ON TOPOLOGICAL PROPERTIES OF SPACES OF POLYTOPES

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INTRODUCTION. The notion of space of polytopes was introduced by A.M.Vershik, and used by him and A.G.Chernjakov in [1] , where they have proved Smale conjecture about the structure of the set of Pareto optimum points. It was shown that this notion may be used in solving different problems of optimal control theory and mathematical economy. Related concept of smooth field of convex polytopes also was discussed. The objective of this paper is to describe some examples of spaces of convex polytopes. We shall give some combinatorial applications.

Let P be a d-dimensional convex polytope which vertices are $a_1, \dots, a_n$. Let $C(P)$ be a set of all n-tuples $(v_1, \dots, v_n)$ where $v_i \in R^d$ and $\psi : a_i \to v_i$ is a combinatorial isomorphism between boundary complexes of polytope P and $conv\{v_1, \dots, v_n\}$, $C(P)$ is topologized as a subspace of R^{dn} . The action of the group of nondegenerate affine transformations on R^d induces a natural action on $C(P)$ as follows:

$$A(v_1 \cdots v_n) = (Av_1 \cdots Av_n) \text{ where } A \in Aff(R^d).$$

DEFINITION. $C(P)$ factorized under action described above is called a space of polytopes combinatorially isomorphic with P . It

is denoted by $C_a(P)$.

In this paper we shall prove the following main result:

THEOREM. If P is a cyclic polytope, then $C_a(P)$ is homeomorphic to the Euclidean space.

§ 1. CYCLIC POLYTOPES

Polytope $P = conv\{v_1 \cdots v_n\}$, where $v_i = (\cos x_i, \sin x_i, \cos 2x_i, \sin 2x_i \cdots \cos dx_i, \sin dx_i)$ is called a cyclic polytope. Combinatorial type of P was calculated for example in [2] .

In this paper we shall give a detailed proof of the result only for a 4-dimensional polytopes, because for another dimension the proof will be completely analogous. When $2d = 4$ vertices v_i, v_j, v_k, v_ℓ form a facet of P iff indexes i, j, k, ℓ may be divided into two disjoint pairs (a_1, a_2) ; (b_1, b_2) , such that

$$|a_1 - a_2| = 1 \ , \ |b_1 - b_2| = 1 \ (\mathrm{mod}\, n).$$

§ 2. THE PROOF OF THE MAIN THEOREM.

LEMMA 1. Let $P_0 = conv\{v_1, \ldots, v_n\}$ be a polytope, combinatorially isomorphic to a cyclic one. Then $P_1 = conv\{v_1, \ldots, v_{n-1}\}$ is a convex polytope, which is combinatorially isomorphic to a cyclic polytope with $n-1$ vertices.

PROOF. Let consider a polytope $P_t = conv\{v_1 \cdots v_{n-1}, (1-t)v_n + tv_{n-1}\}$. If $t \in [0,1)$ then $[v_1, (1-t)v_n + tv_{n-1}]$ is the edge of P_t , because v_1 , v_n , v_{n-1} form a 2-dimensional face of P_0 . If $t \in [0,1)$, then P_t is combinatorially isomorphic to P_0 , because otherwise there would be $t^* \in [0,1)$ and P_{t^*} would possess a non-simplicide facet, and then there would be two vertices with would not be endpoints of common edge. Supposing, that $t \to 1$, we obtain desired result.

LEMMA 2. Let $P^4_{n-1} = conv\{v_1 \ldots v_{n-1}\}$ be a polytope, combinatorially isomorphic to a cyclic polytope. Then there exists a non-empty convex open set $\mathcal{U} \subset R^4$, such that $v_n \in \mathcal{U} \Leftrightarrow P_n^4 = conv\{P, v_n\}$ is combinatorially isomorphic to a cyclic polytope with n vertices.

PROOF. Applying Grünbaum Lemma [2] we conclude that P would have combinatorial type of cyclic polytope iff v_n is beyond facets of P containing $[v_1, v_{n-1}]$ except $(v_{n-1}, v_1, v_{n-2}, v_2)$ and beneath the rest. We shall prove that the set of such points in non-empty. Let $O \in \operatorname{Int} P_{n-1}^4$ and $\mathring{P}_{n-1}^4$ be a dual polytope with the origin in O. The edge figure P_{n-1}^4 of $[v_{n-1}, v_1]$ is a polygon. There exists a hyperplane which separates all the vertices of this polygon except one, corresponding to $(v_1, v_{n-1}, v_2, v_{n-2})$ from the rest vertices of $\mathring{P}_{n-1}^4$. If p is the preimage of this hyperplane, then $p \in \mathcal{U}$. It is clear, that $\mathcal{U}$ is an open convex set.

REMARK. These Lemmas remain true, if we substitute dimension 4 by another even dimension.

THEOREM 1. $C_a(P_n^{2d})$ is homeomorphic to $R^{2d(n-2d-1)}$, where P_n^{2d} is a cyclic $2d$-dimensional polytope with n vertices.

PROOF. When $n = 2d+1$ P_{2d+1}^{2d} is a simplex and $C_a(P_{2d+1}^{2d})$ is a point.

If $n > 2d+1$ let us consider the map ρ defined as follows.

$$\rho(conv\{v_1 \ldots v_n\}) = conv\{v_1 \ldots v_{n-1}\}, \quad \rho : C_a(P_{n+1}^{2d}) \to C_a(P_n^{2d}).$$

Applying Lemma 1 we deduce that ρ is correctly determined. As it follows from Lemma 2 preimage of each point is homeomorphic to R^{2d}. Therefore we have: $C_a(P_n^{2d}) = R^{2d(n-2d-2)} \times R^{2d} = R^{2d(n-2d-1)}$.

§ 3. COROLLARY

Let G be a finite subgroup of the combinatorial automorphisms group of a cyclic polytope P_n^{2d} and $|G| = p$, where p is a prime. Then the set of polytopes $F \subset C(P_n^{2d})$ in each of which

G -symmetry can be realized by an affine transformation has exactly two components of linear connectivity.

PROOF. Let consider the action of G on $C_a(P_n^{2d})$. Let F_A be a set of fixed points of this action. From Smith's inequalities (see, for example, [3]) we get:

$$\beta_0(F_A) \leq \sum_{K=0}^{\infty} \beta_K(C_a(P_n^{2d})),$$

$$\sum_{i=0}^{\infty}(-1)^i\beta_i(F_A) = \sum_{i=0}^{\infty}(-1)^i\beta_i(C_a(P_n^{2d}))$$

where $\beta_i(X) = \dim H_i(X; Z_p)$ are the Betti numbers. From the Theorem 1 we deduce $\beta_0(F_A)=1$, therefore $\beta_0(F)=2$.

REMARK. It is also true if $|G|=p^\alpha$, where p is a prime.

§ 4. GENERAL REMARKS ABOUT "SURGERY" OF POLYTOPES

We are interested in such operations between two polytopes $P_2 \to P_1$ which can be extended to a map $C_a(P_2) \to C_a(P_1)$

Two examples are to be given

1. Let $P_2 = conv\{P_1, v_{n+1}\}$ where v_{n+1} is beyond the only simplicial facet of P and beneath the rest.

Then we have a map $\rho: C_a(P_2) \to C_a(P_1)$ defined as follows:

$$\rho(conv\{v_1 \dots v_{n+1}\}) = conv\{v_1 \dots v_n\}.$$

2. Let polytope P_2 is obtained from P_1 by dual operation , namely first operation was applied to the polar of P_1 and then we have got a polar of the polytope obtained.

In this case we also have a map

$$C_a(P_2) \to C_a(P_1).$$

Using such consideration N.E.Mnev has deduced from Steinlitz theorem the following result.

THEOREM 2. $C_a(P)$, where P is a 3-dimensional polytope, is homeomorphic to R^m.

So we have got the

COROLLARY. The combinatorial automorphism of the prime power order of the 3-dimensional polytope is realized by an affine transformation. Let P be a simplicial polytope. Consider $\overline{C_a}(P) = closure\,(C_a(P))$. Connection between the differential structure of $\partial \overline{C_a}(P)$ and topological properties of $C_a(P)$ is to be considered.

THEOREM 3. If $\overline{C_a(P)}$ is a manifold with angles then $C_a(P) = R^m$, for an appropriate m.

First we shall prove

LEMMA. $C_a(P)$ is contractible to a point in $\overline{C_a(P)}$.

PROOF. Let us consider space of all Gale diagrams (see, for example, [2]), corresponding to the polytopes $M \in C_a(P)$. It is easy to show that this space is homeomorphic to $C_a(P)$. This space consists of all n-tuples $(\hat{v}_1, \dots, \hat{v}_n)$, where $\hat{v}_i \in R^{n-d-1}$

$0 \in \overset{\circ}{conv}\{\hat{v}_i,\ i \in N - I\} \iff conv\{v_i;\ i \in I\}$ is a facet of P and $\hat{v}_1 \dots \hat{v}_{n-d}$ are given.

