FACE NUMBERS AND SUBDIVISIONS OF CONVEX POLYTOPES

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Abstract. The first part of the paper surveys results on f-vectors, flag vectors and h-vectors of convex polytopes. These are combinatorial parameters that have been characterized for simplicial polytopes. Many of the results known in the general case depend on the connection between convex polytopes and toric varieties. The second half of the paper looks at polyhedral subdivisions of convex polytopes. The effect of subdivision on the h-vector is studied. The paper discusses the secondary polytope, which encodes the regular subdivisions of a polytope. Fiber zonotopes and the corresponding hyperplane arrangements, called discriminantal arrangements, are studied.

Key words: convex polytope, f-vector, h-vector, subdivision, triangulation, secondary polytope, fiber polytope, hyperplane arrangement, discriminantal arrangement

The combinatorics of convex polytopes is an active area of research. The most important reference in the subject, in spite of its age, is still Grünbaum's book [18]. Highlighting the accomplishments since then are two recent survey articles, by Klee and Kleinschmidt [25], and by Bayer and Lee [7]. In addition Ziegler has a preliminary version of lecture notes on polytopes [42]. All these offer much more extensive bibliographies than is contained here. This paper focuses on three topics in the combinatorial study of polytopes: numbers of faces, subdivisions, and relation to hyperplane arrangements.

1. Numbers of Faces

1.1. f-VECTOR HISTORY

A polytope is the convex hull of a finite point set in $\mathbf{R}^d$. A d-dimensional polytope has faces of dimension 0 (vertices), 1 (edges), and so on, up to $d-1$ (facets). Write f_i for the number of i-dimensional faces. The f-vector of a polytope is the sequence $(f_0, f_1, \ldots, f_{d-1})$. This has been the subject of much study in this century and before. The characterization of f-vectors of d-polytopes for all d is a major open problem.

At the turn of the century Steinitz [40] gave the complete characterization of f-vectors of polytopes of dimension three.

Theorem 1 (Steinitz) *An integer vector* (f_0, f_1, f_2) *is the* f*-vector of a* *three-dimensional polytope if and only if*

1. $f_0 - f_1 + f_2 = 2$
2. $f_0 \leq 2f_2 - 4$
3. $f_2 \leq 2f_0 - 4$.

No characterization is known for polytopes of any dimension greater than three. For 3-polytopes more detailed combinatorial information has been studied. If P is a 3-polytope let $p_n(P)$ be the number of faces of P which are n-gons, and call the vector $(p_3, p_4, p_5, \ldots)$ the p-vector. Much is known about p-vectors of 3-polytopes, though it falls short of a complete characterization. The basic theorem is due to Eberhard in 1891 [14].

Theorem 2 (Eberhard) *There exists a 3-polytope with p_n n-gons ($n \geq 3$, $n \neq 6$) and some number of 6-gons if and only if the integer $\sum_{n \geq 3}(6-n)p_n$ is even and is at least 12.*

In the 1970s and 1980s Jendrol' and others studied the possible values of p_6 (see [20]).

A polytope is called *simplicial* if all its faces are simplices. For simplicial polytopes Sommerville [33] first observed a useful invertible linear transformation on the f-vector. Extend the definition of f-vector by writing $f_{-1} = 1$. Then the *h-vector* of a simplicial d-polytope is $(h_0, h_1, \ldots, h_d)$, where

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}.$$

In 1971 McMullen [29] conjectured a characterization of h-vectors of simplicial d-polytopes (for all d). The necessity of the "McMullen conditions" was proved in 1980 by Stanley [34], and the sufficiency the same year by Billera and Lee [9]. In 1992 McMullen [28] gave a new proof of necessity, avoiding the algebraic geometry used by Stanley. (Sommerville [33] proved (2) of the theorem below; these equations are known as the Dehn-Sommerville equations.)

Theorem 3 (Stanley; Billera and Lee; McMullen) *An integer vector $(h_0, h_1, \ldots, h_d)$ is the h-vector of a simplicial d-polytope if and only if*

1. $h_i = h_{d-i}$ *for all* i
2. $h_0 = 1$ *and* $h_i \leq h_{i+1}$ *for all* i, $0 \leq i \leq d/2 - 1$
3. $h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$ *for all* i, $0 \leq i \leq d/2 - 1$.

The superscript $\langle i \rangle$ represents the pseudopower operation, defined via binomial coefficients as follows. For any positive integers n and i, n has a unique representation

$$n = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. The ith pseudopower of n is then

$$n^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}.$$

We also define $0^{\langle i \rangle} = 0$ for any positive integer i.

The h-vector is important not only because it gives a convenient way to write down the conditions, but because it is the counting vector for several different sequences of objects associated with the polytope. The discoveries of these fueled much research in polytopes. Some of them have analogues in the nonsimplicial case, and we turn to that now.

1.2. Flag Vectors

It quickly becomes clear in looking at the general polytope case that the f-vector is not the right object of study. It carries too little of the combinatorial information. We introduce instead the flag vector of a polytope, defined as follows. Consider a sequence of distinct proper faces ordered by inclusion, $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k$. This is called an S-flag, where S is the set of dimensions of the faces F_i. The number of S-flags of a polytope is written f_S, and the vector of all such flag numbers is called the *flag vector* of the polytope (here S ranges over all subsets of $\{0, 1, \ldots, d-1\}$). For example, a square based pyramid has flag numbers $f_\emptyset = 1$, $f_0 = f_2 = 5$, $f_1 = 8$, $f_{0,1} = f_{0,2} = f_{1,2} = 16$, and $f_{0,1,2} = 32$.

