen - Macaulay and Eulerian are called *Gorenstein.*

**Question.** What about (49.3.3)?

Shellability of polytopes was established by H. Bruggesser and P. Mani. Further information on shellability see in G.M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1995 (Graduate texts in mathematics, **152**).

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Friday, April 14

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## 50. Flag Vectors

**(50.1) Definition.** Let $P$ be a $d$-dimensional polytope. Suppose that $H$ and $G$ are faces of $P$. We write $H < G$ iff $H \subset G$ and $H \neq G$. A *flag* of $P$ is a chain $\mathcal{F} = F_1 < F_2 < \ldots < F_k < P$ consisting of proper faces of $P$. Let $S \subset \{0, 1 \ldots, d-1\}$ be a set. We say that $S$ is the *index* of a flag $\mathcal{F}$ iff $\mathcal{F}$ contains $|S|$ faces whose dimensions are the numbers from $S$. Then we write $\mathrm{ind}\mathcal{F} = S$. Let $f_S(P)$ be the number of flags of $P$ whose index is $S$. The collection of $2^d$ numbers $\{f_S(P)\}$ is called the *flag vector* of $P$. The set of all $2^d$ subsets $S \subset \{0, \ldots, d-1\}$ we denote by $2^{[0:d-1]}$.

**(50.2) Lemma.** *Let $P$ be a $d$-dimensional polytope and $H \subset G$ be two faces of $P$ with $H \neq G$. Then*

$$\sum_{H \subset F \subset G} (-1)^{\dim F} = 0 \quad \textit{and} \quad \sum_{H < F < G} (-1)^{\dim F} = (-1)^{\dim H + 1} + (-1)^{\dim G - 1}.$$

**Proof.** It is enough to prove the first formula only. Let $G^\circ$ be the dual polytope to $G$ and $\hat{H} \subset G^\circ$ be the face which is dual to $H$ (see Theorem 11.1). Then $\dim \hat{H} = \dim G - \dim H - 1$ and to every face $F$ such that $H \subset F \subset G$ we correspond a face $\hat{F} \subset G^\circ$ such that $\hat{F} \subset \hat{H}$. Besides $\dim \hat{F} = \dim G - \dim F - 1$. Applying the Euler - Poincaré formula (see Theorem 23.1) to $\hat{H}$ we get

$$\sum_{\hat{F} \subset \hat{H}} (-1)^{\dim \hat{F}} = 0,$$

where the sum is taken over all faces of $\hat{H}$. The proof now follows. $\square$

**(50.3) Theorem** [M. Bayer and L. Billera]. *Let $P$ be a $d$-dimensional polytope and $S \subset \{0, 1 \ldots, d-1\}$ be a subset. If $\{i, k\} \subset S \cup \{-1, d\}$, $i < k-1$ and $S$ contains no $j$ such that $i < j < k$, then*

$$\sum_{j=i+1}^{k-1} (-1)^j f_{S \cup j}(P) = f_S(P)\big((-1)^{i+1} + (-1)^{k-1}\big).$$

**Proof.** Let $\mathcal{F}$ be a flag of $P$ whose index is $S$. Let us choose a face $H \in \mathcal{F}$ of dimension $i$ (possibly an empty face for $i = -1$) and a face $G \in \mathcal{F}$ of dimension $k$ (possibly $P$ for $k = d$). Then by Lemma 50.2

$$\sum_{H<F<G} (-1)^{\dim\ F} = (-1)^{i+1} + (-1)^{k-1}.$$

Let sum this equation over all flags $\mathcal{F}$ with the index $S$ (with $H$ and $G$) depending on $\mathcal{F}$. We get

$$f_S(P)\big((-1)^{i+1} + (-1)^{k-1}\big) = \sum_{\mathcal{F}:\mathrm{ind}(\mathcal{F})=S} \ \sum_{H<F<G} (-1)^{\dim\ F} =$$

