Chromatic Polynomials and Hilbert Functions

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1 Introduction

Let G be a connected finite graph without loops or multiple edges, and let $\chi_G(n)$ be the number of proper vertex colorings of G using colors from a set of cardinality n. $\chi_G(n)$ is called the *chromatic polynomial* of G. Chromatic polynomials of graphs, and more generally of matroids, have been studied via the *broken circuit complex* ([Whi32], [Wil76], [Byr77]), which is a simplicial complex whose maximal non-simplices correspond to circuits of G with their last edges deleted. One can think of the broken circuit complex as the Stanley-Reisner complex of a square-free monomial ideal, the *broken circuit*

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ideal. The Hilbert function of the quotient ring by this ideal easily determines $\chi_G(n)$.

The broken circuit ideal varies with the choice of an order on the edges of G, so when one attempts to study colorings of graphs via broken circuits, the intrinsic structure of G is obscured by artifacts of the order used. We exhibit a homogeneous ideal in characteristic 2 which is independent of this choice of order, whose initial ideals with respect to the computation of Gröbner bases are the broken circuit ideals for each possible order. This ideal has the same Hilbert function as the broken circuit ideals, so its free resolutions determine $\chi_G(n)$. It defines a projective algebraic *circuit variety* whose structure we do not understand, but which must be intricately related to the chromatic properties of G. This setup is reminiscent of the construction of canonical curves from graphs studied in [BE91], and can be studied using the computer algebra system *Macaulay* [BS93].

The Four-Color Theorem asserts that if G is a planar graph without loops, then $\chi_G(4) > 0$. Equivalently, the corresponding Hilbert series for such graphs is positive at $-1/3$. We cannot establish this without relying on [AH77], [AHK77].

I would like to thank Jonathan Lennox, Oisin McGuinness, and Peter Woit for their superb implementation of the `MathSci` databases at Columbia; this work benefited from my being able to easily surf the math reviews. I would also like to thank David Eisenbud, Gian-Carlo Rota, Richard Stanley, Bernd Sturmfels, and Herb Wilf for helpful conversations. They haven't seen this draft yet, so any glaring omissions, oversights, or mistakes are my own.

2 Matroids

This section provides an exposition of some standard results on the construction of matroids from graph coloring problems, to make this paper self-contained for algebraists.

Let n be the order of a finite field $K = F_n$. Fix orientations for the edges of the graph G, and fix a vertex of G. Pick a color from K for this vertex. Each way of completing this choice to a proper n-coloring of G using colors from K corresponds to an assignment of nonzero values from K to the edges of

G, giving the differences between adjacent vertices. These edge assignments are characterized by the property that summing around any circuit of G is zero; call such an edge assignment *proper.* $\chi_G(n)$ is n times the number of proper edge assignments of G, which we proceed to count instead.

Let G have e edges. Proper edge assignments correspond to points in K^e away from the coordinate hyperplanes, but contained in the circuit subspace $V \subset K^e$ cut out by the linear equations which assert that summing around any circuit of G is zero. We count such points by inclusion-exclusion: The rank r of V is the number of edges in a spanning tree of G, which is $v - 1$ if G has v edges. So V consists of n^r points. Each edge of G contributes a coordinate hyperplane to be avoided, knocking out n^{r-1} points from V. For each pair of edges, we have knocked out the common intersection of their coordinate hyperplanes with the circuit subspace twice, so we need to add back in these points; there are n^{r-2} such points. Now, the points knocked out by triples of edges need to be subtracted back out. There are n^{r-3} such points if the three edges impose independent conditions on the circuit subspace, and n^{r-2} such points if the three edges instead form a circuit of G. Continuing in a like manner for all larger sets of edges, we end up with a count of the number of proper edge assignments of G.

Define the rank of a set of edges from G to be the number of independent conditions they impose on the circuit subspace V. The set of all edges of G equipped with this rank function is called the *bond matroid* of G.

