DE01212916X

An Upper Bound Theorem for Rational Polytopes

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The upper bound inequality $h_i(P) - h_{i-1}(P) \leqslant \binom{n-d+i-2}{i}$ $(0 \leqslant i \leqslant d/2)$ is proved for the toric h-vector of a rational convex d-dimensional polytope with n vertices. This gives nonlinear inequalities on flag vectors of rational polytopes. © 1998 Academic Press

A major result in polytope theory is the characterization of face vectors of simplicial polytopes, conjectured by McMullen [10] and proved by Stanley (necessity [11]) and Billera and Lee (sufficiency [3]). The "McMullen conditions" in the theorem are stated most easily in terms of the h-vector of the polytope. For P a simplicial d-polytope, write $f_i(P)$ for the number of i-faces of P, and define the h-vector, $h(P) = (h_0, h_1, ..., h_d)$ by $h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}$. Write $g_i = h_i - h_{i-1}$ for $1 \leqslant i \leqslant d/2$; $g_0 = h_0$. For the definition of the nonlinear operator $\langle i \rangle$, see, for example, [11].

THEOREM (Stanley, Billera, and Lee). *An integer vector $(h_0, h_1, ..., h_d)$ is the h-vector of a simplicial polytope if and only if*

(i) $h_i = h_{d-i}$ *for all i*

(ii) $g_0 = 1$, *and* $g_i \geqslant 0$ *for* $1 \leqslant i \leqslant d/2$

(iii) $g_{i+1} \leqslant (g_i)^{\langle i \rangle}$ *for* $0 \leqslant i \leqslant d/2 - 1$.

We present the Upper Bound Theorem as a corollary. McMullen [9] proved parts (ii) and (iii) below ten years before Stanley proved the necessity of the McMullen conditions and hence part (i) of the corollary.

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COROLLARY (Upper Bound Theorem).   (i)   *For every simplicial d-polytope P with n vertices, and for all i, $1 \leqslant i \leqslant d/2$,*

$$g_i(P) \leqslant \binom{n-d+i-2}{i}.$$

(ii)   *For every simplicial d-polytope P with n vertices, and for all i, $1 \leqslant i \leqslant d$,*

$$h_i(P) \leqslant \binom{n-d+i-1}{i}.$$

(iii)   *For every d-polytope P with n vertices and for all i, $0 \leqslant i \leqslant d-1$, $f_i(P) \leqslant f_i(C(n,d))$. (Here $C(n,d)$ is the cyclic d-polytope with n vertices, and explicit formulas for its f-vector are known.)*

(In (i) and (ii), equality holds for $i = 1$.) Part (ii) of the corollary follows by summing the inequalities of part (i). Part (iii) is, for simplicial polytopes, just a translation of part (ii) into f-vectors. The extension to all polytopes uses the fact that the boundary of a polytope can be triangulated to get a simplicial polytope with at least as many faces of each dimension.

The proof of the McMullen conditions [11], but not the original proof of the Upper Bound Theorem, is based on the interpretation of the h-vector of a simplicial polytope as the sequence of Betti numbers of the associated toric variety.

The toric variety is still defined when the polytope is nonsimplicial, as long as it has rational vertices. The ranks of the homology of the toric variety no longer depend only on the face lattice of the polytope (see [8]). However, the middle perversity intersection homology Betti numbers are invariants of the combinatorial type of the polytope. These Betti numbers form the "toric h-vector", which reduces to the previously defined h-vector when the polytope is simplicial. Following Stanley [12] we define the toric h-vector (and g-vector) of any polytope (or Eulerian poset), first encoding the h-vector and g-vector as polynomials: $h(P, t) = \sum_{i=0}^{d} h_i t^{d-i}$ and $g(P, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i t^i$, with the relations $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \leqslant i \leqslant d/2$. Then the h-vector and g-vector are defined by the recursion

1. $g(\varnothing, t) = h(\varnothing, t) = 1$, and

2. $h(P, t) = \sum_{G \text{ face of } P, G \neq P} g(G, t)(t-1)^{d-1-\dim G}$.

The toric h-vector cannot be determined from the f-vector alone. It is a function of the "flag vector". For P a d-polytope and $S \subseteq \{0, 1, ..., d-1\}$, an S-*flag* of P is a sequence $\varnothing \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$, of distinct faces, with $\{\dim F_1, \dim F_2, ..., \dim F_k\} = S$. Write $f_S(P)$ for the number of S-flags of P; the length 2^d vector $(f_S(P))_{S \subseteq \{0, 1, ..., d-1\}}$ is called the *flag vector* of P.

Nonrecursive formulas for the toric h-vector in terms of the flag vector or equivalent parameters have been given by Fine [6], Brenti [5], Bayer and Ehrenborg [14].