Without loss of generality we may assume, that $0 \in \overset{\circ}{conv}\{\hat{v}_1 \cdots \hat{v}_{n-d}\}$

$$h_t : \hat{v}_i \to t\hat{v}_i \qquad i > n-d$$
$$\hat{v}_i \to \hat{v}_i \qquad i \leq n-d$$

give us desired homotopy.

PROOF of the theorem:

Since $\overline{C_a}(P)$ is a manifold with angles, there exists homotopy $g_t\ \overline{C_a}(P) \to \overline{C_a}(P)$, such that $g_0 = id$ $\quad g_1(C_a(P)) \subset C_a(P)$.

Combining it with the previous Lemma we obtain immediately: $\overline{C_a}(P)$ is the manifold with angles contractible to a point. Hence, applying Poincaré conjecture we complete the proof.

ACKNOWLEDGMENTS

I am grateful to A.M.Vershik for his quidance and N.E.Mnev for some helpful discussions.

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CONFIGURATIONS OF SEVEN POINTS IN $\mathbb{R}P^3$

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Let $\mathbb{P}_n$ be the real projective n-space and $S\mathbb{P}_n^m$ be its m-th symmetric power. We call a point of $S\mathbb{P}_n^m$ a nonordered n-configuration of degree m. We consider it as a collection of m points of $\mathbb{P}_n$. Let $GS\mathbb{P}_n^m$ be the subset of $S\mathbb{P}_n^m$ consisting of the configurations with points in general position (see 1.1 below). $S\mathbb{P}_n^m$ has a structure of algebraic variety and $GS\mathbb{P}_n^m$ is its open subset, which consists of several connected components. We call these components the cameras. The structure and the natural stratification of $S\mathbb{P}_n^m$ for $n \leqslant 3$ and $m \leqslant 6$ can be described without difficulties. The main purpose of the present paper is to describe the mutual position of the cameras in $S\mathbb{P}_3^7$. We gain it by three steps. At first we describe the mutual position of the cameras in $S\mathbb{P}_2^7$. Then we introduce the duality which gives in particular biregular isomorphism $GS\mathbb{P}_{n-1}^m/PGL_{n-1} \cong GS\mathbb{P}_{m-n-1}^m/PGL_{m-n-1}$. At last we describe the mutual position of cameras in $GS\mathbb{P}_3^7$, using the Viro theorem on non-amphicheiral configurations. The results given at the first two steps seem to be interesting for their own sake.

Recently new connections between the properties of configurations and non-singular real projective varieties were found (see [14], [17]). This sets new problems and requiers new approaches in the investigation of confugurations. The author's interest to these problems is originated from his studing the real plane algebraic curves [16] and is due to the personal contacts with his teacher V.A.Rohlin and with the participants of the Rohlin seminar in Leningrad. The author's investigation was influenced also by ideas of A.M.Vershik on rigid isotopies and the duality of configurations (see [10]). The author is greatly indebted to O.Ya.Viro for the fruitful discussions and useful advices.

§ 1. BASIC CONSTRUCTIONS

1.1. Combinatorial types of configurations.

A point of m-th power $\mathbb{P}_n^m$ is called an ordered n-configurations of degree m. For $A \in \mathbb{P}_n^m$ we denote the i-th component of A by A_i, so $A = (A_1, \dots, A_m)$. The least projective subspace of $\mathbb{P}_n$ which contains all the points $A_1, \dots, A_m$ will be denoted by $V(A)$. We set

$$def_V(A) = min(m-1, n) - dim\, V(A).$$

For $S = \{ i_1, \dots, i_s \}$, $1 \leqslant i_1 < \dots < i_s \leqslant m$ we set $A_S = (A_{i_1}, \dots, A_{i_s})$. We call the combinatorial characteristic of A the function C_A defined on the set of all subsets of $\{ 1, \dots, m \}$, which set in correspondence to $S \subset \{ 1, \dots, m \}$ number $def_V(A_S)$. Two configurations $A, B \in \mathbb{P}_n^m$ are said to be combinatorialy equivalent if $C_A = C_B$. We call configuration A a generic one if $C_A \equiv 0$. We denote the set of all generic configurations $A \in \mathbb{P}_n^m$ by $G\mathbb{P}_n^m$.

It can be shown that the set of configurations combinatorialy equivalent to a given one is a locally closed subset of $\mathbb{P}_n^m$ in Zariski topology. So we have an algebraic stratification of $\mathbb{P}_n^m$ with strata consisting of combinatorialy equivalent configurations. We call this stratification the natural one. Each stratum is a union if several connected components, which are to be called the partitions. We say that two configurations are rigidly isotopic if they are situated in the same partition. In other words two configurations are rigidly isotopic if they can be joined by isotopy in $\mathbb{P}_n$ consisting of configurations of the same combinatorial type. We call cameras and walls the partitions of codimension 0 (i.e. connected components of $G\mathbb{P}_n^m$) and codimension 1.

Configuration $A \in \mathbb{P}_n^m$ is said to be reducible if there exists a pair of disjoint projective subspaces of $\mathbb{P}_n$, the union of which contains all the elements $A_1, \ldots, A_m$. We denote by $I\mathbb{P}_n^m$ the set of all irreducible configurations from $\mathbb{P}_n^m$. It can be easily seen that $I\mathbb{P}_n^m = \emptyset$ if $m \leq n+1$; $I\mathbb{P}_n^{n+2} = G\mathbb{P}_n^{n+2}$ and $I\mathbb{P}_n^m$ contains all the cameras and walls of $\mathbb{P}_n^m$ if $m \geq n+3$.

1.2. The actions of S_m and PGL_n .

The symmetric group S_m and the projective group PGL_n act naturally on $\mathbb{P}_n^m$: the first one by permutations of the configurations' elements and the second one by applying transformation to each element. Since these two actions commute we have the action of $S_m \times PGL_n$ on $\mathbb{P}_n^m$. Let

$$S\mathbb{P}_n^m = \mathbb{P}_n^m / S_m \ , \quad \overline{\mathbb{P}_n^m} = \mathbb{P}_n^m / PGL_n \ , \quad \overline{S\mathbb{P}_n^m} = \mathbb{P}_n^m / S_m \times PGL_n$$

be the orbit spaces. The space $S\mathbb{P}_n^m$ has an algebraic variety structure as an orbit space of the finite group action, while $\overline{\mathbb{P}_n^m}$ and $\overline{S\mathbb{P}_n^m}$ have not. However their subspaces $\overline{I\mathbb{P}_n^m} = I\mathbb{P}_n^m / PGL_n$ and $IS\mathbb{P}_n^m = I\mathbb{P}_n^m / S_m \times PGL_n$ have the structure of algebraic prevariety, which is the same as the structure of variety without the condition that the space is separated (see [8]).

The actions of S_m and PGL_n on $\mathbb{P}_n^m$ induce actions on strata and partitions of $\mathbb{P}_n^m$. Hence we have stratifications of factorspaces, which we call the natural ones too. Their strata are the locally closed sets in Zariski topology of $S\mathbb{P}_n^m$, $\overline{I\mathbb{P}_n^m}$, $\overline{IS\mathbb{P}_n^m}$. We define the notions of partition, camera, wall, rigid isotopy for $S\mathbb{P}_n^m$, $I\mathbb{P}_n^m$, $IS\mathbb{P}_n^m$ just as for $\mathbb{P}_n^m$. We set $GS\mathbb{P}_n^m = G\mathbb{P}_n^m / S_m$.

1.3. Factorgraph

We shall call a graph a triad $\Gamma = (V, P, I)$ consisting of two sets V and P, which are called the sets of vertices an edges and of map $I : P \to V \times V / S_2$ from P into the set of nonordered pairs of vertices, which is called the incidence map. We call a morphism $\Gamma_1 \to \Gamma_2$ of graph $\Gamma_1 = (V_1, P_1, I_1)$ to graph $\Gamma_2 = (V_2, P_2, I_2)$ pair of maps $f : (f_V, f_P)$, $f_V : V_1 \to V_2$, $f : P_1 \to P_2$ such that the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{I_1} & V_1 \times V_1 / S_2 \\ \Big\downarrow{\scriptstyle f_P} & & \Big\downarrow{\scriptstyle f_V \times f_V / S_2} \\ P_2 & \xrightarrow{I_2} & V_2 \times V_2 / S_2 \end{array}$$

is commutative.

Let we have an equivalence relations on the sets V and P and let $\overline{V}$, $\overline{P}$ be the equivalence classes. If there exists a map $\overline{I} : \overline{P} \longrightarrow \overline{V} \times \overline{V} / S_2$, which makes diagram

$$\begin{array}{ccc} P & \xrightarrow{I} & V \times V / S_2 \\ \Big\downarrow & & \Big\downarrow \\ \overline{P} & \xrightarrow{\overline{I}} & \overline{V} \times \overline{V} / S_2 \end{array}$$

commutative, then we call the graph $\overline{\Gamma} = (\overline{V}, \overline{P}, \overline{I})$ the factorgraph for the given equivalence relation. It is clear that the factorgraph is unique if it exists. The morphism $\Gamma \to \overline{\Gamma}$, which is given by the pair of projections $V \to \overline{V}$, $P \to \overline{P}$ is called the projection of factorization.

