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EQUIDECOMPOSABLE AND WEAKLY NEIGHBORLY POLYTOPES

BY

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ABSTRACT

A polytope is equidecomposable if all its triangulations have the same face numbers. For an equidecomposable polytope all minimal affine dependencies have an equal number of positive and negative coefficients. A subclass consists of the weakly neighborly polytopes, those for which every set of vertices is contained in a face of at most twice the dimension as the set. The h-vector of every triangulation of a weakly neighborly polytope equals the h-vector of the polytope itself. Combinatorial properties of this class of polytopes are studied. Gale diagrams of weakly neighborly polytopes with few vertices are characterized in the spirit of the known Gale diagram characterization of Lawrence polytopes, a special class of weakly neighborly polytopes.

1. Introduction

A general question in combinatorics is how many faces various types of complexes may have. The question has been answered completely for simplicial complexes and for boundary complexes of simplicial polytopes. For boundary complexes

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of arbitrary convex polytopes, however, the question is wide open. It is natural to try to apply results on simplicial complexes to triangulations of a polytope or its boundary to get information about the original polytope. In general the face numbers of a triangulation of a polytope are not functions of the polytope's combinatorial structure. Indeed, different triangulations may have different face numbers. This motivates the study of **equidecomposable** polytopes; these are the polytopes, all of whose triangulations have the same face numbers. We focus on the subclass of weakly neighborly polytopes. For these polytopes the face numbers of a triangulation can be computed from the face lattice of the polytope. The computation involves the toric or generalized h-vector [12]. Perhaps the results here relating h-vectors of triangulations and polytopes will lead to progress in interpreting generalized h-vectors. Background on combinatorial properties of polytopes can be found in [4, 6].

In Section 2 we define equidecomposable polytopes and give a property of their affine dependencies. Section 3 focuses on those triangulations whose h-vectors are the same as the h-vector of the polytope. Sections 4 and 5 deal with weakly neighborly polytopes. In Section 4 we look at general combinatorial properties and 3-dimensional polytopes; section 5 concentrates on weakly neighborly polytopes with few vertices.

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2. Equidecomposable polytopes

A **triangulation** of a d-polytope P is a simplicial complex whose vertex set is the vertex set of P and whose underlying space is P. Write $f_i(C)$ for the number of i-faces of a polyhedral complex C, and call $f(C) = (f_0(C), f_1(C), \ldots, f_d(C))$ the **f-vector** of C. A polytope is **equidecomposable** if all its triangulations have the same f-vector. In this section we show that this implies a certain condition on the affine dependencies among the vertices of the polytope.

A **circuit** of a polytope is a minimal set of affinely dependent vertices. Any circuit supports a unique (up to multiplication by a nonzero scalar) affine dependence. If this affine dependence has an equal number of positive and negative coefficients, the circuit is called **balanced**.

"Circuit" here is a matroid term. We describe what it means in terms of one concrete representation of the matroid, the Gale transform. The reader may consult [4, 9, 14] for background on Gale transforms. A **Gale transform** of a polytope P is the set of columns of a matrix A whose rows form a basis for the affine dependencies on the vertices of P. A Gale transform D of a d-polytope P with n vertices is thus a set of n (not necessarily distinct) points in $\mathbf{R}^{n-d-1}$. To each vertex v of P there corresponds naturally a point $\overline{v}$ of D. A circuit of P corresponds to the complement in D of the set of points contained in a linear hyperplane spanned by points of D. The set of points of a circuit having coefficients of one sign in the affine dependence corresponds to the set of points on one side of the hyperplane. Thus a circuit of P is balanced if and only if there is an equal number of points in either open half-space bounded by the corresponding hyperplane of D.

Some triangulations of a polytope can be constructed using the Gale transform (see Lee [7]). Let $\overline{V}$ be a Gale transform of a d-polytope P and $\overline{z}$ a point of $\mathbf{R}^{n-d-1}$ on no linear hyperplane spanned by elements of $\overline{V}$. The maximal subsets $S \subseteq V$ such that 0 is in the relative interior of the convex hull of $\overline{S}^c \cup \{\overline{z}\}$ are the d-simplices of a triangulation. We call this the Gale triangulation induced by $\overline{z}$. A triangulation obtained in this way is called **regular**. Note that multiplying the points of the Gale transform by positive scalars does not change the sets $\overline{S}^c \cup \{\overline{z}\}$ that capture 0. Thus we can (and often will) normalize the Gale transform to be a subset of $S^{n-d-2} \cup \{0\} \subseteq \mathbf{R}^{n-d-1}$.

THEOREM 1: *If a polytope is equidecomposable, then all its circuits are balanced.*

Proof: Suppose the d-polytope P with n vertices has an unbalanced circuit, $X = X^+ \cup X^-$, where X^+ is the set of points of X having one sign in the affine dependence, X^- the other, and $|X^+| = k < m = |X^-|$. Consider the Gale transform D of P with sets $\overline{X}^+$ and $\overline{X}^-$ in the open half-spaces, H^+ and H^-, bounded by the hyperplane H. If the points of D in H capture 0 (in the relative interior of their convex hull), choose one of these points $\overline{z}$. Otherwise, since the points of D in H span H, there exists a point $\overline{z}$ of H (not in D) such that 0 is in the relative interior of the convex hull of $(D \cap H) \cup \{\overline{z}\}$. Choose points $\overline{z}^+$ and $\overline{z}^-$ close to $\overline{z}$ in H^+ and H^- (here "close" means that the line segment $\operatorname{conv}\{\overline{z}^+, \overline{z}^-\}$ does not intersect any spanned hyperplane of D other than H).