Recall that a matroid is a set equipped with an abstract rank function on subsets, having properties modeled after sets of vectors in vector spaces; see [Aig79] or [Oxl92]. We only consider finite matroids in this paper. Such a matroid M can be understood in terms of a polynomial ring whose variables represent the elements of M, and whose square-free monomials represent the subsets of M. We will use algebraic notation in place of set notation. The dependent subsets of M are the square-free monomials of a monomial ideal generated by the *circuits*, or minimal dependent subsets, of M; this square-free monomial ideal determines the matroid structure on M. Which ideals arise this way? The matroid axioms assert that this is a nontrivial square-free monomial ideal, whose minimal generators, or *circuit monomials*, satisfy one interesting condition, the *exchange axiom* (see [Aig79], p264):

Definition 2.1 *A* matroid *M is a set as above whose circuit monomials* *satisfies either of the equivalent conditions:*

(a) If $C \neq C'$ are circuit monomials, $p \in M$, and $p \mid \gcd(C, C')$, then there exists a circuit monomial D so $pD \mid \operatorname{lcm}(C, C')$.

(b) If $C \neq C'$ are circuit monomials, $p, q \in M$, $p \mid \gcd(C, C')$, and $q \mid C$, $q \nmid C'$, then there exists a circuit monomial D so $q \mid D$ and $pD \mid \operatorname{lcm}(C, C')$.

The *chromatic polynomial* $\chi_M(\lambda)$ of the matroid M is defined by

$$\chi_M(\lambda) \;=\; \sum_{U \subset M} (-1)^{\#(U)}\, \lambda^{r-r(U)}$$

where the sum is over all subsets U of M, $\#(U)$ is the cardinality of U, $r(U)$ is the rank of U, and r is the rank of M. This agrees with the inclusion-exclusion count above, so

$$\chi_G(n) = n\, \chi_M(n)$$

where M is the bond matroid of G. In subsequent sections, we shall study $\chi_M(\lambda)$ for an arbitrary matroid M.

There is a second construction of a matroid useful for studying 4-colorings of planar graphs, which we describe here, but do not use in this paper. This matroid is constructed as follows: To establish the Four-Color Theorem, it suffices to vertex 4-color all graphs G which arise as triangulations of the 2-sphere. View a 4-coloring of an orientable simplicial surface as a non-collapsing simplicial map to a labeled tetrahedron. Choose orientations for both the tetrahedron and the surface, pull the orientation of the tetrahedron back to the surface, and compare the orientations triangle by triangle. The 4-coloring can be recovered from this comparison data alone, up to automorphisms of the tetrahedron.

To yield a 4-coloring, putative comparison data must satisfy both local and global compatibility conditions. The local conditions are seen by considering links of vertices: Each cycle of triangles sharing a given common vertex must map somehow to the three triangles in the tetrahedron sharing the image of the common vertex, so the surplus of orientation matches over clashes must be 0 mod 3. The global condition is homological: The comparison data yields a well-defined map if it is consistent along any closed loop of triangles; this consistency follows from the local conditions for any loop which bounds a surface. Thus, the local conditions suffice for determining 4-coloring of simplicial 2-spheres. This approach can in principal be extended to higher dimensions, but it hits a wall: There exist simplicial 3-spheres with arbitrarily high chromatic number; see [MS71]. Nevertheless, it would be interesting to know if any invariants of higher dimensional manifolds can be derived from matroid structures analogous to those discussed here. For a study of colorings of simplicial complexes, see [Fis89].