$$\sum_{\mathcal{F}:\mathrm{ind}(\mathcal{F})=S} \ \sum_{j=i+1}^{k-1} (-1)^j \sum_{\dim\ F=j} 1,$$

where the last sum is taken over all $j$-dimensional faces $F$ which can be inserted in a given $S$-flag $\mathcal{F}$. Furthermore,

$$\sum_{\mathcal{F}:\mathrm{ind}(\mathcal{F})=S} \ \sum_{j=i+1}^{k-1} (-1)^j \sum_{\dim\ F=j} 1 = \sum_{j=i+1}^{k-1} (-1)^j \sum_{\mathcal{F}:\mathrm{ind}(\mathcal{F})=S} \ \sum_{\dim\ F=j} 1 = \sum_{j=i+1}^{k-1} (-1)^j f_{S\cup j}(P).$$

□

Now we establish some linear relations between $f_S(P)$. Let $\Psi^d$ be the sets of all subsets $S \subset \{0, \ldots, d-2\}$ such that $S$ contains no consecutive integers.

PROBLEM.

1. Prove that $|\Psi^d| = c_d$, the $d$-th Fibonacci number ($c_d = c_{d-1} + c_{d-2} : c_1 = 1, c_2 = 2$.)

**(50.4) Theorem** [M. Bayer and L. Billera]. *For any $d$ there exist numbers $\gamma_{S,T} : T \in \Psi^d, S \in 2^{[0:d-1]}$ such that*

$$f_S(P) = \sum_{T\in\Psi^d} \gamma_{S,T} f_T(P)$$

*for every $d$-dimensional polytope $P$.*

**Proof.** Let us order the subsets $S \in \{0, \ldots d-1\}$ lexicographically. Namely, to each subset $S$ we correspond the number $\sum_{i\in S} 2^i$ and order them accordingly. Let us consider $T \in 2^{[0:d-1]} \setminus \Psi^d$. Then for some $k : 1 \leq k \leq d$ we will have $\{k-1, k\} \subset T \cup \{d\}$. Let $S = T \setminus \{k-1\}$ and $i = \max\{j \in T \cup \{-1\} : j < k-1\}$. Then by Theorem 50.3 we get

$$f_T(P) = (-1)^{k-1}\left( f_S(P)\big((-1)^{i+1} + (-1)^{k-1}\big) - \sum_{j=i+1}^{k-2} (-1)^j f_{S\cup j}(P)\right).$$

Then every subset $S, S \cup \{j\}$ is less than $T$ in the lexicographic order. Iterating this procedure we will get rid of $T \notin \Psi^d$. $\square$

PROBLEMS.

1°. Define the *flag h-vector* as follows: for $S \in 2^{[0:d-1]}$ let

$$h_S(P) = \sum_{T \subset S} (-1)^{|S|-|T|} f_T(P).$$

Prove that

$$f_S(P) = \sum_{T \subset S} h_T(P).$$

2. Prove the *Dehn - Sommerville Equations*:

$$h_S(P) = h_{\overline{S}}(P),$$

where $\overline{S} = \{0, 1, \ldots, d-1\} \setminus S$ is the complement to $S$.

Those are *not* all the linear equations satisfied by the flag vectors of polytopes.

3* [J. Fine]. Let $a$ and $b$ be two non-commuting variables. For $S \in 2^{[0:d-1]}$ let us define a non-commutative monomial $u_S = u_0 u_1 \ldots u_{d-1}$ of length $d$ by letting

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

Let

$$\Gamma_P(a, b) = \sum_{S \in 2^{[0:d-1]}} f_S(P) u_S \qquad \text{and} \qquad \Psi_P(a, b) = \sum_{S \in 2^{[0:d-1]}} h_S(P) u_S.$$

a) Prove that $\Psi_P(a, b) = \Gamma_P(a - b, b)$ and that $\Gamma_P(a, b) = \Psi_P(a + b, b)$.

b) Theorem 50.4 implies that $\Psi_P(a, b)$ is a polynomial $\Phi(c, d)$ in the variables $c = a+b$ and $d = ab + ba$.

