Proc. Natl. Acad. Sci. USA
Vol. 95, pp. 9093–9098, August 1998
Mathematics

Weight functions, double reciprocity laws, and volume formulas for lattice polyhedra

(weights / Euler integration / Dehn–Sommerville / reciprocity law / weight homology)

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Edited by Gian-Carlo Rota, Massachusetts Institute of Technology, Cambridge, MA, and approved May 20, 1998 (received for review March 10, 1998)

ABSTRACT We extend the concept of manifold with boundary to weight and boundary weight functions. With the new concept, we obtained the double reciprocity laws for simplicial complexes, cubical complexes, and lattice polyhedra with weight functions. For a polyhedral manifold with boundary, if the weight function has the constant value 1, then the boundary weight function has the constant value 1 on the boundary and 0 elsewhere. In particular, for a lattice polyhedral manifold with boundary, our double reciprocity law with a special parameter reduces to the functional equation of Macdonald; for a lattice polytope especially, the double reciprocity law with a special parameter reduces to the reciprocity law of Ehrhart. Several volume formulas for lattice polyhedra are obtained from the properties of the double reciprocity law. Moreover, the idea of weight and boundary weight leads to a new homology that is not homotopy invariant, but only homeomorphic invariant.

1. Introduction

Let $\bar{P}$ be a d-dimensional lattice polytope of $\mathbf{R}^m$, i.e., the vertices of $\bar{P}$ are all lattice points. The interior P of $\bar{P}$ in its affine span is called the *relative interior* of $\bar{P}$. For positive integers n, let $i(P; n)$ denote the number of lattice points of $nP = \{nx : x \in P\}$ and $i(\bar{P}; n)$ the number of lattice points of $n\bar{P} = \{nx : x \in \bar{P}\}$. Ehrhart (1, 2) proved that $i(P; n)$ and $i(\bar{P}; n)$ are polynomial functions of n and satisfy the *reciprocity law*

$$i(\bar{P}; n) = (-1)^d i(P; -n). \qquad \textbf{[1.1]}$$

Let M be a lattice polyhedral manifold with boundary ∂M. For positive integers n, the number of lattice points of $nM = \{nx : x \in M\}$, denoted by $L(M; n)$, is a polynomial function of n. Macdonald (3) extended **1.1** to M by showing that the polynomial function $f(z) = L(M; z) - \frac{1}{2}L(\partial M; z)$ satisfies the functional equation

$$f(z) = (-1)^d f(-z). \qquad \textbf{[1.2]}$$

If M is a lattice polytope P, then **1.2** reduces to **1.1**.

In this paper, we present an idea to extend the concept of manifold with boundary to weight and boundary weight functions. With the new concept, we obtain the double reciprocity laws for simplicial complexes, cubical complexes, and lattice polyhedra with respect to weights. Several volume formulas for lattice polyhedra are obtained from the double reciprocity law in the most general context. Moreover, the idea of weight and boundary weight leads to a new homology that is not homotopy invariant but only homeomorphic invariant.

Let X be a compact d-dimensional lattice polyhedron. A *lattice triangulation* of X is a triangulation whose vertices have integral coordinates. We think of every triangulation as a collection of disjoint relatively open simplices. A function ω on X is called a *weight function* if there exists a lattice triangulation Δ of X such that ω is constant on each relatively open simplex of Δ. The weight ω can be viewed as a function on Δ, defined by $\omega(\sigma) = \omega(x)$, $x \in \sigma$. For each integer n, the *nth boundary weight function* $\partial_n\omega$ of ω is defined by

$$\partial_n\omega(\sigma) = \omega(\sigma) - (-1)^n \sum_{\sigma \le \tau \in \Delta} (-1)^{\tau}\omega(\tau).$$

We define for ω the polynomial of two variables t and z,

$$L(X, \omega; t, z) = \sum_{\sigma \in \Delta} (\omega - t\partial_d\omega)(\sigma) i(\sigma, z).$$

For positive integers n, $L(X, \omega; t, n)$ counts the number of lattice points of nX with weight $\omega - t\partial\omega$. Our double reciprocity law is the functional equation

$$L(X, \omega; t, z) = (-1)^d L(X, \omega; 1 - t, -z). \qquad \textbf{[1.3]}$$

If $t = 1/2$, $\omega = 1$ and assuming that X is a manifold with boundary ∂X, then $\partial_d\omega = 1$ on ∂X and $\partial_d\omega = 0$ elsewhere, and **1.3** reduces to **1.2**. By specifying some integral values of z in $L(X, \omega; 1/2, z)$, we found relations between the weighted relative volume of X and the weighted number of lattice points.