Working in F_3, the finite field with 3 elements, label triangles 1 which match orientations with the image tetrahedron, and -1 which clash orientations. Dually, we are assigning nonzero values from F_3 to the vertices of a planar trivalent graph G', so the values sum to zero around faces of G'. The relationship between such assignments and 4-colorings of the regions of G' is known as the Heawood-Tait theorem (see [Aig79], p. 371) and can be derived without the above topological excursion. The corresponding matroid structure on the points of G' varies horridly with the characteristic, but this can be fixed for the application at hand by replacing G' by its edge graph. The resulting bond matroid M has a characteristic-free structure, and satisfies

$$\chi_M(3) > 0 \quad \Leftrightarrow \quad \chi_G(4) > 0$$

for our original triangulation G. We have decremented the λ of interest by one, in exchange for agreeing to work with the edge graph of the dual graph of G. On balance, this doesn't seem to be such a profitable trade.

3 Broken Circuits

There is a useful generating function technique which employs Gröbner bases: Write down a generating polynomial which is too naïve for the counting problem at hand, and then fix it by reducing by an appropriately chosen Gröbner basis. For example, if we want to compute the numerator of the multigraded Hilbert series of a quotient by the square-free monomial ideal $I = (\mathbf{x}^{A_1}, \ldots, \mathbf{x}^{A_j})$, then

$$(1 - \mathbf{x}^{A_1}) \cdots (1 - \mathbf{x}^{A_j}) \tag{1}$$

is too naïve a guess, because when this expression is expanded out, we get products when we want greatest common multiples. However, this effect can be achieved in the square-free case by reducing (1) by the Gröbner basis

$$\{ x_1^2 - x_1, \ldots, x_n^2 - x_n \}$$

where $x_1, \ldots, x_n$ are the variables of the polynomial ring. This technique was commonly used with the computer algebra system *Macaulay*, before the introduction of its `hilb_numer` command.

In the spirit of Definition 2.1, view $x_1, \ldots, x_n$ both as the elements of a matroid M, and as the variables of a polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. Our naïve guess for a generating polynomial is this: If each subset of M has rank equal to its cardinality, then the expression

$$(1 - x_1) \cdots (1 - x_n) \tag{2}$$

can be converted to an expression for $\chi_M(\lambda)$ by substituting

$$x_1 = \ldots = x_n = \lambda^{-1}$$

and multiplying by λ^r, where r is the rank of M. This works because when (2) is expanded out, it has terms representing every subset of M, appropriately signed and with degrees equal to the cardinalities of the corresponding subsets.

For an arbitrary matroid M, expression (2) can be fixed: We need to prune each of the terms in its expansion, removing dependent elements from each monomial until the degrees accurately reflect the ranks. This pruning can be carried out by relations of the form $\mathbf{x}^A - \mathbf{x}^B$, where $\mathbf{x}^A$ is a circuit monomial of M, and $\mathbf{x}^B$ is the same monomial with a variable deleted. Reducing by this binomial has the interpretation: "Whenever a subset of M contains the circuit $\mathbf{x}^A$, throw out the element not in $\mathbf{x}^B$."

We need to choose a consistent set of relations, one for each circuit of M. Such a consistent set will form a Gröbner basis, and reducing by such a Gröbner basis will prune any subset of M down to an independent set with the same span. One consistent set of choices is as follows:

Proposition 3.1 *Let $\{\mathbf{x}^{A_1}, \ldots, \mathbf{x}^{A_m}\}$ be the circuit monomials of a matroid M. For $i = 1, \ldots, m$, let $\mathbf{x}^{B_i}$ be the monomial obtained from $\mathbf{x}^{A_i}$ by deleting its last variable. Then*

$$\{\mathbf{x}^{A_1} - \mathbf{x}^{B_1}, \ldots, \mathbf{x}^{A_m} - \mathbf{x}^{B_m}\} \tag{3}$$

is a Gröbner basis with respect to any monomial order compatible with the variable order $x_1 > x_2 > \ldots > x_n$.