Let $Aut(\Gamma)$ be the group of automorphisms of graph Γ. We say that group G acts on Γ if we have homomorphism $G \to Aut(\Gamma)$. The orbit sets $\overline{V} = V/G$, $\overline{P} = P/G$ are the equivalence classes satisfying the above condition. We denote by Γ/G the factorgraph defined by this equivalence relation.

1.3.1. EXAMPLE. Let Γ be the graph with 2 vetices and 1 edge connecting them. If we take the action of the group $G = Aut(\Gamma)$ (which is isomorphic to symmetric group S_2) then the graph Γ/G is the graph with a single vertex and single edge-loop.

1.3.2. REMARK. We can associate 1-dimensional CW-complex C_Γ with a graph Γ and construct an action of group G on C_Γ according to its action on Γ. The above example shows that CW-complex $C_{\Gamma/G}$ is not necessarily homeomorphic to the factorspace of C_Γ by the action of G.

1.4. Adjacency graph

We describe the mutual position of the cameras in the spaces $\mathbb{P}_n^m$, $S\mathbb{P}_n^m$, $\overline{I\mathbb{P}_n^m}$ and $\overline{IS\mathbb{P}_n^m}$ in terms of graphs. Let X be one of these spaces. We call the adjacency graph of X the graph, which set of vertices is the set of cameras of X, set of edges is the set of walls of X and which incidence map set in correspondence to a wall pair of cameras adjacent to it. Some pair may happen to be a double of one camera:

a) if X has a boundary and the wall is contained in it, as in example 1.4.1 below;

b) if the wall is one-sided subset of X;

c) if the wall is two-sided subset but has the same camera adjacent from each side.

Each of these cases corresponds to a loop in the adjacency graph. We call a wall the inner one in the cases b) and c).

We denote by Γ_n^m, $S\Gamma_n^m$, $\overline{\Gamma_n^m}$ and $\overline{S\Gamma_n^m}$ the adjacency graphs of the spaces $\mathbb{P}_n^m$, $S\mathbb{P}_n^m$, $\overline{I\mathbb{P}_n^m}$ and $\overline{IS\mathbb{P}_n^m}$.

The natural actions of S_m and PGL_n on $\mathbb{P}_n^m$ give actions of these groups on Γ_n^m. It can be shown that $S\Gamma_n^m \cong \Gamma_n^m / S_m$. If $m \geq n+3$ then $\overline{\Gamma_n^m} \cong \Gamma_n^m / PGL_n$ and $\overline{S\Gamma_n^m} \cong \Gamma_n^m / S_m \times PGL_n \cong$ $\cong S\Gamma_n^m / PGL_n$ (if $m \leq n+2$ then the graphs $\overline{\Gamma_n^m}$ and $\overline{S\Gamma_n^m}$ have no edges since $I\mathbb{P}_n^m$ and $IS\mathbb{P}_n^m$ does not contain the walls). For details see [14].

1.4.1. EXAMPLE. It is quite evident that configurations $A, B \in S\mathbb{P}_1^m$ are rigidly isotopic if each of them is generic or if each of them has just one pair of coinciding elements. So the space $S\mathbb{P}_1^m$ contains the only camera (for $m \geq 1$) and the only wall (for $m \geq 2$). So the graph $S\Gamma_1^m$ has a single vertex and a single edge - loop for $m \geq 2$.

1.4.2. EXAMPLE. If is well known that any configuration $A \in G\mathbb{P}_n^{n+2}$ can be transfered into any other configuration $B \in G\mathbb{P}_n^{n+2}$ by the only projective transformation. So the spaces $G\mathbb{P}_n^{n+2}$ and PGL_n are homeomorphic. We can deduce from that fact that:

a) Γ_n^{n+2} has 1 vertex and 1 edge - loop, if n is even and 2 verteces and 1 edge connected them, if n is odd;

b) $S\Gamma_n^{n+2}$, Γ_n^{n+1}, $S\Gamma_n^{n+1}$ are the graphs with 1 vertex and 1 edge for any $n \geq 1$;

c) $\overline{\Gamma_n^{n+2}}$, $\overline{S\Gamma_n^{n+2}}$ and Γ_n^m, $S\Gamma_n^m$ for $m \leq n$, $n \geq 1$ are the graphs with a single vertex and without edges;

d) $\overline{\Gamma_n^m}$, $\overline{S\Gamma_n^m}$ for $m \leq n+1$, $n \geq 1$ are the empty graphs (without vertices and edges).

1.5. The MIRROR INVOLUTION. Since the group PGL_n is connected for even n and so acts trivially on Γ^m_n and S^m_n we have $\Gamma^m_n / PGL_n \cong \Gamma^m_n$, $S\Gamma^m_n / PGL_n \cong S\Gamma^m_n$ and hence $\overline{\Gamma^m_n} \cong \Gamma^m_n$, $\overline{S\Gamma^m_n} \cong S\Gamma^m_n$ for even n and $m \geqslant n+3$. For an odd n PGL_n is a disjoint union of connected components PGL^+_n and PGL^-. The component of identity PGL^+_n acts trivially on Γ^m_n and $S\Gamma^m_n$, while the second component gives involutions on Γ^m_n and $S\Gamma^m_n$, which we call the mirror involutions. We denote them by θ^m_n and ${}^S\theta^m_n$. We have from the definition that $\Gamma^m_n / PGL_n = \Gamma^m_n / \theta^m_n$, $S\Gamma^m_n / PGL_n = S\Gamma^m_n / {}^S\theta^m_n$ and hence for $m \geqslant n+3$: $\overline{\Gamma^m_n} \cong \Gamma^m_n / \theta^m_n$, $\overline{S\Gamma^m_n} \cong S\Gamma^m_n / \theta^m_n$ (it holds, certainly, also for the even n if we set θ^m_n and ${}^S\theta^m_n$, to be the identity involutions).

It is clear that the mirror involutions are induced by the isomorphisms $\mathbb{P}^m_n \longrightarrow \mathbb{P}^m_n$ and $S\mathbb{P}^m_n \longrightarrow S\mathbb{P}^m_n$ defined by the mirror reflection about any hyperplane in $\mathbb{P}_n$.

We say that a configuration (no matter ordered or nonordered) is amphicheiral, if it is rigidly isotopic to its image under the mirror reflection about hyperplane.

1.6. STRUCTURAL GRAPH. Let A be n-configuration (no matter ordered or nonordered), which elements are not contained all in one hyperplane. Each pair of its noncoinciding elements can be connected by 2 linear segments in $\mathbb{P}_n$. We say that a hyperplane crosses the segment if their intersection is non-empty and does not contain an end point of the segment. We can construct a graph which vertices are the elements of A and edges are the segments, not crossed by any of the hyperplanes determined by generic subconfigurations of A of degree n (we add also an edge for each pair of coinciding elements of A). We call this graph the structural one and denote it by Γ_A.

We leave for the reader to prove the next useful property of the structural graph.

1.6.1. BASIC PROPERTY. Let $A \in GS\mathbb{P}_n^m$, $m \geq n+1$ and A_i, A_j be the vertices of Γ_A, connected by an edge. Then the configurations $A_{\hat{i}}, A_{\hat{j}} \in GS\mathbb{P}_n^{m-1}$ which we can get respectively by removing the elements A_i, A_j from A, are rigidly isotopic.

It follows that the configurations $A_{\hat{i}}$ and $A_{\hat{j}}$ are rigidly isotopic also in the case when A_i and A_j belong to the same connected component of Γ_A.

1.7. (n,k) -CONFIGURATIONS. In addition to the configurations of points we shall deal with the configurations of hyperplanes and the configurations of lines in $\mathbb{P}_3$. So it is convenient to introduce a generalizing notion of (n,k)-configuration. Let $\mathbb{P}_{n,k}$ be the manifold of all the linear projective k-subspaces of $\mathbb{P}_n$ (it can be identified with the grassmanian $G_{n+1,k+1}(\mathbb{R})$). A point of its power $\mathbb{P}_{n,k}^m$ is called an ordered (n,k)-configuration of m-th degree. We have $\mathbb{P}_{n,0}^m = \mathbb{P}_n^m$ and $\mathbb{P}_{n,k}^m \cong \mathbb{P}_{n,n-k-1}^m$. For $A \in \mathbb{P}_{n,k}^m$ we denote by $\Lambda(A)$ the intersection of the elements $A_1, \ldots, A_m$ of A and by $V(A)$ the minimal linear subspace of $\mathbb{P}_n$, containing these elements. We set

$$def_V(A) = min(n, m(k+1)-1) - dim\, V(A),$$

$$def_\Lambda(A) = dim\, \Lambda(A) - max(-1, n-m(n-k)),$$

($dim\, \emptyset = -1$). For $S = \{i_1, \ldots, i_s\}$, $1 \leq i_1 < \ldots < i_s \leq m$ we denote by A_S the configuration $\{A_{i_1}, \ldots, A_{i_s}\}$. We call the combinatorial characteristic of A the map from the set of all the subsets of $\{1, \ldots, m\}$ into $\mathbb{Z} \times \mathbb{Z}$ which maps a set $S \subset \{1,\ldots,m\}$ to pair $(def_V(A_S), def_\Lambda(A_S))$ just in the same way as it was for the configurations of points. We can introduce the natural structification of $\mathbb{P}_{n,k}^m$, define actions of S_m and of PGL_n define the factorspaces $S\mathbb{P}_{n,k}^m$, $\overline{\mathbb{P}_{n,k}^m}$, $\overline{S\mathbb{P}_{n,k}^m}$. We introduce also by analogy the notions of rigid isotopy, partition, camera, wall, adjacency graph and define the spaces $G\mathbb{P}^{m}_{n,k}$, $GS\mathbb{P}^{m}_{n,k}$ of generic configurations.