Let Δ^+ and Δ^- be the triangulations of P induced by $\overline{z}^+$ and $\overline{z}^-$. The minimal cofaces that differ for the two triangulations are those for which $\overline{z}^+$ (or $\overline{z}^-$) [illegible] the only point in its open half-space H^+ (or H^-). That is, they are of the form $A \cup \{b, \overline{z}^+\}$ (or $A \cup \{c, \overline{z}^-\}$), where A is a minimal set in H such that $A \cup \{\overline{z}\}$ captures 0, and $b \in \overline{X}^-$ $(c \in \overline{X}^+)$. For each such A there are m of these cofaces for Δ^+ and k for Δ^-, so Δ^+ has more facets (d-faces) than Δ^-. Thus P is not equidecomposable. ∎

It is not clear whether the converse to Theorem 1 holds in general. For d-polytopes with more than $d+3$ vertices, not all triangulations of polytopes are regular. The balanced circuit condition enables us to argue only about regular triangulations.

Proposition 2: *If all circuits of a d-polytope P are balanced, then all regular triangulations of P have the same f-vector.*

Proof: Any two regular triangulations, Δ and Δ', can be connected by a sequence of triangulations, $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_j = \Delta'$, where each adjacent pair Δ_{i-1} and Δ_i is related as Δ^+ and Δ^- are in the proof of Theorem 1. If all circuits are balanced then Δ_{i-1} and Δ_i have the same f-vector for each i, so all regular triangulations have the same f-vector. ∎

By [7] all triangulations of a d-polytope P with at most $d+3$ vertices are regular. Thus we get

Corollary 3: *If P is a d-polytope with at most $d+3$ vertices and all circuits of P are balanced, then P is equidecomposable.*

We close this section with a few examples. The regular octahedron and the triangular prism are both equidecomposable polytopes (whose triangulations have f-vectors $(6,13,12,4)$ and $(6,12,10,3)$, respectively). Note, however, that the octahedron is combinatorially equivalent to polytopes that are not equidecomposable. The bipyramid over a triangle is not equidecomposable: it has one triangulation using two tetrahedra, one using three.

3. Shallow subdivisions and h-vectors

In this section we show how the f-vectors of certain triangulations of a polytope can be computed from combinatorial invariants of the original polytope (or vice versa). The combinatorial invariants form the h-vector of the polytope. The

[illegible]-vector was first defined for simplicial polytopes (its origins go back to Som[illegible]merville's 1927 paper [11]). More recently Stanley extended the definition to Eulerian posets [12]. The form of Stanley's definition comes from algebraic geometry: the h-vector of the boundary ∂P of a rational polytope P is the sequence of intersection homology Betti numbers of the associated toric variety.

Define polynomials $h(C,x)$ for all polyhedral complexes C and $g(S,x)$ for all spheres S by the rules:

1. $h(\emptyset, x) = g(\emptyset, x) = 1$,
2. if $\dim S = d-1$ and $h(S,x) = \sum_{i=0}^{d} k_i x^i$, then $g(S,x) = \sum_{i=0}^{m} (k_i - k_{i-1}) x^i$, where $m = \lfloor d/2 \rfloor$ and $k_{-1} = 0$,
3. if $\dim C = d$, then $h(C,x) = \sum g(\partial F, x)(x-1)^{d - \dim F}$, where the sum is over all faces F of C.

Keeping in mind that the g-polynomial is defined only for spheres, we will often abbreviate $g(\partial P, x)$ to $g(P,x)$ for a polytope P. (Note, however, that we distinguish between $h(\partial P, x)$ and $h(P,x)$; the latter is defined for the polyhedral complex having a single maximal face P.) For a d-complex C with $h(C,x) = \sum_{i=0}^{d+1} k_i x^i$, write $h_i = k_{d+1-i}$ and call $h(C) = (h_0, h_1, \ldots, h_{d+1})$ the **h-vector** of C. If C is a d-sphere the h-vector satisfies the Dehn-Sommerville equations: $h_i = h_{d+1-i}$ for all i. The vector of coefficients of $g(S,x)$ (in standard order) is the **g-vector** of the sphere S. Both the h-vector and the g-vector of the boundary of a rational polytope are nonnegative; the only known proof of this uses the Betti number interpretation of the h-vector.

If F is a simplex then $g(\partial F, x) = 1$. So for a simplicial complex C, the h-vector is a function of the f-vector, namely the usual function defining the h-vector of a simplicial complex:

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}.$$

This relation is invertible, so the h-vector of a simplicial complex determines the f-vector. In general the h-vector is a linear function of the flag vector (but not vice versa) [1]. However, this fact does not lead to a natural interpretation of the h-vector in general. The next theorem may provide a step in that direction.

Generalizing the notion of triangulation, we define a **(polyhedral) subdivision** of a d-polytope P to be a polyhedral complex whose vertex set is that of P and whose underlying space is P. If σ is a face of a polyhedral subdivision Δ of P, the **carrier** $C(\sigma)$ of σ in P is the smallest face of P containing σ. A polyhedral subdivision Δ is **shallow** if and only if for each face $\sigma \in \Delta$, $\dim C(\sigma) \leq 2(\dim\sigma - \deg g(\sigma,x))$. In particular a triangulation Δ is shallow if for each $\sigma \in \Delta$, $\dim C(\sigma) \leq 2\dim\sigma$.

Stanley [13] showed that if P is a rational polytope, then for any subdivision Δ of P, $h(\Delta) \geq h(P)$. We show shallowness gives equality.

THEOREM 4:
(1) *If Δ is a shallow subdivision of a d-polytope P, then $g(\partial P) = h(P) = h(\Delta) = g(\partial\Delta)$.*
(2) *If Δ is a subdivision of a rational d-polytope P and $h(P) = h(\Delta)$, then Δ is shallow.*

We first list some identities satisfied by h-vectors and g-vectors. The second and third are straightforward generalizations of results of [10] for triangulations.