The polynomial $\Phi_P(c, d)$ is called the *cd-index* of $P$.

4* [J. Fine]. Is it true that $\Phi(c, d) \geq 0$? It is known to be true for simplicial polytopes and conjectured to be true for triangulations of spheres, at least.

5°. If $S = \{i\}$ then $f_S(P)$ is the number $f_i(P)$ of $i$-dimensional faces of $P$.

6. Let $\hat{P}$ be the first barycentric subdivision of $P$ (that is, for each proper face of $P$ we add a new point only slightly above the face and take the convex hull of all these points). Then $\hat{P}$ is a simplicial polytope and

$$f_{i-1}(\hat{P}) = \sum_{S \in 2^{[0:d-1]} : |S| = i} f_S(P) \quad \text{and} \quad h_i(\hat{P}) = \sum_{S \in 2^{[0:d-1]} : |S| = i} h_S(P).$$

7°. The equations 2 imply the Dehn - Sommerville equations for the polytope $\hat{P}$.

8 [M. Bayer, L. Billera]. Let us consider two operations on polytopes. If $P \subset \mathbb{R}^d$ is a polytope then $\mathcal{P}(P) \subset \mathbb{R}^{d+1}$ is the *pyramid* over $P$ obtained by adding a new vertex $p \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d$ and $\mathcal{B}(P) \subset \mathbb{R}^{d+1}$ is the *bipyramid* over $P$ obtained by adding two new vertices $p, -p \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d$.

a)°. Suppose that $P \in \mathbb{R}^0$ is a point. Then $\mathcal{P}(P)$ is an interval, $\mathcal{P}^2(P)$ is a triangle, $\mathcal{BP}(P)$ is a square and $\mathcal{B}^2\mathcal{P}(P)$ is an octahedron.

b). Let us interpret a point as a pyramid over the empty polytope. We consider polytopes that can be constructed from the empty set using operations $\mathcal{P}$ and $\mathcal{B}$ and denote such a polytope simply by a string (word) of letters $\mathcal{P}$ and $\mathcal{B}$. Consider the set $\Omega^d$ of $d$-dimensional polytopes encoded by the words of length $d+1$ which end in $\mathcal{P}^2$ and contain no two adjacent $\mathcal{B}$s Prove that $|\Omega_d| = c_d$, the $d$-th Fibonacci number.

c)*. Prove that the flag vectors $f_S(P)$ of polytopes from $\Omega^d$ are affinely independent. So Theorem 50.4 summarizes all linear relations between $f_S(P)$.

9 [J. Fine]. Instead of $\mathcal{B}$ consider the operation $\mathcal{I}$ of taking the *cylinder* $P \times I$ over $P$, where $I$ is the interval.

Prove the equation

$$\mathcal{IIP} + \mathcal{PPI} = \mathcal{IPP} + \mathcal{IPI}$$

interpreted as follows: If $P$ is a polytope then the flag vector of $\mathcal{IIP}(P)$ plus the flag vector of $\mathcal{PPI}(P)$ is the flag vector of $\mathcal{IPP}(P)$ plus the flag vector of $\mathcal{IPI}(P)$.

Prove the result similar to that of problem 8.

The linear relations for flag vectors are found in M. Bayer and L. Billera, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Inventiones Mathematicae*, **79**(1985), 143–157. For *cd*-index see R. Stanley, Flag $f$-vectors and the *cd*-index, *Mathematische Zeitschrift*, **216**(1994), 483–499. See also R. Stanley, A survey of Eulerian posets, in: *Polytopes: Abstract, Convex and Computational*, Kluwer Acad. Publ., Dordrecht, 1994, 301–333 (NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences **440**).