Proof. Let $\mathbf{x}^{A_i} - \mathbf{x}^{B_i}$ and $\mathbf{x}^{A_j} - \mathbf{x}^{B_j}$ be any two binomials from (3), and define $\mathbf{x}^{C_i}$, $\mathbf{x}^{C_j}$, $\mathbf{x}^D$ so

$$\mathbf{x}^{C_i}\mathbf{x}^{A_i} = \mathbf{x}^{C_j}\mathbf{x}^{A_j} = \mathbf{x}^D = \text{lcm}(\mathbf{x}^{A_i}, \mathbf{x}^{A_j}).$$

We need to show that

$$\mathbf{x}^{C_i}(\mathbf{x}^{A_i} - \mathbf{x}^{B_i}) - \mathbf{x}^{C_j}(\mathbf{x}^{A_j} - \mathbf{x}^{B_j}) \tag{4}$$

can be reduced to zero using the relations in (3). Let $a_i = \mathbf{x}^{A_i}/\mathbf{x}^{B_i}$ and $a_j = \mathbf{x}^{A_j}/\mathbf{x}^{B_j}$; we can rewrite (4) after expansion as

$$\mathbf{x}^D/a_j - \mathbf{x}^D/a_i. \tag{5}$$

If $a_j = a_i$ we are done; suppose that $a_j < a_i$, so $\mathbf{x}^D/a_j$ is the lead term of (5). Then a_j is the last variable of $\mathbf{x}^D$, and cannot divide $\mathbf{x}^{A_i}$ (or else it would also be the last variable of $\mathbf{x}^{A_i}$, yielding $a_j = a_i$). Thus $\mathbf{x}^{A_i}$ divides $\mathbf{x}^D/a_j$, so we can reduce (5) by $\mathbf{x}^{A_i} - \mathbf{x}^{B_i}$, yielding

$$-\mathbf{x}^D/a_i + \mathbf{x}^D/a_i a_j. \tag{6}$$

with lead term $-\mathbf{x}^D/a_i$. Now we apply the exchange axiom (b) of Definition 2.1: There exists a circuit monomial $\mathbf{x}^{A_k}$ divisible by a_j, which divides $\mathbf{x}^D/a_i$. Remainder (6) is a multiple of the corresponding binomial $\mathbf{x}^{A_k} - \mathbf{x}^{B_k}$, so it reduces to zero. ∎

Saying that (3) is a Gröbner basis is a whole lot easier to remember than the exchange axiom (b) of Definition 2.1. Unfortunately, these statements aren't equivalent. In the language of Definition 2.1, Proposition 3.1 only checks the exchange axiom for the last element of each $\text{lcm}(C, C')$, and lets putative matroids which fail the exchange axiom for other p slip by.

Now we are ready to reduce the generating polynomial (2) by the Gröbner basis (3), to yield a polynomial from which we can deduce $\chi_M(\lambda)$. If one is bracing for a more complicated answer, it is quite a shock to see the result of this reduction come up on a computer screen: All of the terms which belong to the monomial ideal $J = (\mathbf{x}^{B_1}, \ldots, \mathbf{x}^{B_m})$ simply go away, leaving the other terms intact. J is the square-free monomial ideal corresponding in the Stanley-Reisner sense to the *broken circuit complex* studied by [Whi32], [Wil76], [Byr77] and other authors. Call the $\mathbf{x}^{B_i}$ *broken circuit monomials*; we record the above property of J, due in slightly different language to the cited authors:

Proposition 3.2 *Let $\mathbf{x}^{B_1}, \ldots, \mathbf{x}^{B_m}$ be the broken circuit monomials of a matroid M, and let $J = (\mathbf{x}^{B_1}, \ldots, \mathbf{x}^{B_m})$ be the corresponding broken circuit ideal. Then the chromatic polynomial $\chi_M(\lambda)$ can be computed by substituting $x_1 = \ldots = x_n = \lambda^{-1}$ into the sum*

$$\sum_{\substack{\mathbf{x}^C \text{ square-free} \\ \mathbf{x}^C \notin J}} (-1)^{\#(C)}\, \mathbf{x}^C \tag{7}$$

and multiplying by λ^r, where $\#(C)$ is the degree of $\mathbf{x}^C$, and r is the rank of M.