The toric h-vector of a rational polytope satisfies the same linear equations and linear inequalities as in the simplicial case. However, the intersection homology fails to have the ring structure, which is responsible in the simplicial case for the nonlinear inequalities. Stanley [13] used the decomposition theorem for intersection homology to prove that the h-vector of a rational polytope increases (weakly) under subdivision. He concluded that part (ii) of the Upper Bound Theorem holds for arbitrary rational polytopes. Stanley (private communication) points out that the methods of his paper can be used to prove that the g-vector increases under subdivision and hence that part (i) also holds for rational polytopes. Here is a short alternative proof of part (i) of the Upper Bound Theorem for rational polytopes. It depends on work of Braden and MacPherson [4] on relative g-vectors. Part (ii) of the Upper Bound Theorem follows by summing the inequalities of part (i). A stronger version of part (iii) follows.

There is no reason to think that these inequalities fail for irrational polytopes, but the proofs use results proved only for toric varieties of rational polytopes. The theorem below can also be interpreted as an inequality on intersection homology Betti numbers of toric varieties.

THEOREM.   Let P be a rational d-polytope with n vertices. Then for all i, $1 \leqslant i \leqslant d/2$,

$$g_i(P) \leqslant \binom{n-d+i-2}{i}.$$

(Equality holds for $i = 1$.)

Proof.   The polytope P is a quotient of an $(n+d)$-dimensional Lawrence polytope L with $2n$ vertices (see [2]). (L is the polytope with Gale transform equal to the set $A \cup -A$, where A is a Gale transform of P.) Note that $g_1(P) = n - d - 1 = g_1(L)$. Braden and MacPherson [4] proved Kalai's conjecture, $g(P, t) \geqslant g(F, t)g(P/F, t)$ (coefficientwise) for all faces F of a rational polytope P. This implies that the g-polynomial of any quotient of a rational polytope is bounded above by the g-polynomial of the polytope. So, for $0 \leqslant i \leqslant d/2$, $g_i(P) \leqslant g_i(L)$. Triangulate L (with no new vertices) so that the boundary of the triangulated ball is the boundary of a polytope Q, for example, by pulling vertices (see [7]). Since Lawrence polytopes are weakly neighborly (see [1]), $g(L) = g(Q)$. Thus, $g(P) \leqslant g(Q)$ and $g_1(P) = g_1(Q) = n - d - 1$. The polytope Q is simplicial, so $g(Q)$ satisfies the McMullen conditions. In particular, for all i, $1 \leqslant i \leqslant d/2$, $g_i(P) \leqslant g_i(Q) \leqslant \binom{n-d+i-2}{i}$.   ∎

For $i = 2$ the inequality of the theorem says that $f_{02} - 3f_2 + f_1 \leqslant \binom{f_0}{2}$, which is clearly true for all polytopes, since the left-hand side counts some distinct pairs of vertices. The $i = 3$ inequality is:

$$(f_{024} - 3f_{24} + f_{14} - 4f_{04} + 10f_4) + (f_{03} - 4f_3) - (d - 2)(f_{02} - 3f_2)$$

$$+ f_2 - (d - 1)f_1 + \binom{d}{2}f_0 - \binom{d+1}{3} \leqslant \binom{f_0 - d + 1}{3}.$$

Part (iii) of the Upper Bound Theorem can be strengthened. For simplicial polytopes the definition of the h-vector is inverted to give $f_i = \sum_{j=0}^{i+1} \binom{d-j}{d-i-1} h_j$. This is not valid for the toric h-vector of nonsimplicial polytopes, but the analogue is still nice. For P a d-polytope,

$$\sum_{j=0}^{i+1} \binom{d-j}{d-i-1} h_j(P) = f_i + \sum_{\ell=1}^{i} \sum_{m=2\ell}^{\ell+i} \binom{\ell}{m-i} \sum_{\substack{F \text{ face of } P \\ \dim F = m}} g_\ell(F).$$

Since for a rational polytope P with n vertices, $h_j(P) \leqslant \binom{n-d+j-1}{j} = h_j(C(n, d))$ ($C(n, d)$ being the cyclic polytope), we get

COROLLARY. *Let P be a rational d-polytope with n vertices. Then for all* i. $1 \leqslant i \leqslant d - 1$,

$$f_i(P) + \sum_{\ell=1}^{i} \sum_{m=2\ell}^{\ell+i} \binom{\ell}{m-i} \sum_{\substack{F \text{ face of } P \\ \dim F = m}} g_\ell(F) \leqslant f_i(C(n, d)).$$

Thus the nonnegative terms, $\binom{\ell}{m-i} \sum_{\dim F = m} g_\ell(F)$, measure the unused "space" for i-faces in the polytope.

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