The big enumeration problem for polytopes is then: characterize the flag vectors of d-dimensional polytopes for all d. The problem is trivially solved for dimension at most 3 because the flag vector then depends only on the f-vector. Also, the flag vector of a simplicial polytope depends only on its f-vector, so there is nothing interesting to say in this special case.

The first general result for flag vectors of polytopes is known as the generalized Dehn-Sommerville equations [4].

Theorem 4 (Bayer and Billera) *The affine dimension of the flag vectors of d-polytopes is $e_d - 1$, where (e_d) is the Fibonacci sequence, $e_d = e_{d-1} + e_{d-2}$, $e_0 = e_1 = 1$. The affine hull of the flag vectors is determined by the equations*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = \left(1 - (-1)^{k-i-1}\right) f_S(P),$$

where $i \leq k-2$, $i, k \in S \cup \{-1, d\}$, and S contains no integer between i and k.

The proof that these equations hold for flag vectors of polytopes is the same as Sommerville's proof of the Dehn-Sommerville equations for simplicial polytopes. That they determine the affine hull of flag vectors was proved by exhibiting a basis of polytopes. This was first done in a complicated calculation in [4], and was later done more elegantly by Kalai [23]. In that work Kalai also found the affine hull of flag vectors of k-simplicial d-polytopes (polytopes all of whose k-faces are simplices).

Stanley's proof of the McMullen conditions for f-vectors of simplicial polytopes depends on a connection between rational polytopes and toric varieties. A polytope is rational if all the coordinates of its vertices are rational numbers. Every simplicial polytope is combinatorially equivalent to a rational polytope, but this is not true for nonsimplicial polytopes (see [18, page 94]). For a rational polytope, the affine dependencies among the vertices can be generated by a finite set of affine dependencies with rational coefficients. These are used to define a toric variety. (For an exposition of toric varieties and polytopes see [31].) The h-vector of a simplicial polytope is the sequence of Betti numbers of this variety. For rational, nonsimplicial polytopes, however, the usual Betti numbers no longer depend only on the combinatorial structure of the polytope. The appropriate definition of the h-vector comes from the middle perversity intersection homology. When the polytope is simplicial this definition agrees with the previous definition of h-vector.

A way of calculating the h-vector for general rational polytopes was found independently by several algebraic geometers: Bernstein, Khovanskii and MacPherson. Stanley [35] introduced the formula to combinatorialists; he gives in these proceedings [39] a definition that applies more generally to Eulerian posets. Here is that definition of the *h-vector* of a polytope as a recursion on the face lattice.

For a d-polytope P, from the h-vector $(h_0, h_1, \ldots, h_d) \in \mathbf{N}^{d+1}$ is defined the g-vector $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor}) \in \mathbf{N}^{\lfloor d/2 \rfloor +1}$ by $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq d/2$. The generating functions $h(P,t) = \sum_{i=0}^{d} h_i t^{d-i}$ and $g(P,t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i t^i$ are defined recursively by

1. $g(\emptyset, t) = h(\emptyset, t) = 1$, and
2. $h(P,t) = \displaystyle\sum_{\substack{G \text{ face of } P \\ G \neq P}} g(G,t)(t-1)^{d-1-\dim G}$.

It is easy to check from this recursion that for a fixed dimension d and any i, h_i is a linear function of the flag vector. Fine [16] has a combinatorial interpretation of the coefficients of the flag numbers in this linear function.

If $S \subseteq \{0, 1, \ldots, d-1\}$ the *S-template* consists of d spaces with a vertical bar after the $(j+1)$st space for each $j \in S$. An *admissible pattern* of weight r for the S-template is a placement of an x in each of r spaces so that to the left of each vertical bar (all the way to the left) there are more blank spaces than xs. The unsigned coefficient of f_S in h_i (for d-polytopes) is the number of admissible patterns of weight $d-i$ for the S-template. The sign of the coefficient of f_S in h_i is $(-1)^{d-i+|S|+1}$.

As an example we compute the formula for the h-vector of a 3-polytope. Here are all the admissible patterns, with signs indicated.

```
(+) x x x
(-) x x _
(-) x _ x
(-) _ x x
(+) _ | x x

(+) x _ _
(+) _ x _
(+) _ _ x
(-) _ | x _
(-) _ | _ x
(-) _ _ | x
(-) x _ _ |
(-) _ x _ |
(-) _ _ x |
(+) _ | _ | x
(+) _ | x _ |
(+) _ | _ x |
(+) _ _ | x |
(-) _ | _ | x |

(-) _ _ _
(+) _ | _ _
(+) _ _ | _
(+) _ _ _ |
(-) _ | _ | _
(-) _ | _ _ |
(-) _ _ | _ |
(+) _ | _ | _ |
```

From these we get the following formulas for the h-vector.

$$\begin{aligned} h_0 &= f_\emptyset \\ h_1 &= f_0 - 3f_\emptyset \\ h_2 &= -f_{012} + f_{01} + 2f_{02} + f_{12} - 2f_0 - f_1 - 3f_2 + 3f_\emptyset \\ h_3 &= f_{012} - f_{01} - f_{02} - f_{12} + f_0 + f_1 + f_2 - f_\emptyset \end{aligned}$$