2. Euler Integration over Cell Complexes

Let K be a d-dimensional cell complex. We think of K as a collection of (relatively) open cells, and use $\hat{K}$ to denote the collection with the empty cell $\emptyset$, written $\hat{0}$, jointed to K. If σ and τ are open cells of K, the notation $\sigma \le \tau$, or $\tau \ge \sigma$, means that σ is a face of τ. The topological space underlying K is the geometric realization $|K| = \cup_{\sigma \in K}\sigma$.

Let R be a commutative ring with unity $1 \ne 0$ and let ϕ be a function from $\hat{K}$ to R. For a subset L of $\hat{K}$, define the *Euler integral* as

$$S(L, \phi) = \sum_{\sigma \in L} (-1)^{1+\dim\sigma}\phi(\sigma) = -\int_L \phi(\sigma) \mathrm{d}\chi(\sigma),$$

where the dimension of the empty cell $\hat{0}$ is assumed to be -1. The Euler characteristic χ is a finitely additive measure on cell complexes (4, 5). The idea of integration with respect to the Euler characteristic goes back to Hadwiger (6) and was further developed by Groemer (7), Rota (8), Schanuel (9), and Viro (10).

Let σ be an open cell of $\hat{K}$. Define the *lower closure* $\hat{\sigma} = \{\tau \in \hat{K} : \tau \le \sigma\}$ and the *upper closure* $\check{\sigma} = \{\tau \in \hat{K} : \tau \ge \sigma\}$. We associate with ϕ the R-valued functions $\hat{\phi}$ and $\check{\phi}$ on $\hat{K}$, defined by $\hat{\phi}(\sigma) = S(\hat{\sigma}, \phi)$ and $\check{\phi}(\sigma) = S(\check{\sigma}, \phi)$, respectively. Notice that $\sum_{\sigma \le \tau \le \rho}(-1)^{\dim\tau}$ is equal to $(-1)^{\dim\sigma}$ for $\rho = \sigma$ and equal to 0 for $\rho > \sigma$. We have $\hat{\hat{\phi}} = \phi$ and $\check{\check{\phi}} = \phi$.

This paper was submitted directly (Track II) to the *Proceedings* office.

Let ψ be another R-valued function on $\hat{K}$. The pointwise multiplications $\psi\hat{\phi}$ and $\check{\phi}\psi$ are R-valued functions, and

$$S(\hat{K}, \phi\hat{\psi}) = S(\hat{K}, \check{\phi}\psi). \quad \textbf{[2.1]}$$

This property is essential to obtain the reciprocity law for cell complexes.

For $j = -1, 0, 1, \cdots, d = \dim K$, define $\phi_j(K) = \sum_{\sigma\in\hat{K}, \dim\sigma=j} \phi(\sigma)$. The $(d+2)$-tuple $(\phi_{-1}, \phi_0, \phi_1, \cdots, \phi_d)$ is called the *f-vector of $\hat{K}$ with weight ϕ*. The following theorem is an analog of the *Dehn–Sommerville equations* about f-vectors of $\hat{K}$ with weight ϕ.

THEOREM 2.1. *Let K be a d-dimensional finite simplicial complex. Then for each $i = -1, 0, 1, \cdots, d$,*

$$\check{\phi}_i(\hat{K}) + \sum_{j=i}^{d}(-1)^j \binom{j+1}{i+1} \phi_j(\hat{K}) = 0, \quad \textbf{[2.2]}$$

$$\phi_i(\hat{K}) + \sum_{j=i}^{d}(-1)^j \binom{j+1}{i+1} \check{\phi}_j(\hat{K}) = 0. \quad \textbf{[2.3]}$$