LEMMA 5:
(1) *If C is a $(d-1)$-sphere, then*

$$(x-1)h(C,x) = x^{d+1}g(C,1/x) - g(C,x).$$

(2) *If Q is a polyhedral ball of dimension d, then*

$$h(Q,x) - x^{d+1}h(Q,1/x) = x^{d+1}g(\partial Q,1/x) - g(\partial Q,x).$$

(3) *If Δ is a shallow subdivision of a d-polytope P, then*

$$h(\Delta,x) = x^{d+1}g(\partial\Delta,1/x).$$

We also need the concept of an acceptable function, defined in [12]. Let C be a polyhedral complex, and γ a function that associates a polynomial $\gamma(F,x)$ to every face F of C. The function γ is **acceptable** if for all faces F of C

$$\sum_{G \text{ face of } F} \gamma(G,x)(x-1)^{\dim F - \dim G} = x^{\dim F+1}\gamma(F,1/x).$$

The g-polynomial is the unique acceptable function γ on the polytope such that $\gamma(\emptyset,x) = 1$ and for all other faces F, $\deg\gamma(F,x) \leq \dim F/2$ [12].

Now we define a new polynomial for the faces of a polytope relative to a fixed subdivision. For any polytope P, any subdivision Δ of P, and any face F of P,

let

$$\gamma_\Delta(F,x) = \sum_{\substack{\sigma\in\Delta\\ C(\sigma)=F}} g(\sigma,x)(x-1)^{\dim F - \dim\sigma}.$$

For a face F of P, write $\Delta|_F$ for the subcomplex of Δ that subdivides F and $\partial\Delta|_F$ for its boundary complex (which subdivides the boundary ∂F of F).

PROPOSITION 6: *If P is a polytope and Δ is a subdivision of P, then γ_Δ is acceptable.*

Proof: Fix the polytope P and subdivision Δ. Let F be a face of P of dimension e. Consider the sum

$$S_\Delta(F,x) = \sum_{G \text{ face of } F} \gamma_\Delta(G,x)(x-1)^{e-\dim G}.$$

By the definition of γ_Δ,

$$\begin{aligned} S_\Delta(F,x) &= \sum_{G \text{ face of } F}\ \sum_{\substack{\sigma\in\Delta\\ C(\sigma)=G}} g(\sigma,x)(x-1)^{\dim G-\dim\sigma}(x-1)^{e-\dim G} \\ &= \sum_{\substack{\sigma\in\Delta\\ C(\sigma)\subseteq F}} g(\sigma,x)(x-1)^{e-\dim\sigma}. \end{aligned} \tag{1}$$

Interpreting this sum for the polyhedral complex $\Delta|_F$,

$$\begin{aligned} S_\Delta(F,x) &= \sum_{\sigma\in\Delta|_F} g(\sigma,x)(x-1)^{e-\dim\sigma} \\ &= h(\Delta|_F,x). \end{aligned} \tag{2}$$

On the other hand, by separating the faces of $\Delta|_F$ carried by F in expression (1),

$$\begin{aligned} S_\Delta(F,x) &= \sum_{\substack{\sigma\in\Delta\\ C(\sigma)=F}} g(\sigma,x)(x-1)^{e-\dim\sigma} \\ &\quad + (x-1)\sum_{\sigma\in\partial\Delta|_F} g(\sigma,x)(x-1)^{e-1-\dim\sigma} \\ &= \gamma_\Delta(F,x) + (x-1)h(\partial\Delta|_F,x). \end{aligned}$$

By Lemma 5, parts (1) and (2), we get

$$\begin{aligned} S_\Delta(F,x) &= \gamma_\Delta(F,x) + x^{e+1}g(\partial\Delta|_F,1/x) - g(\partial\Delta|_F,x) \\ &= \gamma_\Delta(F,x) + h(\Delta|_F,x) - x^{e+1}h(\Delta|_F,1/x). \end{aligned} \tag{3}$$

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So, combining (2) and (3),

$$\gamma_\Delta(F,x) = x^{e+1}h(\Delta|_F, 1/x) = x^{e+1}S_\Delta(F,1/x)$$

or

$$S_\Delta(F,x) = x^{e+1}\gamma_\Delta(F,1/x)$$

So γ_Δ is acceptable. ∎

COROLLARY 7: *Let Δ be a subdivision of a d-polytope P. Then Δ is shallow if and only if for every face F of P, $\gamma_\Delta(F,x) = g(F,x)$.*

Proof: As mentioned above, g is the unique acceptable function with value 1 at $\emptyset$ and degree at most $\dim F/2$. Thus $\gamma_\Delta = g$ if and only if $\deg \gamma_\Delta(F,x) \le \dim F/2$ for all faces F of P. This holds if and only if $\dim C(\sigma) - \dim\sigma + \deg g(\sigma,x) \le \dim C(\sigma)/2$ for all faces σ of Δ. ∎

Proof of Theorem 4: (1) Suppose Δ is a shallow subdivision of P. Since each face of Δ is carried by some face of P (possibly P itself),

$$\begin{aligned} h(\Delta,x) &= \sum_{\sigma\in\Delta} g(\sigma,x)(x-1)^{d-\dim\sigma} \\ &= \sum_{F \text{ face of } P}\ \sum_{\substack{\sigma\in\Delta \\ C(\sigma)=F}} g(\sigma,x)(x-1)^{\dim F-\dim\sigma}(x-1)^{d-\dim F} \\ &= \sum_{F \text{ face of } P} \gamma_\Delta(F,x)(x-1)^{d-\dim F} \end{aligned}$$

By Corollary 7

$$\begin{aligned} h(\Delta,x) &= \sum_{F \text{ face of } P} g(F,x)(x-1)^{d-\dim F} \\ &= h(P,x). \end{aligned}$$

Applying Lemma 5 (3) gives $g(\partial P) = h(P) = h(\Delta) = g(\partial\Delta)$.