In summary, associated with any rational d-polytope P is an h-vector $h(P) = (h_0, h_1, \ldots, h_d)$, which depends linearly on the flag vector of P. As in the case of simplicial polytopes, this h-vector is positive, symmetric and unimodal. We don't know if the nonlinear inequalities satisfied by simplicial h-vectors hold in general. Note that the flag vector cannot be computed just from the h-vector; the flag vector contains a Fibonacci number of pieces of information (exponential in dimension d), while the h-vector contains only $d/2$ pieces of information. The unimodality of the h-vector or, equivalently, the nonnegativity of the g-vector gives $d/2$ linear inequalities satisfied by flag vectors of rational d-polytopes. It is believed that these are sharp and that they are satisfied by flag vectors of irrational polytopes. It should be noted that the first two inequalities are known for irrational polytopes as well: one is trivial ($f_0 \geq d+1$), and the other ($f_{02} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} \geq 0$) was proved by Kalai [22] using rigidity.

Kalai [23] made a wonderful observation that enabled him to extend the h-vector to a vector linearly equivalent to the flag vector, and to get many more linear inequalities on flag vectors. This is that the convolution of two linear forms on flag vectors is also a linear form on flag vectors. First we define the convolution of two flag numbers. For $S \subseteq \{0,1,\ldots,d-1\}$, $T \subseteq \{0,1,\ldots,e-1\}$, and P a $(d+e+1)$-polytope, let

$$f_S * f_T(P) = \sum_{F \text{ } d\text{-face of } P} f_S(F) f_T(P/F) = f_{S\cup\{d\}\cup(T+d+1)}(P).$$

(Here P/F is the quotient polytope of F, an e-polytope whose face lattice is the interval $[F,P]$ in the face lattice of P.) Repeat to get the convolution of any finite sequence of flag numbers, and extend linearly to define the convolution of a sequence of linear forms in flag numbers. In the following we write g_i^d for the linear form in flag numbers that calculates g_i ($= h_i - h_{i-1}$) for d-polytopes.

Theorem 5 (Kalai) *The following linear forms on flag numbers form a length e_d vector linearly equivalent to the flag vector:*

$$g_{\ell_1}^{s_1} * g_{\ell_2}^{s_2} * \cdots * g_{\ell_k}^{s_k},$$

where $d = k - 1 + \sum s_i$, $0 \le \ell_k \le s_k/2$ and for $1 \le i < k$, $1 \le \ell_i \le s_i/2$.

The convolution of a sequence of linear forms that are nonnegative for all polytopes is itself a linear form that is nonnegative for polytopes of the appropriate dimension. Thus convolutions of g_is are nonnegative for rational polytopes, and hence give linear inequalities on flag numbers that hold for all rational polytopes.

Here is an example. Consider the convolution $g_1^2 * g_1^2$ for 5-polytopes; here $g_1^2(Q) = f_0(Q) - 3 \ge 0$ for a 2-polytope Q. Then for any 5-polytope P

$$g_1^2 * g_1^2(P) = \sum_{F \text{ 2-face of } P} (f_0(F) - 3)(f_0(P/F) - 3) \ge 0.$$

This is equivalent to the linear inequality

$$f_{023}(P) - 3f_{23}(P) - 3f_{02}(P) + 9f_2(P) \ge 0,$$

which holds for all 5-polytopes P. (Note: the inequality $f_0 - 3 \ge 0$ holds for all 2-polytopes, so the resulting inequality holds for irrational as well as rational 5-polytopes.) Each linear form in the flag numbers has a dual linear form, obtained by replacing f_S by $f_{\tilde{S}}$, where $\tilde{S} = \{d-1-s : s \in S\}$. By polytope duality the dual of a linear form that is nonnegative for all (rational) d-polytopes is also nonnegative for all (rational) d-polytopes. We thus can use the duals of the g_i^d in convolutions to generate nonnegative linear forms in flag numbers. The following appears in [23].

Conjecture 6 (Kalai) *The nonnegativity of convolutions of the g_i and their duals imply all linear inequalities on the flag numbers of polytopes.*

Meisinger [30] showed that the conjecture is false, because starting in dimension six these inequalities do not imply all the following trivial inequalities: $f_i \geq \binom{d+1}{i+1}$ (that is, the fact that every d-polytope has at least as many i-faces as the d-simplex).

A particular choice of convolutions of g_i^d form a Fibonacci-length vector linearly equivalent to the flag vector. Another such vector, called the *cd*-index, was introduced by Fine (but was first published in [6]). Stanley [38] proved that the *cd*-index is nonnegative for shellable regular CW-spheres (thus proving an extended version of a conjecture of Fine). The resulting inequalities on flag numbers are in particular true for polytopes, but we can expect them to be weaker than the g-vector convolution inequalities, which are special to (rational) polytopes. See the paper by Stanley in these proceedings [39] for more information about the *cd*-index.

Finally let me observe that the intersection homology picture only tells us linear inequalities. We know by convexity considerations that the flag vectors (in dimension four and higher) cannot be characterized only by linear equations and linear inequalities. But we have no conjecture for a set of nonlinear inequalities that are tight and hold for all d-polytopes.

We end this section with a brief look at two special classes of polytopes.

1.3. Cubical Polytopes

A polytope is called *cubical* if and only if all its proper faces are combinatorially equivalent to cubes. As in the simplicial case, the flag vectors of cubical polytopes depend linearly on the f-vectors. Also, the f-vectors of cubical polytopes satisfy $\lceil d/2 \rceil$ linear equations, analogous to the Dehn-Sommerville equations [18]. Adin (in work as yet unpublished) has defined a special h-vector for cubical polytopes (different from the "toric variety" h-vector), which is symmetric, nonnegative, and computable by shelling the polytope. The symmetry of Adin's h-vector gives the linear equations on the f-vector mentioned above.