Since for $(n,0)$-configurations the numbers $def_{\Lambda}(A_5)$ are determined by the numbers $def_{V}(A)$, the new definitions coincide with the former ones in this case. It is clear also that the natural isomorphisms $\mathbb{P}^{m}_{n,k} \cong \mathbb{P}^{m}_{n,n-k-1}$ and $S\mathbb{P}^{m}_{n,k} \cong S\mathbb{P}^{m}_{n,n-k-1}$ keep the strutifications.

.8. CONFIGURATIONS OF HYPERPLANES (ARRANGEMENTS). The nonordered $(n,n-1)$- configurations which have no coinciding elements are traditionally called n-arrangements. n-arrangement is said to be simple one if it is generic and have no points common for its hyperplanes. We say that configurations $A,B \in \mathbb{P}^{m}_{n,n-1}$ are homeomorphic if there exists a homeomorphism $\mathbb{P}_n \longrightarrow \mathbb{P}_n$ which maps A_i into B_i. If we omit the order preserving condition we get the definition of homeomorphism for n-arrangements. The rigidly isotopic $(n,n-1)$-configurations are clearly homeomorphic (the opposite statement is not true as it was shown by N.E.Mnev [11]).

The hyperplanes of n-arrangement if they have no common points give a cellular decomposition of $\mathbb{P}_n$. It is easy to see that two such n-arrangements are homeomorphic if and only if they give combinatorially isomorphic cellular decompositions.

We denote by $p_{\tau}(A)$ the number of n-cells with τ facets in the cellular decomposition defined by the hyperplanes of simple n-arrangement A of degree m. We have clearly $p_{\tau}(A)=0$ if $\tau \leqslant n$ or if $\tau > m$. We call the sequence $p(A) = (p_{n+1}(A),\ldots,p_{m}(A))$ the spectrum of A.

More information about topological and combinatorial properties of arrangements can be found in [5] and [1].

§ 2. PLANAR CONFIGURATIONS

2.1. Topological classification of simple arrangements for $m \leqslant 8$.

The classification of simple 2-arrangements for $m \leqslant 4$ is a trivial problem because of their projective equivalence. We can see that $\rho(A)=(4)$ if $A \in GS\mathbb{P}^3_{2,1}$ and $\rho(A)=(4,3)$ if $A \in GS\mathbb{P}^4_{2,1}$. The problem of classification for $m=5$ and $m=6$ also can be solved without any troubles. There exists the only topological type of simple 2-arrangements for $m=5$ (spectrum $5,5,1$) and 4 topological types for $m=6$ (spectra $(6,9,0,1)$, $(6,8,2)$, $(7,6,3)$, $(10,0,6)$.

For $m=7$ the topological classification of simple 2-arrangements was given by H.S.White [11] and L.D.Cummings [2]. There exist 11 topological types of such arrangements (see dia. 1 in appendix). For $m \leqslant 6$ we could see that the spectrum of simple arrangement determines its topological type. For $m \geqslant 7$ it does not. There exist a pair of topological types with spectrum $(7,12,3)$ and a pair with spectrum $(8,10,4)$. In each pair one type gives arrangements which have two pentagons with common edge and the other type gives arrangements without such pairs of pentagons. We would denote topological types by their spectra supplying the spectra $(7,12,3)$ and $(8,10,4)$ with indices A or B : A for types with a pair of adjacent pentagons and B for types without such pairs. The problem of classification for $m \geqslant 8$ seems to be too difficult to deal with without computers. E.R.Halsey [6] and R.I.Canham [1] solved this problem for $m=8$ using 2 diffirent computer algorithms. It was proved to exist 135 topological types of simple 2-arrangements of degree 8.

2.2. Isotopic classification.

The topological type of 2-arrangement for $m \leqslant 7$ determines its rigid isotopy type (see [14]). So for $m \leqslant 5$ the space $S\mathbb{P}^m_{2,1}$ contains 1 camera, $S\mathbb{P}^6_{2,1}$ contains 4 cameras and $S\mathbb{P}^7_{2,1}$ contains

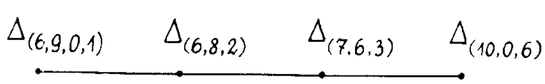
11 cameras. It may seem to be likely that the homeomorphic 2-arrangements are rigidly isotopic for any m, but it isn't true even for case of simple arrangements as results of N.E.Mnev [14] show. His simplest example contains 19 lines and the question whether or not homeomorphic 2-arrangements are always rigidly isotopic is open for $8 \leq m \leq 18$

We call an arrangement k-donal if k is the maximal number of sides, which have the polygons of this arrangement (so k is a number of the latest position in the spectrum of arrangement where a nonzero integer stands). It is difficult to prove that any two m-donal 2-arrangements of degree m can be connected by a rigid isotopy for any m (see [9], [16]). It can be proved also that for any m homeomorphic $(m-1)$-donal 2-arrangements of degree m are rigidly isotopic (see [16]).

Let $\check{A} \in S\mathbb{P}_2^m$ is a dual configuration for $A \in S\mathbb{P}_{2,1}^m$. We call $\check{A}$ the k-gonal configuration if A is k-donal arrangement. If $\check{A}$ is m-gonal then we can remove some line from $\mathbb{P}_2$ to get an affine plane, on which the elements of $\check{A}$ become the vertices of convex m-polygon. By analogy we can give description of $\check{A}$ if it is k-donal.

2.3. The graphs $S\Gamma_2^m$ for $m \leq 7$.

We use the description of the adjacency graphs $S\Gamma_2^m$ for $m \leq 7$, which is given in [16]. For $m \leq 4$ the graph $S\Gamma_2^m$ is described in 1.4.2, graph $S\Gamma_2^5$ has a single vertex and a single edge-loop. The graph $S\Gamma_2^6$ is shown on the diagram below.



The structure of the graph $S\Gamma_2^7$ is much more complicated. So it has loops with centres $\Delta_{(8,11,2,1)}$, $\Delta_{(11,5,5,1)}$, $\Delta_{(8,10,4)B}$ (pair of loops for each vertex) and a double edge between vertices $\Delta_{(9,8,5)}$ and $\Delta_{(8,10,4)_A}$ (see dia. 2 in appendix).

One of the loops with centre $\Delta_{(8,10,4)_B}$ corresponds to a two-sided inner wall in $S\mathbb{P}^7_4$ and the other correspond to one-sided inner walls.

§ 3. EXTRINSIC DUALITY

3.1. The history of construction

The subject of this Section is the duality between configurations of $\mathbb{P}^m_{n-1}$ and of $\mathbb{P}^m_{m-n-1}$ which induces also duality between configurations of $S\mathbb{P}^m_{n-1}$ and $S\mathbb{P}^m_{m-n-1}$. The idea of this duality seems to belong to H.Whitney. He investigated its combinatorial properties which can be formulated in terms of matroids (see [12]). D.Gale studied the geometrical properties of the duality construction and used this construction in the theory of politopes with great success (Gale transformations, Gale diagrams, see [3] , [4]). P.McMullen was the first who applied this construction to arrangements [7] . R.I.Canham [1] and E.R.Halsey [6] carried further investigation of the McMullen's construction and clarify its topological nature.

In this article the duality is introduced in algebro-geometric terms as a biregular isomorphism $\overline{I\mathbb{P}^m_{n-1}} \cong \overline{I\mathbb{P}^m_{m-n-1}}$ and its symmetrization $\overline{IS\mathbb{P}^m_{n-1}} \cong \overline{IS\mathbb{P}^m_{m-n-1}}$. The author is greatly indebted to A. Vershic and N.E.Mnev, who acquainted him with their considerations conserning this subject. Now we give two definitions of the extrinsic duality, which we hope can help the reader to grasp the geometrical idea of the construction.

Let $A_1, \dots, A_m \in \mathbb{P}_{m-1}$ be the point with coordinates represented by the lines of the identity matrix (δ_{ij}) of dimention $m \times m$.