(2) Now suppose P is rational and the triangulation Δ of P is not shallow. Then for some face G of P, $\deg\gamma_\Delta(G,x) > \deg g(G,x)$. (This is equivalent to $\gamma_\Delta(G,x) \ne g(G,x)$.) For any face F of P, write $r(F) = d - \dim F + \deg\gamma_\Delta(F,x)$. Let $r = \max\{r(F) : F \text{ is a face of } P \text{ with } \deg\gamma_\Delta(F,x) > \deg g(F,x)\}$. The coefficient of x^s in $h(\Delta,x) - h(P,x)$ is the coefficient of x^s in

$$\sum_{\substack{F \text{ face of } P \\ s\le r(F)\le r}} (\gamma_\Delta(F,x) - g(F,x))(x-1)^{d-\dim F}.$$

Thus the coefficient of x^s in $h(\Delta,x) - h(P,x)$ is 0 if $s > r$. So $\deg(h(\Delta,x) - h(P,x)) \le r$. The coefficient of x^r in $h(\Delta,x) - h(P,x)$ is the coefficient of x^r in

$$\sum_{\substack{F \text{ face of } P \\ r(F)=r}} (\gamma_\Delta(F,x) - g(F,x))(x-1)^{d-\dim F},$$

which is

$$\begin{aligned} &\sum_{\substack{F \text{ face of } P \\ r(F)=r}} \text{coefficient of } x^{r-d+\dim F} \text{ in } \gamma_\Delta(F,x) \\ &= \sum_{\substack{F \text{ face of } P \\ r(F)=r}} \text{coefficient of } x^{\deg\gamma_\Delta(F,x)} \text{ in } \gamma_\Delta(F,x) \\ &= \sum_{\substack{F \text{ face of } P \\ r(F)=r}} \text{leading coefficient of } \gamma_\Delta(F,x). \end{aligned}$$

If P is rational, each term in the last sum is positive, so $h(\Delta,x) - h(P,x) \ne 0$. So the toric h-vector of Δ does not equal the toric h-vector of P. ∎

Note that this implies that if P is a rational, equidecomposable polytope, then P has a shallow triangulation if and only if all subdivisions of P are shallow. Also, the proof of (1) shows that if the faces of Δ up to dimension k satisfy the shallowness condition, then $h(\Delta)$ will agree with $h(P)$ up to index $\lfloor k/2\rfloor + 1$.

For a simplicial complex Δ write $h(\Delta \setminus \partial\Delta, x) = h(\Delta,x) - (x-1)h(\partial\Delta,x)$ and $f_i(\Delta\setminus\partial\Delta) = f_i(\Delta) - f_i(\partial\Delta)$. A result of [10] shows that for Δ a shallow triangulation of a polytope P, $g(\partial\Delta,x) = h(\Delta\setminus\partial\Delta,x)$. Combining with Theorem 4 this gives

COROLLARY 8: *If Δ is a shallow triangulation of a d-polytope P, then*

(1) $g_i(P) = \sum_{j=\lceil d/2\rceil+1}^{d+1-i} (-1)^{d+1-i-j}\binom{d+1-j}{i} f_{j-1}(\Delta\setminus\partial\Delta)$ *for* $0 \le i \le d/2$,

(2) $g_{\lfloor d/2\rfloor}(P) = f_{\lceil d/2\rceil}(\Delta\setminus\partial\Delta)$,

(3) $f_j(\Delta\setminus\partial\Delta) = \sum_{i=\lceil d/2\rceil+1}^{j+1} \binom{d+1-i}{d-j} g_{d+1-i}(P)$ *for* $0 \le j \le d$,

(4) $f_{\lceil d/2\rceil+1}(\Delta\setminus\partial\Delta) = \lfloor d/2\rfloor g_{\lfloor d/2\rfloor}(P) + g_{\lfloor d/2\rfloor-1}(P)$.

It would be valuable to extend this to an interpretation of the g-vector of a polytope P in terms of face numbers of an arbitrary triangulation.

4. Weakly neighborly polytopes

A polytope P is **weakly neighborly** if and only if every set of $k+1$ vertices is contained in a face of dimension at most $2k$, for all k. Among the weakly neighborly polytopes are the *even-dimensional* neighborly polytopes, those $2m$-dimensional polytopes for which every set of at most m vertices is the vertex set of a face.

PROPOSITION 9: *A polytope is weakly neighborly if and only if all its triangulations are shallow.*

Proof: The forward implication is clear. The reverse implication follows from the observation that every affinely independent set of vertices of P is the vertex set of a face of some triangulation of P. ∎

COROLLARY 10: *Every weakly neighborly polytope is equidecomposable.*

Note that by definition, weak neighborliness, unlike equidecomposability, is a combinatorial property.

The Hirsch conjecture states that for any d-polytope P with n facets, the edge-distance between any two vertices is at most $n-d$ (see [5]).

PROPOSITION 11: *Weakly neighborly polytopes satisfy the Hirsch conjecture.*

Proof: We observe, more generally, that for two vertices x and y on a 2-face of any d-polytope with n facets, the distance, $d(x,y)$, between x and y is at most $n-d$. To see this, suppose x and y are on a 2-face F and F is an m-gon. Then clearly the distance between x and y is at most $m/2$. We estimate the number of facets n of the polytope P. F is the intersection of $d-2$ facets of P. Each edge on F is the intersection of these $d-2$ facets with an additional distinct facet. Thus P has at least $d-2+m \geq d+\lfloor m/2 \rfloor$ facets. So $d(x,y) \leq \lfloor m/2 \rfloor \leq n-d$, as desired. ∎

Although we are most interested in using shallow triangulations to study nonsimplicial polytopes, we first look briefly at the simplicial case. A simplicial d-polytope is **k-stacked** if and only if P has a triangulation in which every $(d-k-1)$-face is a face of P (see [10]).