Proof. It is enough to show that (7) is the result of reducing (2) by the Gröbner basis (3). The terms $\mathbf{x}^C \notin J$ can never be reached by this reduction process; we need to show that all the other terms cancel out.

It is easy to see how all multiples of a single $\mathbf{x}^{B_i}$ are canceled out by reduction by $\mathbf{x}^{A_i} - \mathbf{x}^{B_i}$: Let $a_i = \mathbf{x}^{A_i}/\mathbf{x}^{B_i}$ as before, and pair square-free multiples of $\mathbf{x}^{B_i}$ which differ only by the presence or absence of a_i. The terms of these pairs have opposite signs in the expansion of (2), so after reduction, the terms divisible by a_i cancel the corresponding terms not divisible by a_i.

To continue this argument for the other binomials of (3), we need to insure that the terms already canceled out pair up in the above sense with respect to each subsequent binomial under consideration. It suffices to sequence the set of binomials so the a_i are considered in the order $x_1, \ldots, x_n$. ∎

From Proposition 3.2, we can see that the chromatic polynomial of M is closely related to the Hilbert function of the quotient by J: The chromatic polynomial is derived from an alternating sum of square-free monomials not in J, whereas the Hilbert function is derived from the ordinary sum of all monomials not in J. This observation yields

Proposition 3.3 *Let M be a matroid of rank r with broken circuit ideal J. If S/J has the Hilbert series*

$$F(t) \;=\; \frac{f(t)}{(1-t)^n} \;=\; \sum_{d=0}^{\infty} t^d \dim(S/I)_d$$

then the chromatic polynomial of M is given by

$$\chi_M(\lambda) \;=\; \lambda^r \, F\left(\frac{-1}{\lambda-1}\right) \;=\; \lambda^{r-n}\,(\lambda-1)^n\, f\left(\frac{-1}{\lambda-1}\right). \qquad (8)$$

Proof. Various methods for computing the Hilbert series of S/J build up $F(t)$ by adding or subtracting terms of the form

$$\frac{t^d}{(1-t)^n}$$

obtained by substituting $x_1 = \ldots = x_n = t$ into the sum of all monomial multiples of certain degree d monomials $\mathbf{x}^A \in J$ for various d. These $\mathbf{x}^A$ and their eventual coeffiecients as used in $F(t)$ can also be obtained from a free resolution of S/J, or by appying Möbius inversion to the characteristic function of the monomial complement of J. (This is what the terms of $f(t)$ "mean" to an algebraist; we are likely to be much more comfortable looking at $f(t)$ than $\chi_M(\lambda)$.)

When J is square-free, all such $\mathbf{x}^A$ used in constructing $F(t)$ will also be square-free. To instead build up $\chi_M(\lambda)$, we need for each $\mathbf{x}^A$ to substitute $x_1 = \ldots = x_n = \lambda^{-1}$ into the sum

$$\sum_{\substack{\mathbf{x}^C \text{ square-free} \\ \mathbf{x}^A | \mathbf{x}^C}} (-1)^{\#(\mathbf{x}^C)}\, \mathbf{x}^C$$

and multiply by λ^r, yielding

$$\lambda^r(-1)^d\lambda^{-d}(1-\lambda^{-1})^{n-d} \;=\; \lambda^r\,(1-\lambda)^{-d}(1-\lambda^{-1})^n.$$

Substituting $t = -1/(\lambda-1)$ into $t^d/(1-t)^n$, we get

$$\frac{(\frac{-1}{\lambda-1})^d}{(1+\frac{1}{\lambda-1})^n} = \frac{(-1)^d(\lambda-1)^{n-d}}{\lambda^n} = (1-\lambda)^{-d}(\frac{\lambda-1}{\lambda})^n = (1-\lambda)^{-d}(1-\lambda^{-1})^n.$$

The formula for $\chi_M(\lambda)$ in terms of $F(t)$ follows by linearity, and the formula in terms of $f(t)$ then follows by a routine calculation. ∎

We can use Proposition 3.3 to define the chromatic polynomial of any projective algebraic variety.