For each open d-simplex σ of $\hat{K}$, $\check{\phi}(\sigma) = (-1)^{d+1}\phi(\sigma)$. Let K_{d-1} denote the subcomplex of K consisting of the open simplices of dimensions less than or equal to $d-1$. Then

$$\begin{aligned}\check{\phi}_i(\hat{K}) &= \check{\phi}_i(\hat{K} - \hat{K}_{d-1}) + \check{\phi}_i(\hat{K}_{d-1}) \\ &= (-1)^{d+1}\phi_i(\hat{K} - \hat{K}_{d-1}) + \check{\phi}_i(\hat{K}_{d-1}) \\ &= -(-1)^d\phi_i(\hat{K}) + [\check{\phi} + (-1)^d\phi]_i(\hat{K}_{d-1}).\end{aligned}$$

Plugging this into **2.2** and rearranging the terms properly, then

$$\begin{aligned}&[\phi + (-1)^d\check{\phi}]_i(\hat{K}_{d-1}) \\ &\quad= [1 - (-1)^{d-i}]\phi_i(\hat{K}) \\ &\qquad + \sum_{j>i}^{d}(-1)^{d-j-1}\binom{j+1}{i+1}\phi_j(\hat{K}).\end{aligned} \quad \textbf{[2.4]}$$

Let us introduce the $(d+2)\times(d+2)$ matrix $\hat{D}(d)$, called the *extended Dehn–Sommerville matrix*, whose (i, j)-entry is

$$\hat{D}(d)_{i,j} = \begin{cases} 1 - (-1)^{d-i} & \text{for } i = j \\ (-1)^{d-j-1}\binom{j+1}{i+1} & \text{for } i < j \\ 0 & \text{otherwise,}\end{cases}$$

$-1 \le i, j \le d$. The difference between $\hat{D}(d)$ and the $d \times (d+1)$ matrix $D(d)$ in ref. 11 is that the Euler equation is included in $\hat{D}(d)$. More precisely,

$$\hat{D}(d) = \begin{bmatrix} 1 + (-1)^d & * \\ 0 & D(d)\end{bmatrix},$$

where $* = (-1)^{d-1}(1, -1, \cdots, (-1)^d)$ is a $(d+1)$-dimensional vector. Thus **2.4** can be written as

$$\hat{D}(d)\phi(\hat{K}) = [\phi + (-1)^d\check{\phi}](\hat{K}_{d-1}), \quad \textbf{[2.5]}$$

where $\phi(\hat{K}) = (\phi_{-1}, \phi_0, \phi_1, \cdots, \phi_d)^T$ is a column vector. Because $\phi + (-1)^d\check{\phi} = 0$ on $\hat{K} - \hat{K}_{d-1}$, $\hat{K}_{d-1}$ can be replaced by $\hat{K}$ on the right side of **2.5**. The extended Dehn–Sommerville matrix $\hat{D}(d)$ then transforms an f-vector with weight ϕ to an f-vector with weight $\phi + (-1)^d\check{\phi}$, which vanishes on top-dimensional open simplices of $\hat{K}$.

Definition 2.2: For each integer n, the nth boundary $\partial_n\phi$ of ϕ is the R-valued function

$$\partial_n\phi = \phi + (-1)^n\check{\phi}. \quad \textbf{[2.6]}$$

The collection $\partial_n(K, \phi)$ of cells, on which $\partial_n\phi$ is nonzero, is called the nth boundary of K with respect to the weight ϕ.

Because $\check{\check{\phi}} = \phi$, we have $\partial_{n-1}\partial_n\phi = 0$. One can define the *$n$th coboundary operator* δ_n by $\delta_n\phi = \phi + (-1)^n\hat{\phi}$ and have the same property. Thus

$$\partial_{n-1}\partial_n = 0, \qquad \delta_{n+1}\delta_n = 0. \quad \textbf{[2.7]}$$

With the operators ∂ and δ, Eq. **2.1** is equivalent to the following theorem.

THEOREM 2.3. (*Discrete Analog of Stokes Theorem*)

$$\int_{\hat{K}}(\partial_n\phi)\psi = \int_{\hat{K}}\phi(\delta_n\psi). \quad \textbf{[2.8]}$$

Eq. **2.1** also implies the following prototype of the double reciprocity law for finite cell complexes.