A seemingly basic fact was proved only recently by Blind and Blind [12].

Theorem 7 (Blind and Blind) *If P is a cubical d-polytope, then for all i, $f_i(P) \geq f_i(d\text{-cube})$. Equality in a single i implies that P is a d-cube.*

Several people had conjectured that a cubical polytope always has at least as many vertices as facets. This has been disproved recently by Jockusch, who constructs a sequence of cubical polytopes of increasing dimension for which the ratio of number of facets to number of vertices increases without bound [21]. Jockusch also conjectures a lower bound on f_i for cubical polytopes in terms of the number of vertices and the dimension.

1.4. Centrally Symmetric Polytopes

In 1982 Bárány and Lovász conjectured lower bounds on the number of faces of simplicial, centrally symmetric polytopes [2]. This conjecture was later strengthened by Björner. The stronger version was proved by Stanley by studying the effect of symmetries on h-vectors [36].

Theorem 8 (Stanley) *If P is a simplicial, centrally symmetric d-polytope, then for all i, $1 \leq i \leq d/2$,*

$$h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1}.$$

Adin has given analogous results for rational simplicial polytopes with other types of symmetries [1].

We mention finally a conjecture of Kalai [24] on centrally symmetric polytopes, nonsimplicial as well as simplicial.

Conjecture 9 (Kalai) *For any centrally symmetric polytope*

$$f_0 + f_1 + \cdots + f_{d-1} \geq 3^d - 1.$$

2. Subdivision

We now turn to polyhedral subdivisions of polytopes. Initially we will be interested in the effect of subdivision on flag numbers. It is easy to see that as you subdivide the faces of a polytope the flag numbers increase. Also the boundary of an arbitrary polytope can be subdivided to get (combinatorially) a simplicial polytope. This can be done, for example, by barycentric subdivision, but it can also be done without adding any new vertices. The f-vectors and flag vectors of simplicial polytopes are characterized. How can we use a simplicial subdivision to give conditions on flag vectors?

Define a *polyhedral subdivision* of a d-polytope P to be a polyhedral complex whose vertex set is that of P and whose underlying space is P. Note that here we're considering a subdivision of the solid polytope, not of its boundary, but the subdivision of P induces a subdivision of its boundary. Also we assume that no new vertices occur in the subdivision. The subdivision is a *triangulation* if all its faces are simplices. Earlier we defined the h-vector of a d-polytope. More properly we should have called this the h-vector of the boundary complex ∂P of P, i.e., the polyhedral complex consisting of the faces of the boundary of P. The computation of an h-vector from a flag vector can be performed for any polyhedral complex. When this is done for the polyhedral complex consisting of the boundary complex of P plus the one d-face P itself, we find that $h_0(P) = h_0(\partial P) = 1$, $h_i(P) = h_i(\partial P) - h_{i-1}(\partial P)$ for $1 \leq i \leq d/2$, and $h_i(P) = 0$ for $i \geq d/2$. (This is the g-vector of the boundary of P.) Stanley showed the following [37].

Theorem 10 (Stanley) *If P is a rational convex polytope and Δ is a (rational) polyhedral subdivision of P, then for all i, $h_i(\Delta) \geq h_i(P)$ and $h_i(\partial\Delta) \geq h_i(\partial P)$.*

This gives a limited amount of inequality information. For example, if you subdivide a rational polytope adding no new vertices and then apply the upper bound theorem, you get inequalities on the h-vector, and thus the flag vector, of the original polytope.

2.1. Shallow Subdivisions

To help understand how subdivision changes the h-vector of a polytope, we study those subdivisions that leave the h-vector unchanged. For ease of presentation we will restrict the discussion to triangulations. (The situation for other subdivisions is understood as well but it's harder to state. See [3] for details.) Let Δ be a triangulation of a polytope P. For a face σ of Δ, define the *carrier* $C(\sigma)$ of σ to be the smallest face of P containing σ. A triangulation Δ of P is *shallow* if and only if for all faces σ of Δ, $\dim C(\sigma) \leq 2\dim\sigma$. Note that the condition for shallowness applied to vertices of Δ says that every vertex of Δ must be a vertex of P, which we have assumed anyway. A triangulation of a 3-polytope is shallow if and only if every edge of the triangulation is contained in a 2-face, i.e., if and only if there are no interior edges.

Here are a few examples. The bipyramid over a triangle has two triangulations, one into two simplices, the other into three. The first is shallow; the second is not. The regular octahedron has only one combinatorial type of triangulation, into four simplices around an edge connecting opposite vertices. It is not shallow. The triangular prism has one combinatorial type of triangulation, into three simplices. It is shallow.

Shallow triangulations are important because of the following theorem [3].

Theorem 11 *1. If Δ is a shallow subdivision of a polytope P, then $h(\Delta) = h(P)$ and $h(\partial\Delta) = h(\partial P)$.*

2. If Δ is a subdivision of a rational *polytope P and $h(\Delta) = h(P)$, then Δ is shallow.*

Thus if P has a shallow triangulation whose boundary is polytopal, then the h-vector of the boundary of P satisfies the McMullen conditions (nonlinear inequalities as well as the linear conditions). It would be interesting, then, to have a nice characterization of polytopes that have shallow subdivisions.