4 Circuit Varieties

The following assertion is supported by substantial computer evidence and meager intuition, but has not yet been established.

Conjecture 4.1 *Let M be a matroid. For each circuit $\mathbf{x}^A$ of M, define the corresponding* circuit form *in characteristic 2 to be the sum*

$$\sum_{x_i | \mathbf{x}^A} \mathbf{x}^A / x_i$$

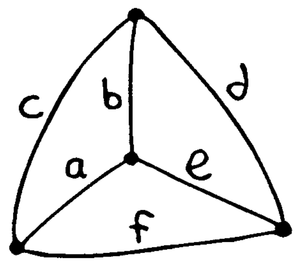
of all broken circuits obtainable from $\mathbf{x}^A$. This set of circuit forms is a Gröbner basis with respect to any monomial order, with initial ideal given by the corresponding broken circuit ideal for that variable order. In particular, if I is the ideal generated by these forms, then S/I has the same chromatic polynomial as M, in the sense of Proposition 3.3.

Apparently, no similar result holds in other characteristics, with exceptions such as for the bond matroid of a 2-colorable graph. The quotient S/I can be thought of as a commutative analogue to the exterior algebras studied in [Arn69], [OS80]. Thus, the use of characteristic 2 is not surprising; it is a haven for constructions that want to alternate.

Define the *circuit variety* of a matroid M to be the variety $X_M \subset \mathbf{P}^{n-1}$ over a field of characteristic 2, cut out by the ideal I of Conjecture 4.1. We would like to better understanding the properties of X_M, and how they affect $\chi_M(\lambda)$.

A noteworthy feature of the ideal I is that its minimal generating set is substantially smaller than the full set of circuits of the matroid M. For example, for the bond matroid of a triangulated 2-sphere, it appears that one only needs forms corresponding to face circuits, and to circuits such that the dual graph of each half of the sphere cut out by the circuit is 2-connected.

Example 4.2 Let the graph G be the tetrahedron shown below:



The circuits of the corresponding bond matroid M are abc, bde, aef, cdf, $acde$, $bcef$, and $abdf$. The ideal $I \subset S = k[a,b,c,d,e,f]$ defining the circuit variety X_M is generated by the circuit forms corresponding to the face circuits, so

$$I = (ab + ac + bc,\ bd + be + de,\ ae + af + ef,\ cd + cf + df).$$

The minimal free resolution of I has the betti numbers given by the following *Macaulay* output,

```
% betti c
total:      1    4    5    2
--------------------------------
    0:      1    -    -    -
    1:      -    4    2    -
    2:      -    -    3    2
```

where the rows and columns of this table are numbered starting from zero, and the number of ith syzygies of degree d is given in row $d-i$, column $i+1$. The Hilbert series for S/I is therefore

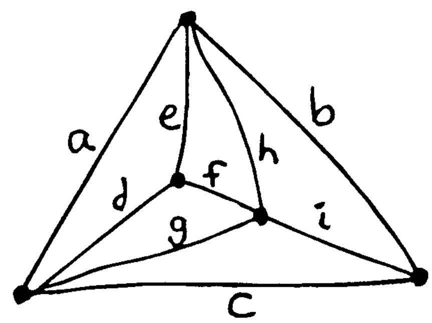
$$F(t) = \frac{1 - 4t^2 + 2t^3 + 3t^4 - 2t^5}{(1-t)^6}$$

so $F(-1/3) = 3/32$. The rank of M is 3. Thus, the number of 4-colorings of the vertices of G is given by

$$4\,\chi_M(4) \;=\; 4 \cdot 4^3\, F(-1/3) = 24,$$

which can be checked by inspection.