PROPOSITION 12: *Let P be a simplicial d-polytope. Then P has a shallow triangulation if and only if P is $\lfloor d/2 \rfloor$-stacked.*

Proof: $\Rightarrow$ Let Δ be a shallow triangulation of the simplicial d-polytope P. If $\dim\sigma < d/2$, then the carrier of σ is some proper face F of P. This face F is a simplex, because P is simplicial, so every subset of its vertices determines a face of P. Thus σ is a face of P. So every face of Δ of dimension less than $d/2$ is a face of P, so P is $\lfloor d/2 \rfloor$-stacked.

$\Leftarrow$ Suppose the simplicial d-polytope P is $\lfloor d/2 \rfloor$-stacked. Then P has a triangulation Δ for which every face of dimension less than $d/2$ is a face of P. The triangulation Δ is clearly shallow. ∎

PROPOSITION 13: *A simplicial polytope is weakly neighborly if and only if it is a simplex or an even-dimensional neighborly polytope.*

Proof: Simplices and even-dimensional neighborly polytopes are clearly weakly neighborly. Now suppose P is a simplicial weakly neighborly d-polytope. Then every set of $m = \lfloor (d+1)/2 \rfloor$ vertices of P is contained in a facet. The facets are all simplices, so any set of m vertices of P is the vertex set of an $(m-1)$-face of P. Thus P is m-neighborly. If d is even, this says that P is neighborly; if d is odd, it implies P is a simplex. ∎

Among nonsimplicial polytopes there are two classes known to be weakly neighborly. The first of these is the class of Lawrence polytopes [2]. A Lawrence polytope is a polytope with an even number of vertices and a centrally symmetric (normalized) Gale transform. Equivalently, it is a polytope with vertex set $\{u_1,\ldots,u_n,v_1,\ldots,v_n\}$ such that the complement of each pair $\{u_i,v_i\}$ is the vertex set of a face. The flag vector of a Lawrence polytope depends only on the underlying matroid. It is open whether this is true for all weakly neighborly polytopes. We shall return to this question in the next section.

Another example of weakly neighborly polytopes is given by the Cartesian product of two simplices (of any dimension). Billera, Cushman and Sanders [3] showed that these polytopes are equidecomposable. They described certain regular triangulations of the product $T^m \times T^n$ of an n-simplex and an m-simplex, and computed the h-vector of such a triangulation. This gives the g-vector of the product of two simplices: $g_k(T^m \times T^n) = \binom{m}{k}\binom{n}{k}$ for $0 \leq k \leq (m+n)/2$.

Which operations on polytopes preserve weak neighborliness?

PROPOSITION 14: *Let Q be a weakly neighborly polytope. Then*

(1) *The pyramid PQ over Q is weakly neighborly.*

(2) *Any subpolytope (that is, the convex hull of any subset of the vertices of Q) is weakly neighborly. In particular, any face of Q is weakly neighborly.*

Proof: (1) Let S be a set of $k+1$ vertices of PQ. If S is contained in the vertex set of Q, then S is contained in a face of Q of dimension at most $2k$. This face is also a face of PQ. Now suppose S contains the pyramiding vertex v of PQ. Then $S \setminus \{v\}$ is a set of k vertices of Q, and hence is contained in a face F of Q of dimension at most $2k-2$. Then S is contained in the pyramid over F, a face of PQ of dimension at most $2k-1$.

(2) Let P be a subpolytope of Q, and let S be a set of $k+1$ vertices of P. Then S is contained in a face F of Q of dimension at most $2k$. Now let $G = F \cap P$; G is a face of P containing S and of dimension at most $2k$. ∎

This section closes with the characterization of weakly neighborly 3-polytopes.

THEOREM 15: *The only weakly neighborly 3-polytopes are the prism over a triangle and the pyramids over 2-polytopes.*

Proof: Suppose P is a weakly neighborly 3-polytope, and hence is equidecomposable. Then every set of five vertices, being dependent, contains a circuit of size four. The affine span of these four vertices is 2-dimensional; its intersection with P is a quadrilateral. The endpoints of a diagonal of this quadrilateral must be in a 2-face, so the four vertices are in a 2-face. Thus, if P is a weakly neighborly 3-polytope, then every five vertices contain four that are contained in a face of P.

Recall that the only simplicial weakly neighborly 3-polytope is a simplex (which is a pyramid over a triangle). Suppose now that P has a face with $k \geq 6$ vertices, $u_1, u_2, \ldots, u_k$ in cyclic order. If P has only one other vertex, then P is a pyramid over a k-gon. If P has at least two other vertices, v_1 and v_2, then $\{u_1, u_3, u_5, v_1, v_2\}$ does not contain four vertices in a 2-face. This contradicts the weak neighborliness of P.

So let k be the maximum number of vertices of a face of P, $k = 4$ or $k = 5$. Let $u_1, u_2, \ldots, u_k$ be the vertices of the face F_1 of P. If P has only one other vertex, then P is a pyramid over a k-gon. Otherwise, let v_1 and v_2 be two other vertices of P. Then $\{u_1, u_2, u_3, v_1, v_2\}$ contains four vertices in a 2-face. This 2-face F_2 cannot contain both u_1 and u_3, so without loss of generality say the four vertices are u_1, u_2, v_1, and v_2. Let w_1 (respectively, w_2) be the vertex of F_2 other than u_2 (respectively, other than u_1) adjacent to u_1 (respectively, u_2). See Figure 1.