A polytope is *weakly neighborly* if all its triangulations are shallow. (The triangular prism is an example of a weakly neighborly polytope.) There is a nice combinatorial characterization of weakly neighborly polytopes [3]; it explains the choice of the term.

Theorem 12 *A polytope P is weakly neighborly if and only if every set of $k + 1$ vertices is contained in a face of dimension at most $2k$ for all k.*

The three-dimensional examples of weakly neighborly polytopes are not very interesting: they are just the triangular prism and all 3-dimensional pyramids. Among simplicial d-polytopes, having a shallow triangulation is equivalent to being $\lfloor d/2 \rfloor$-stacked. The only simplicial, weakly neighborly polytopes are simplices and even-dimensional neighborly polytopes. A large class of weakly neighborly polytopes is the set of Lawrence polytopes. These are the polytopes having symmetric Gale diagrams. Perhaps weakly neighborly polytopes can be characterized as having nearly symmetric Gale diagrams, but a precise statement is known only for polytopes with few vertices [3]. One other example of a weakly neighborly polytope is the Cartesian product of two simplices (of any dimension).

Stanley introduced local h-vectors, which serve as measures of nonshallowness at faces of a polytope [37].

The *local h-vector* of a subdivision Δ of a d-polytope P is the vector of coefficients of the degree $d+1$ polynomial $\ell_P(\Delta, x)$ that satisfies the recursion

1. $\ell_\emptyset(\emptyset, x) = 1$
2. $h(\Delta, x) = \sum_{F \text{ face of } P} \ell_F(\Delta|_F, x) h(P/F, x)$.

(Here $h(P/F, x)$ is the polynomial whose coefficients form the h-vector of the solid polytope with face lattice equal to the interval $[F, P]$ in the face lattice of P.)

The local h-vector of a subdivision Δ depends on the numbers of flags of Δ and on the dimensions of the carriers of the faces; an explicit formula for the local h-vector reflecting this fact is still lacking.

Here is the most important theorem about local h-vectors, from [37].

Theorem 13 (Stanley) *Let Δ be a subdivision of a d-polytope P, and let $\ell_P(\Delta, x) = \ell_0 + \ell_1 x + \cdots + \ell_{d+1} x^{d+1}$.*

1. *If P is rational, then for all i, $\ell_i = \ell_{d+1-i}$ and $\ell_i \geq 0$.*
2. *If P is rational and Δ is regular, then for $0 \leq i < d/2$, $\ell_i \leq \ell_{i+1}$.*

The first of these is what is used to prove that the h-vector increases under subdivision. We will discuss the regularity condition on subdivisions in the next section.

2.2. Secondary Polytopes

A polyhedral subdivision of a polytope can be obtained in the following way. Given a d-polytope in $\mathbf{R}^d$, embed $\mathbf{R}^d$ in $\mathbf{R}^{d+1}$, and "lift" the polytope by assigning heights $h(x)$ to each of the vertices x. Thus we get a set of points $\{y \in \mathbf{R}^{d+1} : x = (y_1, y_2, \ldots, y_d)$ is a vertex of P and $y_{d+1} = h(x)\}$; call its convex hull Q. Let Q_{top} be the subcomplex of ∂Q that is visible from way out on the $(d+1)$-coordinate axis. Projecting Q_{top} onto $\mathbf{R}^d$ gives a subdivision of P. A subdivision obtained in this way is called *regular.*

Consider the partially ordered set (poset) of regular subdivisions ordered by refinement. The following remarkable result is from [17].

Theorem 14 (Gel'fand, Kapranov and Zelevinsky) *Given any polytope P there exists a polytope $\Sigma(P)$ whose face lattice is isomorphic to the poset of regular subdivisions of P.*

The polytope $\Sigma(P)$ is called the *secondary polytope* of P. Note that its vertices correspond to regular triangulations of P. The case where P is 2-dimensional was proved independently by Carl Lee [26] and Mark Haiman [19]. They called the secondary polytope of a polygon an *associahedron.* For example, the pentagon has five subdivisions into a quadrilateral and a triangle, and five triangulations. The refinement poset is easily seen to be the face lattice of a pentagon. Thus the secondary polytope (associahedron) of a pentagon is a pentagon. The secondary polytope of a hexagon is a simplicial 3-polytope with 14 vertices, 36 edges and 24 2-faces. See [8] for several equivalent descriptions of secondary polytopes.

The secondary polytope can be constructed from a projection of a simplex onto the polytope P. This construction was generalized by Billera and Sturmfels in their definition of *fiber polytopes* [10]. Given polytopes $P \subseteq \mathbf{R}^n$, $Q \subseteq \mathbf{R}^d$, and a projection $\pi : P \to Q$, consider all measurable functions $\gamma : Q \to \mathbf{R}^n$ such that for all x, $\pi \circ \gamma(x) = x$. Define $\Sigma(P,Q) = \{\int_Q \gamma(x)\,dx\} \subseteq \mathbf{R}^n$. Thus $\Sigma(P,Q)$ is an average of the fibers $\pi^{-1}(x)$, as x ranges over the points of Q; it is called the fiber polytope. Billera and Sturmfels proved that $\Sigma(P,Q)$ is a convex polytope of dimension $\dim P - \dim Q$, and that its face lattice is isomorphic to a poset of certain kinds of subdivisions of Q induced from P. We will look at the special case where P is a cube and Q is a zonotope.