Let π be projective $(n-1)$-subspace of $\mathbb{P}_{m-1}$ and π' be $(m-n-1)$-subspace dual to it. Let $\rho : (\mathbb{P}_{m-1} \setminus \pi') \to \pi$ and $\rho' : (\mathbb{P}_{m-1} \setminus \pi) \to \pi'$ be the natural projections. If no one of the points $A_1, \dots, A_m$ is contained in π and π' then we get two configurations $(\rho(A_1), \dots, \rho(A_m))$ and $(\rho'(A_1), \dots, \rho'(A_m))$, which we call the extrisicly dual ones. It is projective types between which this duality is defined, since π and π' can be identified with $\mathbb{P}_{n-1}$ and $\mathbb{P}_{m-n-1}$ by different ways.

The definition is also can be given in terms of matrix. Let $A \in \mathbb{P}^m_{n-1}$ be configuration with $V(A) = \mathbb{P}_{n-1}$ and $B \in \mathbb{P}^m_{m-n-1}$ be configuration with $V(B) = \mathbb{P}_{m-n-1}$. Let A_i has coordinates $(a_{i1} : \dots : a_{im})$ and B_j has coordinates $(b_{j1} : \dots : b_{j(m-n)})$. Configuration B is extrinsicly dual to A if $a_{1s} \cdot b_{1t} + \dots + a_{ms} \cdot b_{mt} = 0$ for any $s \in \{1, \dots, n\}$, $t \in \{1, \dots, m-n\}$ (in other words if matrix (b_{ij}) is an orthogonal complement to (a_{ij})).

3.2. The extrinsic duality in vector form.

We would use symbol $\mathbb{R}_n$ to denote the space $\mathbb{R}^n$ and symbol $\mathbb{R}^m_n$ to denote the m-th power of $\mathbb{R}_n$, so an element $A \in \mathbb{R}^m_n$ is a sequence $(A_1, \dots, A_m)$ where $A_i \in \mathbb{R}_n$. An element of $\mathbb{R}^m_n$ is said to be a vector configuration. Two vector configurations $A, B \in \mathbb{R}^m_n$ are called linear equivalent if there exists a linear isomorphism $\mathbb{R}_n \to \mathbb{R}_n$, which maps A_i into B_i for $i = 1, \dots, m$.

Let $L_A : \mathbb{R}_m \to \mathbb{R}_n$ be the linear transformation, which maps the vectors of standart basic $e_1, \dots, e_m$ of $\mathbb{R}_m$ into the element $A_1, \dots, A_m$ of a configuration $A \in \mathbb{R}^m_n$. We denote by $Lin(A_1, \dots, A_m)$ the linear subspace of $\mathbb{R}_n$ generated by the vectors $A_1, \dots, A_m$. We have clearly $\dim(\ker L_A) = m - n$, if $Lin(A_1, \dots, A_m) = \mathbb{R}_n$ and $\ker L_A = \ker L_B$ if B is linearly equivalent to A. Let $C\mathbb{R}^m_n = \{A \in \mathbb{R}^m_n : Lin(A_1, \dots, A_m) = \mathbb{R}_n\}$ and $\overline{C\mathbb{R}^m_n} = C\mathbb{R}^m_n / GL_n$. We have a map from $\overline{C\mathbb{R}^m_n}$ into the grassmanian $G_{m,m-n}$, which maps a linear equivalence class of A into $\ker L_A$. We leave for the reader to prove some properties of extrinsic duality (complete proofs can be found in [16]).

3.2.1. LEMMA. The map $\overline{C\mathbb{R}^m_n} \longrightarrow G_{m,m-n}$ defined above is a bijective one.

3.2.2. LEMMA. The composition $C\mathbb{R}^m_n \to \overline{C\mathbb{R}^m_n} \to G_{m,m-n}$ is a regular morphism of algebraic varieties.

So we get a factorstructure of algebraic variety on $\overline{C\mathbb{R}^m_n}$. We call the composition of isomorphisms $\overline{C\mathbb{R}^m_n} \cong G_{m,m-n} \cong G_{m,n} \cong \overline{C\mathbb{R}^m_{m-n}}$ the outer duality isomorphism. The function C_A from the set of all the subsets of $\{1,\dots,m\}$ into $\mathbb{Z}$, which maps a set $S=\{i_1,\dots,i_s\}$ into the integer $C_A(S)=\min(s,n)-\dim(\mathrm{Lin}(A_{i_1},\dots,A_{i_s}))$ is called the combinatorial characterictic of configuration $A \in \mathbb{R}^m_n$.

3.2.3. LEMMA. Let $A \in C\mathbb{R}^m_n$, $B \in C\mathbb{R}^m_{m-n}$ be extrinsicly dual configurations (e.d.c.). Then $C_A(S)=C_B(\hat{S})$ for any set $S \subset \{1,\dots,m\}$ and for $\hat{S}=\{1,\dots,m\}\setminus S$.

3.2.4. COROLLARY. Let $A \in C\mathbb{R}^m_n$, $B \in C\mathbb{R}^m_{m-n}$ be e.d.c. Then $A_i=0$ if and only if $\mathrm{Lin}(B_1,\dots,\hat{B}_i,\dots,B_m) \neq \mathbb{R}_{m-n}$ (symbol ^ over B_i means that B_i is omitted).

3.2.5. COROLLARY. Let $A \in C\mathbb{R}^m_n$, $B \in C\mathbb{R}^m_{m-n}$ be e.d.c. Let $1 \le i < j \le m$ and $A_i, A_j \neq 0$. Then A_i, A_j are collinear if and only if $\mathrm{Lin}(B_1,\dots,\hat{B}_i,\dots,\hat{B}_j,\dots,B_m) \neq \mathbb{R}_{m-n}$.

3.2.6. LEMMA. Let $A \in C\mathbb{R}^m_n$, $B \in C\mathbb{R}^m_{m-n}$ be e.d.c., $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then configurations $(A_1,\dots,A_{i-1},\alpha A_i,A_{i+1},\dots,A_m)$ and $(B_1,\dots,B_{i-1},\alpha^{-1}\cdot B_i, B_{i+1},\dots,B_m)$ are also e.d.c.

We call configuration $A \in \mathbb{R}^m_n$ reducible if the elements of A can be divided into two nonempty subsets, such that the intersection of two subspaces of $\mathbb{R}_n$ linearly generated by their elements is $\{0\}$

3.2.7. LEMMA. Let A and B be e.d.c. Then A is reducible if and only if B is reducible.

We let $I\mathbb{R}^m_n=\{A \in \mathbb{R}^m_n : A$ is irreducible$\}$, $\overline{I\mathbb{R}^m_n}=I\mathbb{R}^m_n/GL_n$ and denote by

$$\alpha^m_m : \overline{I\mathbb{R}^m_n} \longrightarrow \overline{I\mathbb{R}^m_{m-n}}$$

restriction of the extrinsic duality isomorphism.

3.3. Extrinsic duality in projective form

The natural projection $\mathbb{R}_n \setminus \{0\} \to \mathbb{P}_{n-1}$ gives the projection $I\mathbb{R}^m_n \longrightarrow I\mathbb{P}^m_{n-1}$ (since irreducible configurations have no elements equal to 0), which induces $\overline{I\mathbb{R}^m_n} \to \overline{I\mathbb{P}^m_{n-1}}$. According to the lemma 3.2.6 we can construct a map $\beta^m_n : \overline{I\mathbb{P}^m_{n-1}} \longrightarrow \overline{I\mathbb{P}^m_{m-n-1}}$ which makes diagram

$$\begin{array}{ccc} \overline{I\mathbb{R}^m_n} & \xrightarrow{\ \alpha^m_n\ } & \overline{I\mathbb{R}^m_{m-n}} \\ \downarrow & & \downarrow \\ \overline{I\mathbb{P}^m_{n-1}} & \xrightarrow{\ \beta^m_n\ } & \overline{I\mathbb{P}^m_{m-n-1}} \end{array}$$

commutative. It follows that β^m_n is a morphism of prevarieties and since $\beta^m_{m-n} = \left(\beta^m_n\right)^{-1}$ it is a biregular isomorphism. Factorization by the action of S_m gives isomorphism ${}^S\beta^m_n : \overline{IS\mathbb{P}^m_{n-1}} \longrightarrow \overline{IS\mathbb{P}^m_{m-n-1}}$. According to the lemma 3.2.3 isomorphisms β^m_n and ${}^S\beta^m_n$ preserve the natural stratifications. We call these isomorphisms the extrinsic duality isomorphisms. So we get the next theorem.

3.3.1. THEOREM. The extrinsic duality isomorphisms $\beta^m_n : \overline{I\mathbb{P}^m_{n-1}} \to \overline{I\mathbb{P}^m_{m-n-1}}$ and ${}^S\beta^m_n : \overline{IS\mathbb{P}^m_{n-1}} \longrightarrow \overline{IS\mathbb{P}^m_{m-n-1}}$ are the biregular isomorphisms of prevarieties preserving their natural stratifications.