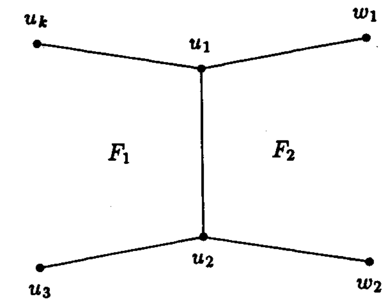

Figure 1: Two 2-faces of a weakly neighborly polytope

Now suppose P has another vertex w_3, not in F_1 or F_2. Consider the set $\{u_1, u_2, u_k, w_2, w_3\}$. Neither of the pairs $\{u_2, u_k\}$ and $\{u_1, w_2\}$ can be in a 2-face containing four of these. So there is no set of four of the vertices contained in a 2-face.

So all vertices of P are in $F_1 \cup F_2$. Since $\{u_2, u_3, u_k, w_1, w_2\}$ contains four vertices in a 2-face, and neither of the pairs $\{u_2, u_k\}$ and $\{u_2, w_1\}$ can be in such a 2-face, $\{u_3, u_k, w_1, w_2\}$ must be in a 2-face. Such a 2-face cannot contain both u_3 and u_5, so $k = 4$. Then $\{u_3, u_4, w_1, w_2\}$ is the set of vertices of a 2-face. Thus P has vertex set $\{u_1, u_2, u_3, u_4, w_1, w_2\}$ and three of the 2-faces of P are $\{u_1, u_2, u_3, u_4\}$, $\{u_1, u_2, w_1, w_2\}$, and $\{u_3, u_4, w_1, w_2\}$. Clearly P is a prism over a triangle. ∎

5. Weakly neighborly polytopes with few vertices

5.1 DESCRIPTION OF THE GALE TRANSFORMS. In this section we use Gale transforms to study weakly neighborly d-polytopes with at most $d+3$ vertices. First consider the case of d-polytopes with $d+2$ vertices. The Gale transform consists of $d+2$ points distributed between +1 and -1. A Gale triangulation is induced by adding one more point. If the new point has the sign of the larger of the two sets, then the induced triangulation is shallow. If the new point has the sign of a strictly smaller set, then the induced triangulation is not shallow. Thus all polytopes with $d+2$ vertices have shallow triangulations. All triangulations of the polytope are shallow if and only if the two sets of its Gale transform are the same size; in this case the Gale transform is centrally symmetric. Thus we conclude:

Proposition 16: *Let P be a d-polytope with $d+2$ vertices. Then the following are equivalent.*

(1) *P is equidecomposable.*
(2) *P is weakly neighborly.*
(3) *P is an r-fold pyramid over a Lawrence polytope, for some $r \geq 0$.*
(4) *P is an r-fold pyramid over an even-dimensional cyclic polytope, for some $r \geq 0$.*

We can also characterize the Gale transforms of weakly neighborly d-polytopes with $d+3$ vertices. First we consider equidecomposable polytopes. Recall that the condition that all circuits are balanced is equivalent, in the Gale transform, to every spanned linear hyperplane bounds two open half-spaces with the same number of points (counting multiplicity).

Theorem 17: *Let P be a d-polytope with $d+3$ vertices. Then P is equidecomposable if and only if P is an r-fold pyramid ($r \geq 0$) over a polytope whose Gale transform satisfies*

(1) *there exists a nonnegative integer c such that each diameter of the Gale transform has exactly c more points on one end than on the other; and*
(2) *if $c \neq 0$, then the number of diameters is odd, and in the cyclic order the diameter ends alternate between having more points and having fewer points.*

Proof: $\Rightarrow$ Suppose P is an equidecomposable d-polytope with $d+3$ vertices. Since a pyramid is equidecomposable if and only if its base is, we may assume P is not a pyramid. The normalized Gale transform of P is a circle with numbers of points at the ends of the k diameters, $a_1^+, a_2^+, \ldots, a_k^+, a_1^-, a_2^-, \ldots a_k^-$. P being equidecomposable means that for every j, $1 \leq j \leq k$,

$$\sum_{i=1}^{j-1} a_i^+ + \sum_{i=j+1}^{k} a_i^- = \sum_{i=1}^{j-1} a_i^- + \sum_{i=j+1}^{k} a_i^+. \tag{4}$$

For $1 \leq j \leq k-1$, subtract Equation 4 for $j+1$ from Equation 4 for j. Then for $1 \leq j \leq k-1$, $a_{j+1}^+ - a_j^- = a_{j+1}^- - a_j^+$, or $a_{j+1}^+ - a_{j+1}^- = a_j^- - a_j^+$. Letting $c = |a_1^+ - a_1^-|$, we get condition 1, and we see that if $c \neq 0$, the excess must alternate between positive and negative. Now observe that Equation 4 for $j=1$ says $\sum_{i=2}^{k}(a_i^+ - a_i^-) = 0$. If k is even, the sum on the left can be 0 only if $c = 0$.

$\Leftarrow$ Now suppose conditions 1 and 2 are satisfied. Thus we may assume for every i, $a_i^- = a_i^+ + (-1)^i c$. It is easy to check that Equation 4 is satisfied for every j, so P is equidecomposable. ∎

For an equidecomposable polytope P, call the constant c given by Theorem 17 the **sign difference** of P.

Theorem 18: *Let P be a d-polytope with $d+3$ vertices. Then P is weakly neighborly if and only if P is equidecomposable and the sign difference of P is 0 or 1.*

Proof: The theorem follows easily from the case where P is not a pyramid. So suppose P is a d-polytope with $d+3$ vertices that is not a pyramid. Consider the Gale transform D of P as described in the proof of Theorem 17.