2.3. Fiber Zonotopes and Discriminantal Arrangements

This section contains joint work with Brandt [5]. A *zonotope* is the Minkowski sum of a set of intervals $[-a_i, a_i]$. A zonotope has a natural dual object, the central hyperplane arrangement whose hyperplanes have normals a_i. The face lattice of the hyperplane arrangement is dual to the face lattice of the zonotope. We'll study the fiber polytope of a zonotope and see what it says about the hyperplane arrangement.

Fix a set of n nonzero vectors a_i in $\mathbf{R}^d$, no two parallel. Let Z be the zonotope generated by the corresponding intervals $[-a_i, a_i]$. Billera and Sturmfels give an explicit description of the fiber zonotope [10].

Theorem 15 (Billera and Sturmfels) *The fiber polytope $\Sigma(C_n, Z)$ is an $(n-d)$-dimensional zonotope,*

$$\Sigma(C_n, Z) = \frac{1}{vol\,Z} \sum_J [-E_J, E_J],$$

where the sum is over all $(d+1)$*-subsets of* $\{1,2,\ldots,n\}$*, and for* $J = \{j_1, j_2, \ldots, j_{d+1}\}$

$$E_J = \sum_{i=1}^{d+1} (-1)^i \det(a_{j_1}, \ldots, a_{j_{i-1}}, a_{j_{i+1}}, \ldots, a_{j_{d+1}}) \cdot e_{j_i}.$$

The face lattice of $\Sigma(C_n, Z)$ *is isomorphic to the poset of regular zonotopal subdivisions of* Z.

Note that the vectors E_J do not have to be nonzero and nonparallel, so the fiber zonotope may have fewer than $\binom{n}{d+1}$ zones.

It remains to define regular zonotopal subdivisions. A *zonotopal subdivision* of the zonotope Z is a polyhedral subdivision of Z each of whose edges is a translation of one of the intervals $[-a_i, a_i]$. Zonotopal subdivisions of Z are ordered by refinement. The most refined subdivide Z into (combinatorial) cubes. Some zonotopal subdivisions of Z can be obtained in the following way. "Lift" the zones of Z by assigning each of them a height h_i. Thus we get a set of vectors $(a_i, h_i) \in \mathbf{R}^{d+1}$, and we take, in addition, the standard unit vector e_{d+1}. Let Z' be the zonotope defined by these vectors, and let Z'_{top} be the subcomplex of $\partial Z'$ that is visible from way out on the $(d+1)$-coordinate axis. Projecting Z_{top} onto $\mathbf{R}^d$ gives a subdivision of Z. A zonotopal subdivision obtained in this way is called *regular*.

Example 1. Take four vectors in $\mathbf{R}^2$. The zonotope is an octagon. It has eight "cubical" subdivisions, having six quadrilaterals each. It has eight subdivisions into one hexagon and three quadrilaterals. All zonotopal subdivisions are regular. Thus the fiber zonotope is also an octagon. A portion of the face lattice of the fiber zonotope is shown in Figure 1.

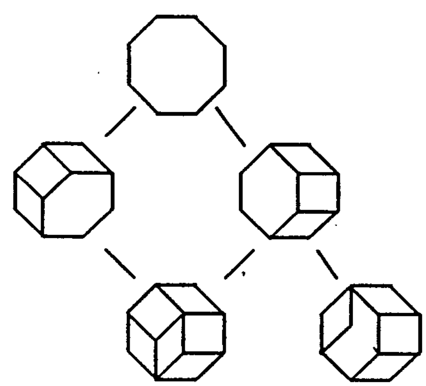

Fig. 1. Portion of face lattice of fiber zonotope

Example 2. Consider the five vectors in $\mathbf{R}^3$: $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (1,1,0)$, $a_4 = (0,0,1)$ and $a_5 = (1,0,1)$. The zonotope $Z = Z(\mathcal{A})$ is a 3-polytope with four hexagonal faces and eight quadrilateral faces. The drawing in Figure 2 shows the combinatorial structure of one side of this polytope, with half of the 2-faces showing. (Since zonotopes are centrally symmetric, this shows enough to determine the whole polytope.) The fiber zonotope $\Sigma(C_5, Z)$ is a hexagon. The original zonotope has twelve regular zonotopal subdivisions. Four of them each cut Z into two hexagonal prisms and two cubes. Two of them each cut Z into a twelve-sided zonotope and four cubes. Refining these are six cubical subdivisions of Z.

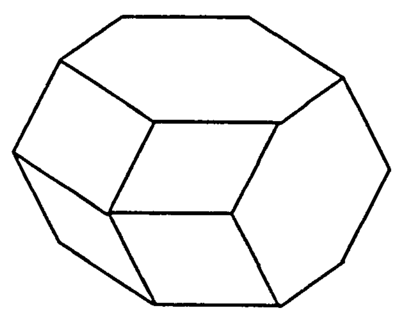

Fig. 2. 3-dimensional zonotope

Write $\mathcal{A}$ for the central hyperplane arrangement, $\mathcal{A} = \{H_1^0, H_2^0, \ldots, H_n^0\}$, where $H_i^0 = \{x \in \mathbf{R}^d : a_i \cdot x = 0\}$. There are two lattices associated with $\mathcal{A}$. The *lattice of intersections* $L(\mathcal{A})$ consists of all distinct intersections $\bigcap_{i\in I} H_i^0$, ordered by reverse inclusion, with a least element $\hat{0}$ adjoined.