3.3.2. COROLLARY. Isomorphisms β^m_n and ${}^S\beta^m_n$ give 1-1 correspondence between cameras and walls of $\overline{I\mathbb{P}^m_{n-1}}$, $\overline{IS\mathbb{P}^m_{n-1}}$ and $\overline{I\mathbb{P}^m_{m-n-1}}$, $\overline{IS\mathbb{P}^m_{m-n-1}}$. This correspondence keeps adjacencies , so we have $\overline{\Gamma^m_{n-1}} \cong \overline{\Gamma^m_{m-n-1}}$, $\overline{S\Gamma^m_{n-1}} \cong \overline{S\Gamma^m_{m-n-1}}$ for $3 \leqslant n \leqslant m-2$ (we need the last condition for the configurations of cameras and walls to be irreducible).

3.3.3. COROLLARY. For odd $n \geqslant 3$ and even $m \geqslant n+3$ we have $\Gamma^m_{n-1} \cong \Gamma^m_{m-n-1}$, $S\Gamma^m_{n-1} \cong S\Gamma^m_{m-n-1}$. In particular graph $S\Gamma^{n+3}_n$ for even n is a graph with a single vertex and single edge-loop (see 1.4.2).

3.3.4. COROLLARY. The graph $\overline{S\Gamma_3^7}$ is isomorphic to the graph $S\Gamma_2^7$ (see dia. 2 in appendix).

3.3.5. THEOREM. Let $A \in G\mathbb{P}_{n-1}^m$, $B \in G\mathbb{P}_{m-n-1}^m$ be e.d.c. Then the structural graphs Γ_A and Γ_B are isomorphic.

PROOF. If A_i is connected with A_j by edge of Γ_A then a segment connecting A_i with A_j in $\mathbb{P}_{n-1}$ is not crossed by any of hyperplanes determined by elements of A (see 1.6). Let moving A_i along this segment toward A_j , all the other elements of A keeping on their places. This deformation of A is a rigid isotopy (exept of the end of movement). We get configuration A' at the end of movement which has the i-th element coiciding with the j-th one. Let B' be extrinsicly dual to A' . According to 3.2.5 all the elements of B' exept of B_j' and B_j' lie in the same hyperplane, and so one of the segments connecting B_i' with B_j' is not crossed by it. It is clear that after a small deformation of B' this segment gives an edge of structural graph of the result of deformation. Hence B_i and B_j should be connected by an edge of Γ_B .

3.3.6. COROLLARY. Any configuration $A \in GS\mathbb{P}_2^6$ is extrinsicly dual to itself since the structural graphs of configurations from different cameras of $GS\mathbb{P}_2^6$ are different (see dia. 4 in appendix).

§ 4. ADJACENCY GRAPH

4.1. Linking number for (3,1)-configurations of degree 3

In the next two sections we describe the construction of O.Ya.Viro [13] , which shows that all the generic nonordered (3,0)-configurations of degrees 6 and 7 are nonamphicheiral.

For two nonintersecting oriented lines L_1, L_2 in $\mathbb{P}_3$ the linking number $lk(L_1, L_2)$ is defined. It can be equal to ± 1*. We can determine it for example by the next way. Let π is a plane in $\mathbb{P}_3$ such that $L_1 \subset \pi$. The orientation of L_1 gives the orientation of the complement $\pi \setminus L_1$. The intersection number of this complement with L_2 is just equal to the linking number $lk(L_1, L_2)$. In this definition we need the orientation of $\mathbb{P}_3$ to be fixed. If we change the orientation the value of the linking number should be changed too, just as if we change the orientation of one of the lines. The number $lk(L_1, L_2)$ is not changing in the process of rigid isotopy and when we change the order of lines. Now let $L = \{L_1, L_2, L_3\}$ be a set of mutually nonintersecting lines in $\mathbb{P}_3$ (in other words a generic non-ordered (3,1)-configuration of degree 3). Let L_1^*, L_2^*, L_3^* be the same lines with some orientations fixed. We denote the product $lk(L_1^*, L_2^*) \cdot lk(L_2^*, L_3^*) \cdot lk(L_3^*, L_1^*)$ by $lk(L)$ or by $lk(L_1, L_2, L_3)$. It can be easily proved that this product does not depend on the choise of the orientations and order of the lines, that it is changed if we change the orientation of $\mathbb{P}_3$ and that it isn't changed in the process of rigid isotopy.

4.2. The invariants of generic (3,0)-configurations of degrees 6 and 7

Let A be a generic (3,0)-configuration of degree 6 (no matter whether it is ordered or not). We denote by $s(A)$ the sum of the numbers $lk(L)$ for all the generic (3,1)-configurations L of degree 3 defined by partitions of A into pairs. The number $s(A)$ is not changed in the process of rigid isotopy of A and is multiplied

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*) According to the definition, which is generally used in literature the linking number is $\pm\frac{1}{2}$. We regard the double of the linking number.

by -1 if we change the orientation of $\mathbb{P}_3$. This sum consists of 15 summands which are equal to ± 1, hence $S(A) \neq 0$.

Let B be a generic (3,0)-configuration of order 7. If we remove some element B_i from B we get in result generic (3,0)-configuration $B_{\hat{i}}$ of degree 6. We associate with element B_i number +1, if $S(B_{\hat{i}}) > 0$ and number -1, if $S(B_{\hat{i}}) < 0$. So we supply the configuration B with the canonical orientation (as a manifold of dimension 0). This orientation isn't changed in process of rigid isotopy and is replaced by the opposite one, if we change the orientation of $\mathbb{P}_3$. Let $\sigma(B)$ be the signature of B (which is equal to the sum of the numbers associated with elements of B).

We can prove now the next theorem of O.Ya.Viro.

4.2.1. THEOREM. All the generic nonordered (3,0)-configurations of degrees 6 and 7 are not amphicheiral.

PROOF. For generic nonordered (3,0)-configuration A of degree 6 we have $S(A) \neq 0$ and so the value of $S(A)$ is changed if we change the orientation of $\mathbb{P}_3$. Hence A can't be amphicheiral. For configurations of degree 7 the proof is analogous to the above one with replacing S by σ.

This theorem shows that the mirror involutions on the graphs $S\Gamma_3^6$ and $S\Gamma_3^7$ does map no vertex into itself. So the numbers of cameras in $S\mathbb{P}_3^6$ and $S\mathbb{P}_3^7$ are the doubled numbers of cameras in $\overline{S\mathbb{P}_3^6}$ and $\overline{S\mathbb{P}_3^7}$ (see 1.4). Together with the theorem 3.3.1, example 1.4.1, and isotopy classification of generic nonordered (2,0)-configurations (see 2.2) it gives the next result.

4.2.2. THEOREM. The space $S\mathbb{P}_3^6$ contains 2 cameras, and the space $S\mathbb{P}_3^7$ contains 22 cameras.

In other words there exist 2 isotopy types of generic nonordered (3,0)-configurations of degree 6, and 22 isotopy types of generic nonordered (3,0)-configurations of degree 7. Now we can describe the graph $S\Gamma_3^6$. It has 2 vertices by Theorem 4.2.2. It is not difficult to prove also, that the stratum of condition 1 in $S\mathbb{P}_3^6$ is connected, so this graph has the only edge which should connect its two vertices.

Combination of the results of 1.4 and Theorem 3.3.1 gives isomorphisms $S\Gamma_3^7/s\theta_3^7 \cong \overline{S\Gamma_3^7} \cong \overline{S\Gamma_2^7} \cong \overline{S\Gamma}_2^7$. We denote by $\eta=(\eta_v,\eta_p)$ the composition of the projection $S\Gamma_3^7 \to S\Gamma_3^7/s\theta_3^7$ and these isomorphisms.

Since $\eta_v^{-1}(\Delta)$ consists of two elements for each vertex Δ of $S\Gamma_2^7$, $\eta_p^{-1}(\pi)$ also consists of two elements for each edge π of $S\Gamma_2^7$ which connects two different vertices. The problem is to determine how does this pair of edges connects two pairs of vertices.

4.3. Orientations of generic (2,0)-configurations of order 7

Let $s(t)$ be a path in $S\mathbb{P}_3^7$ which connects configurations $A, B \in G S\mathbb{P}_3^7$ from the adjacent cameras. Let this path crosses the wall between this cameras in a single point $C = s\left(\frac{1}{2}\right)$. Then 4 elements of A which turn to be contained in the same plane during the deformation, preserve their orientations after crossing the wall. The other 3 elements change their orientations. To prove this we should remove one element from A and look at $s(t)$. If we remove one of the four elements indicated above, the deformation $s(t)$ turns to be a rigid isotopy of generic (3,0)-configuration of degree 6, while if we remove any other element the deformation turns to be connecting configurations from different cameras of $S\mathbb{P}_3^6$.

We can see that from this rule of changing the orientations it follows that the signatures of configurations from the adjycent cameras of $S\mathbb{P}_3^7$ should differ by 2 or by 6.