$\Rightarrow$ Suppose P is weakly neighborly with sign difference c. Suppose without loss of generality that $a_1^+ - a_1^- = c$, so $a_j^- = a_j^+ + (-1)^j c$ for all j. Let $\overline{S}$ be the set of $\sum_{i=1}^k a_i^-$ points on the half-open semicircle of D. Then S is contained in no proper face of P. Now

$$\begin{aligned} 2|S| - 2 &= 2\sum_{i=1}^{k} a_i^- - 2 \\ &= \sum_{i=1}^{k} a_i^- + \sum_{i=1}^{k}(a_i^- - (-1)^i c) - c - 2 \\ &= d + 3 - c - 2 = d + 1 - c \end{aligned}$$

Since P is weakly neighborly, $2|S| - 2 \geq d$, so $c \leq 1$.

$\Leftarrow$ Assume P is an equidecomposable nonpyramid with sign difference $c = 0$ or $c = 1$. Note that P has (possibly) two kinds of facets: simplices (whose cofacets are two-dimensional in the Gale transform), and equidecomposable $(d-1)$-polytopes with $d+1$ vertices (whose cofacets are located on one diagonal). Both types of facet are weakly neighborly polytopes. Suppose S is a set of vertices of P. If S is contained in a facet, then because the facet is weakly neighborly, S is contained in a face of the facet of at most twice its dimension. But this face is also a face of P, so S is contained in a face of P of at most twice its dimension. If S is not contained in a facet, then there exists j such that $\overline{S}$ contains $\{a_1^+, a_2^+, \ldots, a_j^+, a_{j+1}^-, \ldots, a_k^-\}$. So

$$\begin{aligned}
2|S|-2 &\geq 2\sum_{i=1}^{j} a_i^+ + 2\sum_{i=j+1}^{k} a_i^- - 2 \\
&= \sum_{i=1}^{j} a_i^+ + \sum_{i=1}^{j}(a_i^+ + (-1)^i c) - \sum_{i=1}^{j}(-1)^i c \\
&\quad + \sum_{i=j+1}^{k} a_i^- + \sum_{i=j+1}^{k}(a_i^- - (-1)^i c) + \sum_{i=j+1}^{k}(-1)^i c - 2 \\
&= \sum_{i=1}^{k} a_i^+ + \sum_{i=1}^{k} a_i^- + \sum_{i=2}^{k}(-1)^i c - (-1)^j c - 2 \\
&= d+3-(-1)^j c - 2 = d+1-(-1)^j c \\
&\geq d+1-c \geq d.
\end{aligned}$$

So S is contained in a face of dimension at most twice the dimension of S. Thus P is weakly neighborly. ∎

5.2 SHELLINGS AND h-VECTORS. The h-vectors of weakly neighborly d-polytopes with $d+2$ vertices are the vectors of the form $(1,2,3,\dots,k,k,\dots,k,k-1,\dots,3,2,1)$ (here $k = 1/2(d-r+2)$ if P is an r-fold pyramid). This is the entire set of possible h-vectors for d-polytopes with $d+2$ vertices. We shall see that this fails for d-polytopes with $d+3$ vertices: not all h-vectors of d-polytopes with $d+3$ vertices are h-vectors of weakly neighborly polytopes.

It is well known that the h-vector of a shellable simplicial complex can be calculated from an explicit shelling. Lee [8] used this to compute the h-vector of a simplicial polytope from its Gale diagram. We apply this technique to weakly neighborly polytopes.

A **shelling** of a pure simplicial d-complex Δ is an ordering $F_1, F_2, \dots, F_m$ of its facets so that for each j, $2 \leq j \leq m$, $F_j \cap (\cup_{i<j} F_i)$ is a pure $(d-1)$-dimensional subcomplex of F_j. If $F_1, F_2, \dots, F_m$ is a shelling of Δ, then $h_k(\Delta)$ is the number of j such that k is the cardinality of the minimal face of F_j not contained in any previous F_i [9].

Suppose a d-polytope P with $d+3$ vertices has a (normalized) Gale transform D. We say a ray r from 0 is in **general position** with respect to D if $-r$ contains no point of D and r does not meet the intersection of any two diagonals of D (a diagonal of D is a line segment connecting two points of D). The pairs $\{\overline{x},\overline{y}\}$ of points of D such that the open ray $r \setminus \{0\}$ intersects the relative interior of conv $\{\overline{x},\overline{y}\}$ are the cofaces of the maximal simplices of a triangulation Δ of P. (This is the Gale triangulation induced by a point $\overline{z}$ on $-r\setminus\{0\}$.)

We describe a shelling order for Δ (this follows [7], where a certain degree of affine independence is assumed). First choose an ordering of the points at each point location. Now order the diagonals in order of decreasing distance from 0 along r. Let $\overline{X_i^+}$ and $\overline{X_i^-}$ be the points at the two ends of the ith diagonal. Order the pairs in $\overline{X_i^+} \times \overline{X_i^-}$ consistently with the orderings of $\overline{X_i^\pm}$. This gives an ordering of all pairs $\{\overline{x},\overline{y}\}$ whose convex hulls intersect $r\setminus\{0\}$. It is straightforward to check that it gives a shelling of Δ.