To describe the *face lattice* of $\mathcal{A}$, consider the complement of $\mathcal{A}$, that is, $\mathbf{R}^d \setminus \bigcup_{i=1}^{n} H_i^0$. This is a set of open cones. The face lattice $\mathcal{F}(\mathcal{A})$ consists of all faces of the closures of these cones, ordered by inclusion, also with $\hat{0}$ adjoined. The face lattice of $\mathcal{A}$ is dual to the face lattice of the zonotope Z.

For example, for the standard unit vectors $e_1, e_2, \ldots, e_n$ in $\mathbf{R}^n$, Z is the n-cube, $L(\mathcal{A})$ is the Boolean lattice, and $\mathcal{F}(\mathcal{A})$ is the face lattice of the crosspolytope.

How does the fiber zonotope relate to the hyperplane arrangement? Define a new central hyperplane arrangement $B(\mathcal{A})$ by taking as normal vectors a maximal set of nonzero, pairwise nonparallel vectors among the E_J (the vectors of the fiber zonotope). Thus $B(\mathcal{A})$ is the hyperplane arrangement dual to the fiber zonotope. It is an arrangement of hyperplanes in $\mathbf{R}^n$ but it is of essential dimension $n-d$ (that is, all the hyperplanes intersect on a d-dimensional subspace). In the case where the original vectors a_1, a_2, ..., a_n are in general position, this is the *discriminantal arrangement* introduced by Manin and Schechtman; we will extend that term to the case of arbitrary a_i. We study the complement and the intersection and face lattices of the discriminantal arrangement.

Consider all *affine* hyperplane arrangements that have normal vectors a_i. These can be parametrized by points of $\mathbf{R}^n$. An easy way to do this is as follows: for $b \in \mathbf{R}^n$, let $\mathcal{A}_b$ be the affine arrangement with hyperplanes $H_i = \{x \in \mathbf{R}^n : a_i \cdot x = b_i\}$. Call $\mathcal{A}_b$ a *parallel translation* of $\mathcal{A}$. It is in *general position* if and only if for all $I \subseteq \{1, 2, \ldots, n\}$, $\bigcap_{i \in I} H_i = \emptyset$ if $|I| > d$, and $\dim \bigcap_{i \in I} H_i = d - |I|$ if $I \leq d$. The central arrangement $\mathcal{A}$ is *generic* if and only if some parallel translation $\mathcal{A}_b$ is in general position. This is the case treated in [27].

Theorem 16 (Manin and Schechtman) *If $\mathcal{A}$ is generic, then the complement of the discriminantal arrangement $B(\mathcal{A})$ is the set of b for which $\mathcal{A}_b$ is in general position.*

To see what the complement is for an arbitrary arrangement $\mathcal{A}$, we need a definition. The affine arrangement $\mathcal{A}_b$ is in *relatively general position* if and only if for all $I \subseteq \{1, 2, \ldots, n\}$, $\bigcap_{i \in I} H_i = \emptyset$ if $|I| > d$, and $\dim \bigcap_{i \in I} H_i \leq d - |I|$ if $I \leq d$. (The idea here is that the high dimensional intersections, but not the parallelisms, of a nongeneral position arrangement can be eliminated by parallel translation of the hyperplanes.)

Theorem 17 *The complement of the discriminantal arrangement of $\mathcal{A}$ is the set of b for which $\mathcal{A}_b$ is in relatively general position.*

Now a hyperplane of the discriminantal arrangement has a vector

$$E_J = \sum_{i=1}^{d+1} (-1)^i \det(a_{j_1}, \ldots, a_{j_{i-1}}, a_{j_{i+1}}, \ldots, a_{j_{d+1}}) \cdot e_{j_i}$$

as normal. Such a vector represents a minimal dependency among the a_i. Corresponding to a minimal dependency among the a_i is a minimal possible violation of relatively general position in the parallel translations of $\mathcal{A}$. Suppose $\{a_i : i \in S\}$ is a minimal dependent set. Define H_S to be the set of points $b \in \mathbf{R}^n$ such that in the affine arrangement $\mathcal{A}_b$, $\dim \bigcap_{i \in S} H_i = d - |S| + 1$.

Theorem 18 *The hyperplanes of the discriminantal arrangement based on $\mathcal{A}$ are the H_S, for all sets S that index minimal dependent subsets of $\{a_i : 1 \leq i \leq n\}$.*

This generalizes the observation of [27] that the hyperplanes of the discriminantal arrangement of a generic arrangement correspond to the $(d+1)$-sets of $\{1, 2, \ldots, n\}$.

We have described the hyperplanes and the complement of the discriminantal arrangement. We now turn to the whole face structure. Every parallel translation $\mathcal{A}_b$ of $\mathcal{A}$ has its own face poset, and we can label each open face with a sign vector in $\{-, 0, +\}$, which indicates its position with respect to each of the n hyperplanes of $\mathcal{A}_b$. This gives a description of the face lattice of the discriminantal arrangement.

Theorem 19 *Two points b and b' are in the same (open) face of the discriminantal arrangement $B(\mathcal{A})$ if and only if the two affine arrangements $\mathcal{A}_b$ and $\mathcal{A}'_b$ have the same labeled face poset.*

Note that each face of the discriminantal arrangement corresponds to a face of the fiber zonotope, which in turn represents a regular zonotopal subdivision of the original zonotope Z. There is a natural way of assigning sign vectors to the faces of a regular zonotopal subdivision, and under the duality between the face lattice of the fiber zonotope and that of the discriminantal arrangement, corresponding faces are labeled with the same sign vectors. The reason for this is that both situations are different interpretations of the same oriented matroid. The base zonotope Z and hyperplane arrangement $\mathcal{A}$ represent a single oriented matroid. Regular zonotopal subdivisions of Z come from projections of a higher dimensional zonotope. Affine parallel translations of $\mathcal{A}$ come from sections of a higher dimensional central hyperplane arrangement. The higher dimensional zonotope and central hyperplane arrangement are duals; that is, they are represented by the same oriented matroid, called a *lifting* of the original oriented matroid. (See [11].)