THEOREM 2.4. *Let ϕ and ω be R-valued functions on a d-dimensional finite cell complex K. Then for any elements a and b of R such that $a + b = 1$,*

$$S(\hat{K}, \hat{\phi}(\omega - a\partial_d\omega)) + (-1)^d S(\hat{K}, \phi(\omega - b\partial_d\omega)) = 0. \quad \textbf{[2.9]}$$

If $|K|$ is a d-manifold with boundary and $\omega = 1$, $a = b = 1/2$, then $\partial_d\omega = 1$ on the boundary $\partial|K|$ and $\partial_d\omega = 0$ on $|K| - \partial|K|$, and **2.9** reduces to an equivalent form of **2.1** in ref. 3. All results of this section can be extended to semi-Eulerian posets (12, 13).

3. Weight Functions

Let X be a compact subset of a manifold M without boundary. We always assume that X is *finitely stratified*, i.e., X can be decomposed into a disjoint union of finitely many connected submanifolds X_i without boundary, such that, whenever $X_i \cap \bar{X}_j \neq \emptyset$, then $X_i \subset \bar{X}_j$, and in this case we say that X_i is a *face* of X_j, written $X_i \le X_j$. The collection $\{X_i\}$ is called a *stratification* of X and each X_i is called a *stratum*. The *dimension* of X is the highest dimension of its strata. The set of all strata forms a poset by the face ordering. There is an *intrinsic stratification* for X that can be constructed as follows: first, set $X_d = X$, then remove the set of d-dimensional points (the points that have an open neighborhood homeomorphic to a d-dimensional open ball) from X_d, and denote the leftover by X_{d-1}. Second, remove the set of $(d-1)$-dimensional points from X_{d-1} and denote the leftover by X_{d-2}. Repeat this process until all the points of X have been removed. We then have the *intrinsic filtration* (see refs. 14 and 15)

$$X = X_d \supset X_{d-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset. \quad \textbf{[3.1]}$$

Strictly speaking, the stratification we defined above may be called a topological stratification. One can define smooth, analytic, subanalytic, semialgebraic, and piecewise linear stratifications. A more general treatment of Eulerian stratification is studied in ref. 16.

Definition 3.1: Let X be a stratified space with strata X_i. A real valued function ω on X is called a weight function if ω is constant on each stratum of X.

Example 3.2: (*i*) Any constant function on a stratified space X is a weight function.

(*ii*) The *local Euler characteristic* $\chi(x) = \sum_{\sigma\le\tau\in\Delta}(-1)^{\dim\tau}$, where Δ is a triangulation of X and σ is the unique open simplex containing x, defines a weight function on X with respect to the intrinsic stratification.

(*iii*) The *local link number* $\ell(x) = \chi(\mathrm{lk}(x, X))$ also defines a weight function on X with respect to the intrinsic stratification.

Let Δ be a *stratified triangulation* of X, i.e., each stratum of X is a disjoint union of open simplices of Δ. For each open simplex $\sigma \in \Delta$ and $x \in \sigma$, the local Euler characteristic and the local link number of x are related by

$$\chi(x) = (-1)^{\dim\sigma}[1 - \chi(\mathrm{lk}(\sigma, \Delta)],$$
$$\chi(\mathrm{lk}(x, X)) = 1 - (-1)^{\dim\sigma} + (-1)^{\dim\sigma}\chi(\mathrm{lk}(\sigma, \Delta)).$$

For each weight function ω on X, we define a function $\check{\omega}$ on X by

$$\check{\omega}(x) = \check{\omega}(\sigma) = \sum_{\sigma \le \tau \in \Delta} (-1)^{1+\dim \tau} \omega(\tau), \qquad \textbf{[3.2]}$$

where σ is the unique open simplex such that $x \in \sigma$. For a fixed x, let Δ' denote the triangulation refined from Δ by adding x as a vertex. Then

$$\check{\omega}(x) = \sum_{\sigma \le \tau \in \Delta} \omega(\tau) \sum_{\substack{\rho \subset \tau \\ x \le \rho \in \Delta'}} (-1)^{1+\dim \rho}$$
$$= \sum_{x \le \rho \in \Delta'} (-1)^{1+\dim \rho} \omega(\rho).$$