The extrincis duality defines the orientations on the generic (2,0)-configurations of order 7. Since the extrinsicly dual configuration for $A \in G S P_2^7$ is defined modulo projective transformation we have really a pair of orientations on A, which are opposite one to another.

If we take a dual property for the rule of changing the orientations given above, then we find what happens when we go from one camera of $S\mathbb{P}_2^7$ to another one through a wall. The orientations are changed in three elements which turn to be at the same line in the process of deformation. In the other 4 elements the orientations are not changed.

4.3.1. LEMMA. The orientation of 7-gonal configuration $A \in GS\mathbb{P}_2^7$ has equal values at each element (the values are all +1 or all -1).

PROOF. Let $B \in GS\mathbb{P}_3^7$ be extrinsicly dual configuration to A. We have isomorphism of the structural graphs $\Gamma_A \cong \Gamma_B$ according to the Theorem 3.3.5. Graph Γ_A is connected (since it forms 7-gon), so Γ_B is connected too. It follows from the property 1.6.1 that all subconfigurations of B of degree 6 are rigidly isotopic. Hence all elements of B have the same value of orientation.

Now we know the orientations of configurations from one camera of $S\mathbb{P}_2^7$. We know also how the orientations are changed if we go from some camera to the adjacent one. So we can determine the orientations of configurations from each camera step by step. The result of the calculations is shown on diagram 4 in appendix.

4.4. The edges of the graph $S\Gamma_3^7$

Let Δ_1 and Δ_2 be two different vertices of $S\Gamma_2^7$ connected by an edge π . Let $\eta_V^{-1}(\Delta_1) = \{\Delta_1^+, \Delta_1^-\}$, $\eta_V^{-1}(\Delta_2) = \{\Delta_2^+, \Delta_2^-\}$, where "+" stands if camera consists of the configurations with positive signature and "-" stands if their signature is negative.

Since signatures of configurations from the adjacent cameras should differ by 2 or by 6, we can determine whether Δ_1^+ is connected by the edge with Δ_2^+ (and Δ_1^- with Δ_2^-) or with Δ_2^- (and Δ_1^- with Δ_2^+). There we use that we know values of signatures and that the values are odd.

Since the signatures of adjacent cameras in $S\mathbb{P}_3^7$ should be different, the graph $S\Gamma_3^7$ have no loops. So the only question remained to construct the graph $S\Gamma_3^7$ is whether $\eta_p^{-1}(\pi)$ consists of 1 element ot of 2 elements for the loop π of the graph $S\Gamma_2^7$.

4.4.1. LEMMA. $\eta_\rho^{-1}(\pi)$ consists of 1 element if π is one-sided wall and consists of 2 elements if π is two-sided wall.

PROOF. Let $\pi' \in \eta_\rho^{-1}(\pi)$. By the definition π' is the only element of $\eta_\rho^{-1}(\pi)$ if and only if the mirror involution $^S\theta_3^7 : S\Gamma_3^7 \longrightarrow S\Gamma_3^7$ maps π' into itself or in other words iff π' consists of amphicheiral configurations. If π' consists of amphecheiral configurations then π'/PGL_3 is on one-sided wall in $S\mathbb{P}_3^7$ because the action of any element $\xi \in PGL_3^-$ maps π' into itself but transpose 2 cameras adjacent to π'. If π' is not the mirror image of itself than π'/PGL_3 is a two-sided wall in $S\mathbb{P}_3^7$ because any desorienting loop in π'/PGL_3 is covered by a desorienting loop in π'. To conclude the proof we note that a wall π is one-sided if and only if π/PGL_2 is a one-sided wall in $S\mathbb{P}_2^7$.

Now the information about the graph $S\Gamma_2^7$ (see 2.1) and signatures of the generic (3,0)-configurations of degree 7 enable us to construct the graph $S\Gamma_3^7$. It is shown on diagram 5 in appendix.

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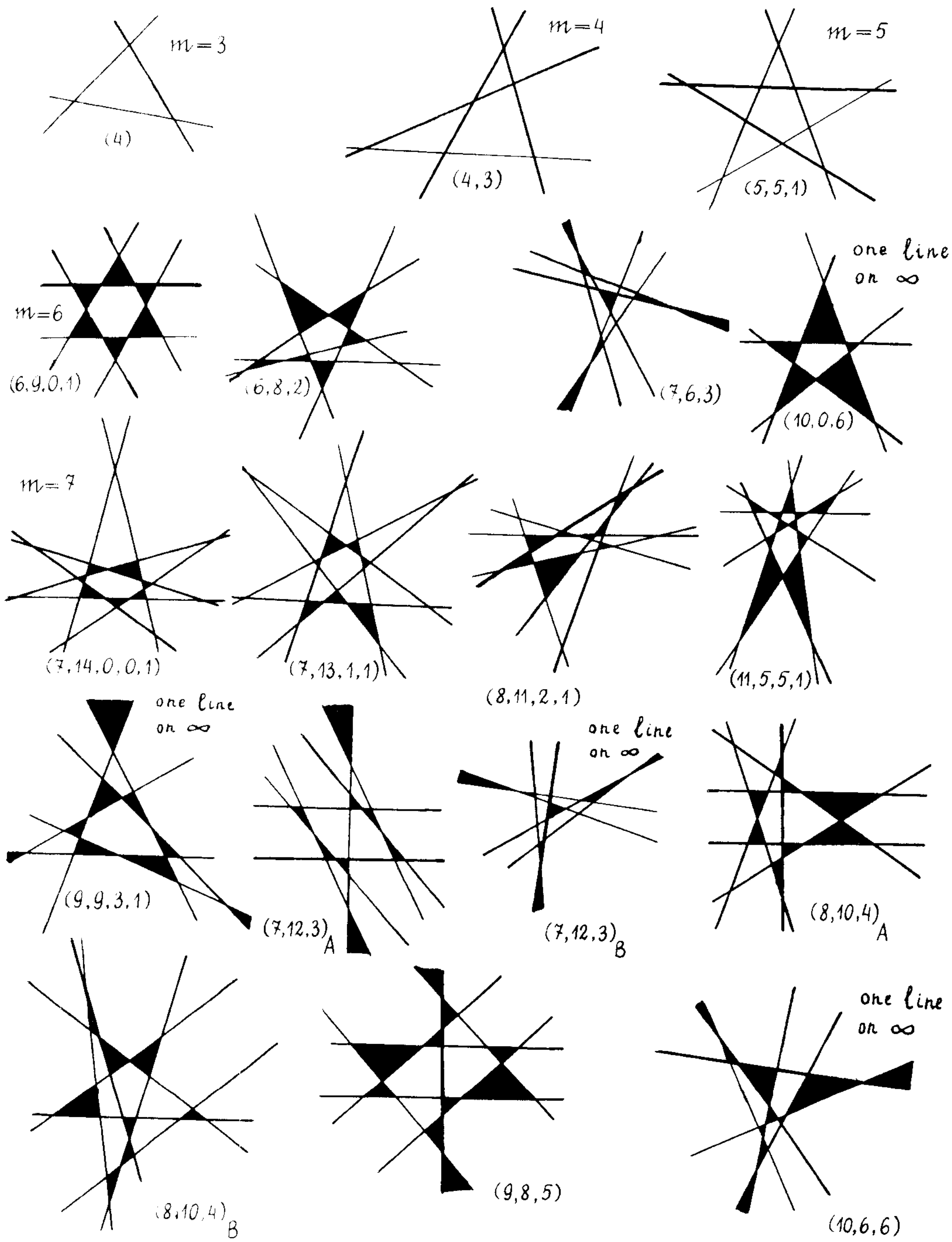
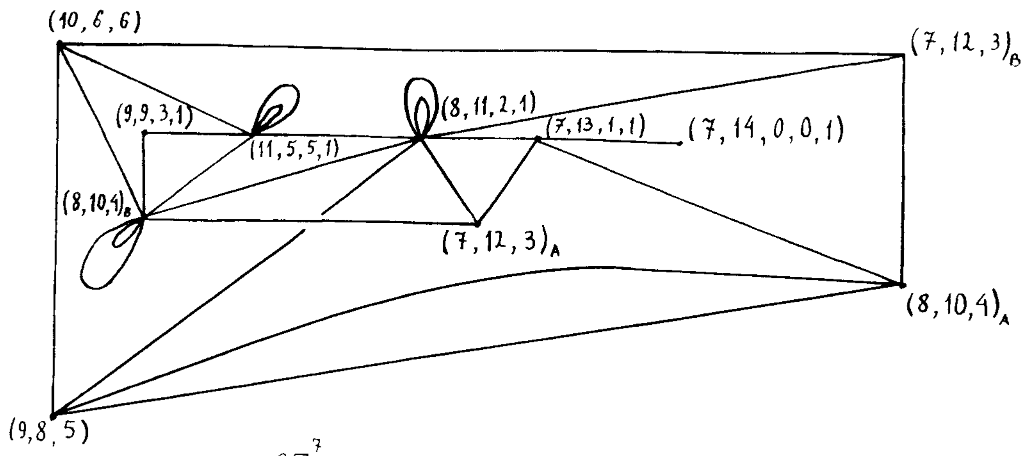