> **Example 1** continued. For the arrangement $\mathcal{A}$ of four lines in $\mathbf{R}^2$ choose normals $a_1 = (1,0)$, $a_2 = (0,1)$, $a_3 = (1,-1)$, and $a_4 = (1,1)$. Let $b = (0,0,2,4)$. In Figure 3 we show $\mathcal{A}_b$ with maximal faces labeled, and the corresponding zonotopal subdivision with vertices labeled.

Manin and Schechtman [27] described the intersection lattice of discriminantal arrangements coming from arrangements of $d+3$ hyperplanes in $\mathbf{R}^d$ in the "most generic" case, that is, for an open Zariski dense subset of all d-arrangements with $d+3$ hyperplanes. Falk [15] showed that not all generic arrangements have an intersection lattice of this type. We conjecture a condition on the a_i guaranteeing "most generic" status, and a description of the intersection lattice $L(B(\mathcal{A}))$ of the discriminantal arrangement for arbitrary n.

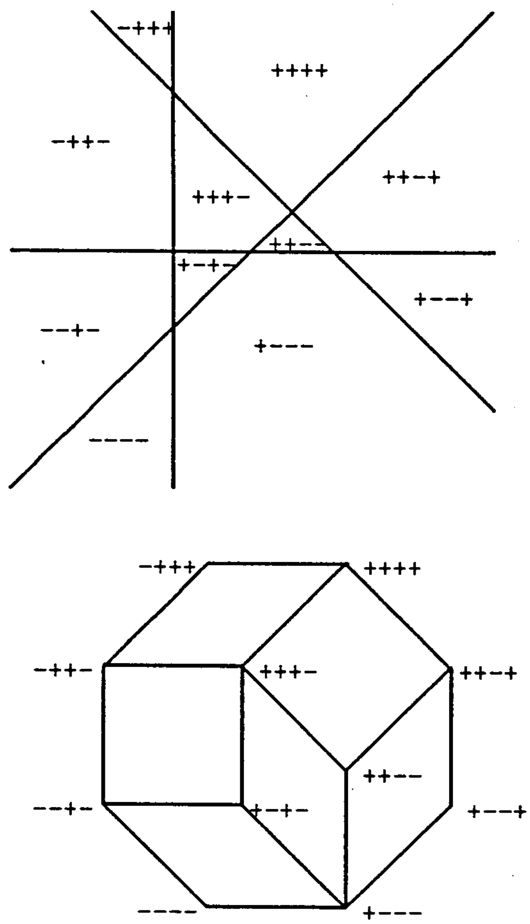

Fig. 3. Labeled faces of an affine arrangement and labeled vertices of the zonotopal subdivision

For $n \geq d+1 \geq 2$ let $P(n,d)$ be the following poset. The elements are sets $\{S_1, S_2, \ldots, S_m\}$ of subsets of $\{1, 2, \ldots, n\}$ satisfying

1. for each i, $|S_i| \geq d+1$
2. for each i, j, $i \neq j$, $|S_i \cap S_j| < d$
3. for each $I \subseteq \{1, 2, \ldots, m\}$, $|\bigcup_{i \in I} S_i| > d + \sum_{i \in I}(|S_i| - d)$.

The ordering is given by $\{S_1, S_2, \ldots, S_m\} \preceq \{T_1, T_2, \ldots, T_p\}$ if and only if for each i there exists j such that $S_i \subseteq T_j$. This is a ranked poset, with $\text{rank}\{S_1, S_2, \ldots, S_m\} = \sum_{i=1}^{m}(|S_i| - d)$.

Conjecture 20 *1. If $\mathcal{A}$ is a generic arrangement of n hyperplanes in $\mathbf{R}^d$ with algebraically independent normal vectors, then $L(B(\mathcal{A}))$ is isomorphic to $P(n, d)$.*

2. Among arrangements $\mathcal{A}$ of n hyperplanes in $\mathbf{R}^d$ the number of rank i elements of $L(B(\mathcal{A}))$ is maximized for all i by the arrangements described in (1).

We mention briefly the issue of freeness of discriminantal arrangements. Terao [41] defined a central arrangement to be free if its module of derivations is free (the definition has its source in singularity theory). There has been much interest in determining which arrangements are free. Manin and Schechtman [27] computed the Möbius function for the intersection lattice of the discriminantal arrangement coming from "most generic" arrangements of $d+3$ hyperplanes in $\mathbf{R}^d$. From this Orlik and Terao [32] observed that these discriminantal arrangements are not free. No other general results on freeness of discriminantal arrangements are known, but we know of no examples that are free. However, known examples have exhibited a high degree of formality [13], which is a necessary condition for freeness.

2.4. Conclusion

This paper has touched on three interesting topics in the combinatorial study of convex polytopes: numbers of faces, subdivision, and relation to hyperplane arrangements. Results range from the very old to the very new, and we hope the reader is left with an interest in the many open questions in the subject.

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