Example 4.3 Let the graph G be the triangulated 2-sphere shown below:



The ideal $I \subset S = k[a,b,c,d,e,f,g,h,i]$ defining the circuit variety X_M is generated by the circuit forms corresponding to the face circuits, and by one additional circuit form corresponding to the circuit agh, so

$$\begin{aligned} I \;=\; & (ab+ac+bc,\; ad+ae+de,\; ef+eh+fh, \\ & df+dg+fg,\; cg+ci+gi,\; bh+bi+hi,\; ag+ah+gh). \end{aligned}$$

The minimal free resolution of I has the betti numbers given by the following *Macaulay* output,

```
% betti c
total:      1     7    19    25    16     4
--------------------------------------------
   0:       1     -     -     -     -     -
   1:       -     7     4     -     -     -
   2:       -     -    15    16     4     -
   3:       -     -     -     9    12     4
```

which takes close to 3 megabytes of memory to compute. The Hilbert series for S/I is therefore

$$F(t) = \frac{1 - 7t^2 + 4t^3 + 15t^4 - 16t^5 - 5t^6 + 12t^7 - 4t^8}{(1-t)^{9}}$$

so $F(-1/3) = 3/128$. The rank of M is 4. Thus, the number of 4-colorings of the vertices of G is given by

$$4\,\chi_M(4) \;=\; 4 \cdot 4^4\, F(-1/3) = 24,$$

which again can be checked by inspection.

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The theory of Hilbert functions lies at the borderline in Commutative Algebra, Algebraic Combinatorics and Algebraic Geometry. In this workshop we will treat different topics connected to this theory. Particular emphasis will be placed on computational issues.

Hilbert Functions and Related Topics

October 28-30, 1993

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Mathematical

Thursday, October 28

9:00	Welcome
9:15-10:00	**Tony Geramita**, Hilbert functions of points in P^n
10:00-10:30	**Mid-morning Break**
10:30-11:15	**Lorenzo Robbiano,** Computation of Hilbert functions
11:30-12:15	**Mark Green,** Cook's generalization of the connectedness of the numerical character for space curves
12:15-2:00	**LUNCH**
2:00-2:45	**Giuseppe Valla,** The h-vector of a Gorenstein codimension three graded domain
3:00-3:45	**Alfio Ragusa**, Hilbert functions of points on a quadric in P^3
3:45-4:15	Afternoon Break
4:15-5:00	**Dave Bayer**, Hilbert functions and chromatic polynomials
8:30	PROBLEM SESSION

Friday, October 29

9:00-9:45	**David Eisenbud,** Hilbert functions of ideals containing a regular sequence
9:55-10:40	**Francesco Brenti**, Unimodal h-vectors
10:40-11:05	**Midmorning Break**
11:05-11:50	**Tony Iarrobino,** Derivatives of homoge neous forms and determinantal ideals of catalecticant matrices
12:00-12:45	**Luchezar Avramov**, Resolutions of modules over complete intersections

Afternoon Free: Weather permitting, some of us will go for a hike.

Saturday, October 30

9:15-10:00	**Mark Haiman,** Diagonal harmonics and q,t-Catalan numbers
10:00-10:30	**Mid-morning break**
10:30-11:15	**Craig Hunecke,** Hilbert functions and rational singularities
11:30-12:15	**Vladimir I. Arnold,** Graded rings with simplest Poincare series, continued f fractions, and simple Young dirgrams
12:15-2:00	**LUNCH**
2:00-2:20	**Ed Davis**, Subcanonical curves with gaps in the Rao module: the numerical possibilities
2:25-2:45	**Juan Migliore,** On the Cohen-Macaulay type of the general hypersurface section
2:50-3:10	**TBA**
3:10-3:30	**Afternoon Break**
3:30-3:50	**TBA**
3:55-4:15	**TBA**
4:20-4:40	**TBA**

7:30 Dinner Rulloff's

All talks will be held in Room 212, Mathematical Sciences Institute, 409 College Avenue, Cornell University