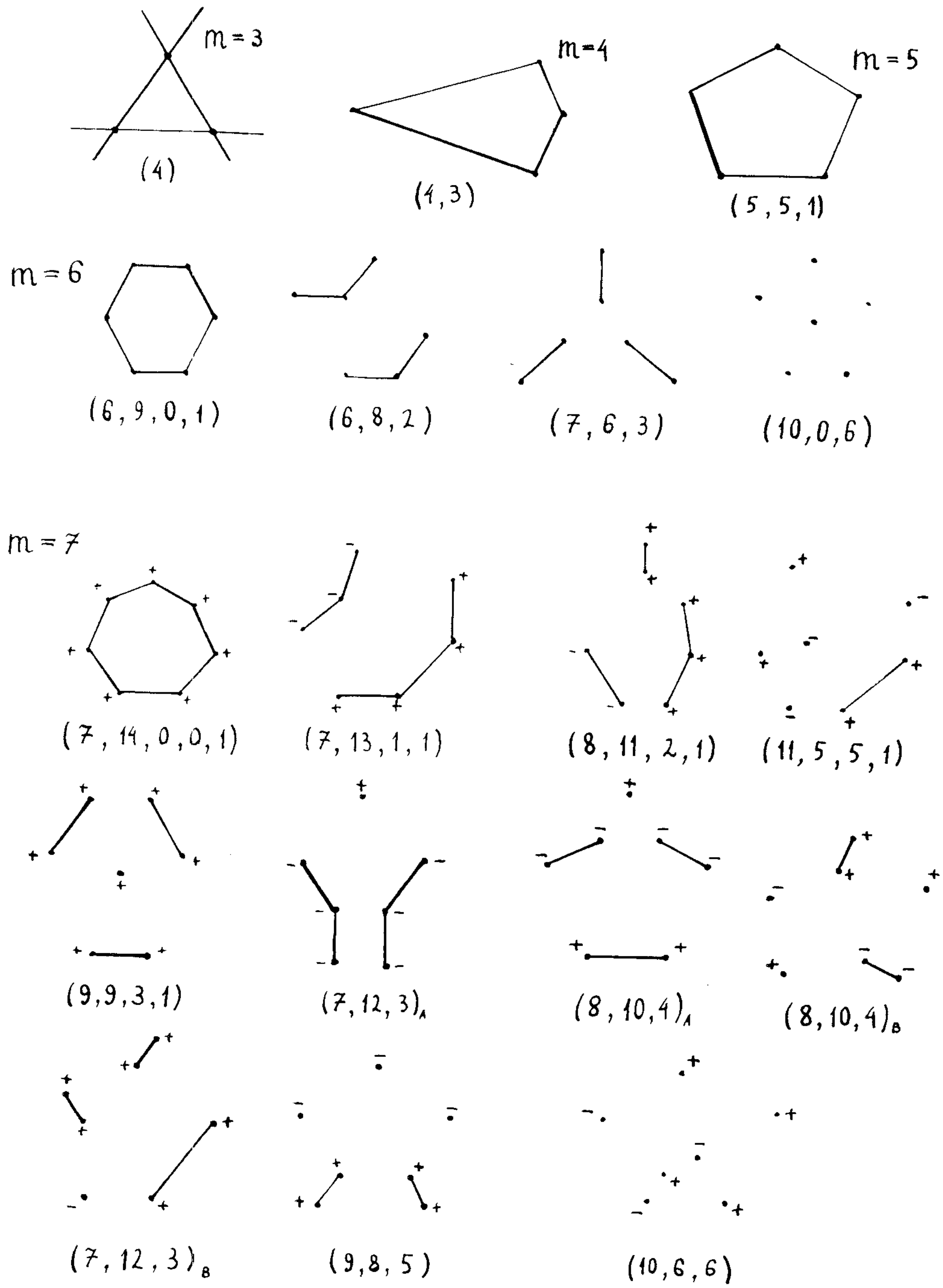
Dia. 1. Simple 2-arrangements of degree $\leqslant 7$.



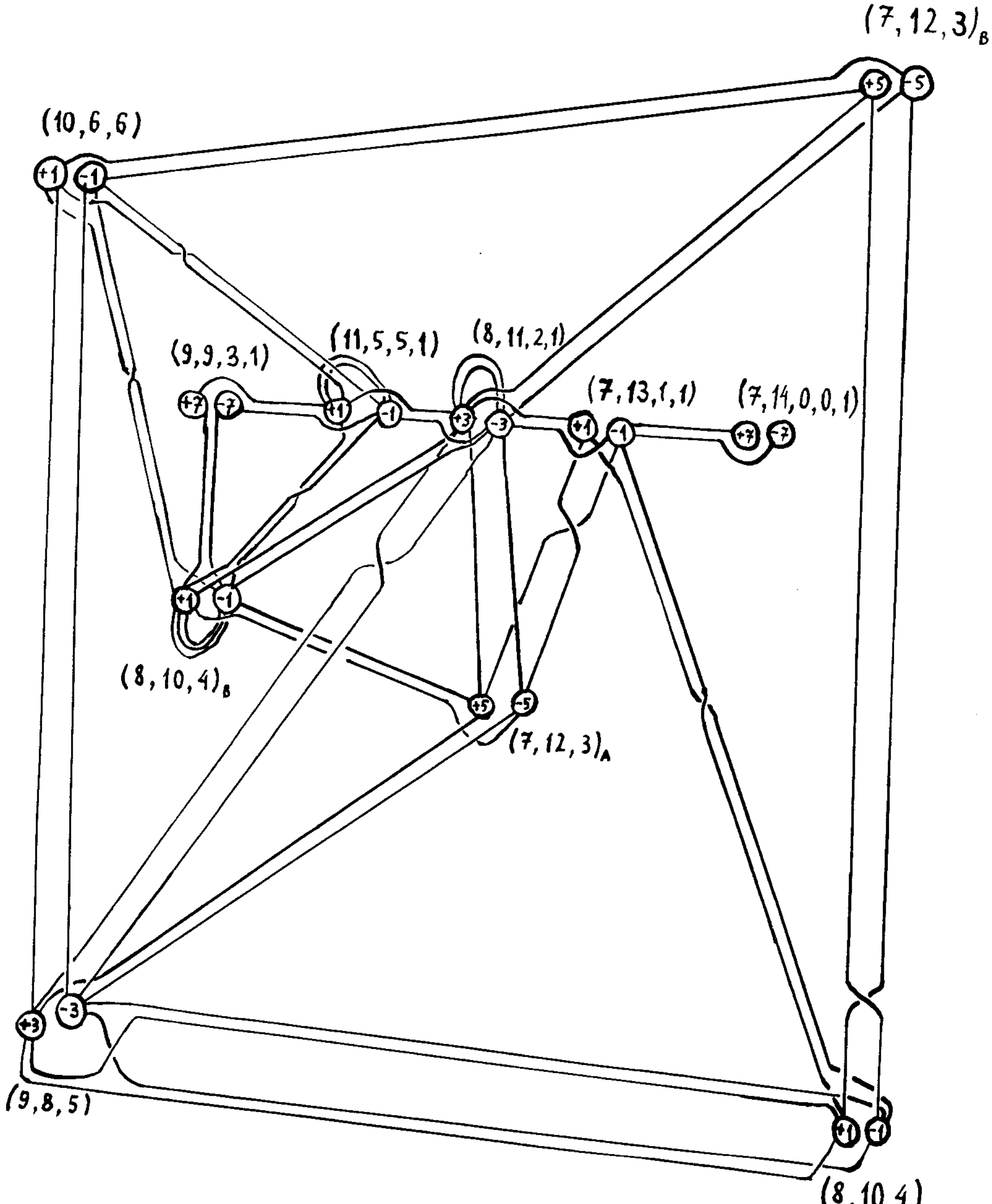
Dia. 2. Graph $S\Gamma_2^7$

Simple 2-arrangements of degree 7	Simple 3-arrangements of degree 7
(7, 14, 0, 0, 1)	(7, 21, 7, 7)
(7, 13, 1, 1)	(7, 19, 11, 5)
(8, 11, 2, 1)	(8, 17, 12, 5)
(9, 9, 3, 1)	(9, 15, 13, 5)
(11, 5, 5, 1)	(11, 11, 15, 5)
$(7, 12, 3)_A$	$(7, 18, 13, 4)_A$
$(7, 12, 3)_B$	$(7, 18, 13, 4)_B$
$(8, 10, 4)_A$	$(8, 16, 14, 4)_A$
$(8, 10, 4)_B$	$(8, 16, 14, 4)_B$
(9, 8, 5)	(9, 14, 15, 4)
(10, 6, 6)	(10, 12, 16, 4)

Dia. 3. Spectra of extrinsicly dual simple 2- and 3-arrangements of degree 7.



Dia. 4. (2,0)-configurations of degree $m \leq 7$, structural graphs and orientations for $m=7$.



Dia. 5. Graph

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