Say that a point $\overline{v}$ is **weakly separated** from 0 by $\{\overline{x},\overline{y}\}$ if either $\overline{v}$ is in the open half-space bounded by the line aff$\{\overline{x},\overline{y}\}$ not containing 0, or $\overline{v}$ is in the same set $\overline{X_i^+} \cup \overline{X_i^-}$ as $\overline{x}$ and $\overline{y}$ and occurs before $\overline{x}$ or $\overline{y}$ in the chosen orderings of $\overline{X_i^\pm}$. If F is a facet of Δ with coface $\{\overline{x},\overline{y}\}$ then the minimal face of F not contained in a previous facet of the shelling is spanned by the vertices v such that $\overline{v}$ is weakly separated from 0 by $\{\overline{x},\overline{y}\}$. Thus $h_i(\Delta)$ is the number of pairs $\{\overline{x},\overline{y}\}$ whose convex hull intersects $r\setminus\{0\}$ and which weakly separate exactly i points from 0. Applying Theorem 4 gives the following.

PROPOSITION 19: *Let P be a d-polytope with $d+3$ vertices; let D be a Gale transform of P; let r be a ray from 0 in general position with respect to D; and let Δ be the triangulation of P induced by r. If Δ is shallow, then $g_i(P)$ is the number of pairs $\{\overline{x},\overline{y}\}$ whose convex hull intersects $r\setminus\{0\}$ and which weakly separate exactly i points from 0.*

For a weakly neighborly polytope this implies that no diagonal of a Gale transform separates more than $d/2$ points from 0.

For Lawrence polytopes the flag vector (and hence, f-vector and h-vector) depends only on the underlying matroid (the affine matroid on the vertices), not on the oriented matroid. The next theorem gives a partial extension to equidecomposable and weakly neighborly polytopes.

THEOREM 20: *If P and Q are equidecomposable polytopes with at most $d+3$ vertices having the same matroid, then all triangulations of P and Q have the same f-vector. If P and Q are weakly neighborly polytopes with at most $d+3$ vertices having the same matroid, then $h(\partial P) = h(\partial Q)$.*

Proof: For equidecomposable polytopes with at most $d+2$ vertices, the matroid determines the combinatorial type of the polytope. Now suppose P is an equidecomposable d-polytope with $d+3$ vertices, and let D_P be a normalized Gale transform for P. Let Q be a polytope with normalized Gale transform D_Q differing from D_P in that two diameters are exchanged (and flipped to preserve equivariance).

Consider a ray r from the origin in general position, and the induced triangulations Δ_P and Δ_Q of P and Q, respectively. A straightforward calculation shows that the diagonals crossed by r in the two Gale transforms make the same contribution to the h-vectors of Δ_P and Δ_Q.

Since P and Q are equidecomposable, all triangulations of the two polytopes have the same h-vector. The h-vector of a simplicial complex determines its f-vector, so the first statement of the theorem is proved. Finally, for weakly neighborly polytopes, the h-vector of a triangulation determines the h-vector of the boundary of the polytope, so the second statement holds. ∎

CONJECTURE 21: *The flag vector and h-vector of (the boundary of) a weakly neighborly d-polytope depend only on the underlying matroid.*

By examining Gale transforms of 6-polytopes with nine vertices it can be shown that there is no equidecomposable polytope with h-vector $(1,3,4,5,4,3,1)$. Thus not all h-vectors of d-polytopes with $d+3$ vertices can be realized by equidecomposable or weakly neighborly polytopes.

6. Further problems

Analogues of Theorems 17 and 18 hold only for "generic" d-polytopes with $d+4$ vertices. Here we call a polytope generic if the distinct diameters of its Gale transform are in general position, while any number of points may occur on each diameter. An equidecomposable generic d-polytope with $d+4$ vertices has a Gale transform satisfying

(1) there exists a nonnegative integer c such that each diameter of the Gale transform has exactly c more points at one end than at the other; and

(2) if $c \neq 0$, then the number of diameters is even.

Furthermore, a generic equidecomposable d-polytope with $d+4$ vertices is weakly neighborly if and only if this sign difference c is 0 or 1.

Finally we mention some open questions for weakly neighborly polytopes of any dimension and any number of vertices.

1. Are all weakly neighborly polytopes rigid? Here a polytope is called rigid if its face lattice uniquely determines its oriented matroid.
2. Can a proper subdivision (with no new vertices) of a polytope be weakly neighborly?
3. Does every even-dimensional weakly neighborly polytope have f-vector minimal among polytopes with the same h-vector?

The interpretation of the h-vector for nonsimplicial polytopes is important for further progress on facial enumeration questions. We hope that the study of weakly neighborly polytopes will suggest ideas in that direction.

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PLANE STRUCTURES IN THERMAL RUNAWAY

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ABSTRACT

We consider the problem

(1) $u_t = u_{xx} + e^u$ when $x \in \mathbb{R}$, $t > 0$,

(2) $u(x,0) = u_0(x)$ when $x \in \mathbb{R}$,

where $u_0(x)$ is continuous, nonnegative and bounded. Equation (1) appears as a limit case in the analysis of combustion of a one-dimensional solid fuel. It is known that solutions of (1), (2) blow-up in a finite time T, a phenomenon often referred to as thermal runaway. In this paper we prove the existence of blow-up profiles which are flatter than those previously observed. We also derive the asymptotic profile of $u(x,T)$ near its blow-up points, which are shown to be isolated.

1. Introduction

Consider the semilinear parabolic equation

$$u_t - u_{xx} = e^u; \qquad x_0 \in (a,b), \quad t > 0, \tag{1.1}$$

where $-\infty \leq a < b \leq +\infty$. Equation (1.1) is one of the simplest models arising from combustion theory. Indeed, it is well known (cf. for instance [BE], Chapter I) that thermal reaction of a one-dimensional solid fuel can be described by the system

$$T_t = T_{xx} + \delta\varepsilon c \exp\left[\frac{T-1}{\varepsilon T}\right], \tag{1.2a}$$

$$c_t = -\varepsilon\delta\Gamma c \exp\left[\frac{T-1}{\varepsilon T}\right] \tag{1.2b}$$

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