Notice that $\{\rho : x \le \rho \in \Delta'\}$ is a cone decomposition of the star open neighborhood $\mathrm{st}(x)$ of x. We see that $\check{\omega}$ is independent of cone decompositions. In fact, for two cone decompositions Δ_1 and Δ_2 of $\mathrm{st}(x)$, there is a common refined cone decomposition Δ_3. Then

$$\sum_{x \le \tau \in \Delta_3} (-1)^{1+\dim \tau} \omega(\tau) = \sum_{x \le \sigma \in \Delta_1} \sum_{\tau \subset \sigma} (-1)^{1+\dim \tau} \omega(\tau)$$
$$= \sum_{x \le \sigma \in \Delta_1} \sum_{\tau \subset \sigma} (-1)^{1+\dim \tau} \omega(\sigma)$$
$$= \sum_{x \le \sigma \in \Delta_1} (-1)^{1+\dim \sigma} \omega(\sigma).$$

Because $\check{\omega}(x)$ is defined via arbitrary stratified triangulations, $\check{\omega}$ is constant on each stratum.

PROPOSITION 3.3. *For any weight function ω on a d-dimensional stratified space X, $\check{\omega}$ is a weight function. Thus the nth boundary function $\partial_n \omega = \omega + (-1)^n \check{\omega}$ is a weight function. So is $\partial \omega = \partial_d \omega$.*

Stratified spaces with weight functions share some properties of manifolds with boundary. For instance, the boundary of a weighted space is a weighted manifold with the boundary weight. For a nonzero weight function ω on X, denote $d(\omega) = \dim\{x \in X : \omega(x) \neq 0\}$. Write $m = d(\omega)$; if $\partial_m \omega = 0$, we call (X, ω) an *m-dimensional weighted manifold with weight* ω. The closure of $\{x \in X : \partial\omega(x) \neq 0\}$ is called the *boundary* of (X, ω) and is denoted by $\partial(X, \omega)$. Now if $\partial_n \omega = 0$, then for a $d(\omega)$-dimensional open cell σ such that $\omega(\sigma) \neq 0$, we have $\partial_n \omega(\sigma) = \omega(\sigma) + (-1)^n (-1)^{d(\omega)+1} \omega(\sigma) = [1 - (-1)^{n+d(\omega)}]\omega(\sigma) = 0$. Thus n and $d(\omega)$ must have the same parity. If, on the other hand, $\partial_n \omega \neq 0$, then for a $d(\partial_n \omega)$-dimensional open cell τ such that $\partial_n \omega(\tau) \neq 0$, we have $\partial_{n-1}\partial_n \omega(\tau) = \partial_n \omega(\tau) + (-1)^{n-1}(-1)^{d(\partial_n \omega)+1} \partial_n \omega(\tau) = [1 + (-1)^{n+d(\partial_n \omega)} \partial_n \omega(\tau) = 0$. This means that n and $d(\partial_n \omega)$ must have different parity.

THEOREM 3.4. *Let ω be a nonzero weight function. If $\partial_n \omega = 0$, then n and $d(\omega)$ have the same parity. If $\partial_n \omega \neq 0$, then n and $d(\partial_n \omega)$ have different parity.*

Let ϕ be a real-valued function on Δ. Define $\phi(\Delta) = (\phi_0, \phi_1, \cdots, \phi_d)$, where ϕ_i is the number of i-dimensional open simplices. Then $D(d)\phi(\Delta) = \partial_d \phi(\Delta)$ by **2.5**. Define the linear function χ on $\mathbf{R}^{d+1}$ by $\chi(v) = \sum_{i=0}^{d} (-1)^i v_i$, $v = (v_0, v_1, \cdots, v_d)$. Given a weight function ω on X, $\chi(X, \omega) = \chi(\omega(\Delta))$ is called the *ω-weighted Euler characteristic* of X. It has been shown in ref. 11 that $\chi(D(d)v) = [1 - (-1)^d]\chi(v)$. This gives Theorem 3.5.

THEOREM 3.5. *For any d-dimensional space X with a weight function ω,*

$$\chi(X, \partial_d \omega) = [1 - (-1)^d]\chi(X, \omega).$$

If $\partial_d \omega = 0$ and d is odd, then $\chi(X, \omega) = 0$. We thus have Corollary 3.6.

COROLLARY 3.6. *Odd-dimensional weighted manifolds have the weighted Euler characteristic zero.*

If X has only odd-dimensional strata and $\omega = 1$, then by Theorem 3.4, $\partial\omega = 0$, i.e., X is an odd-dimensional weighted manifold with weight 1. By Corollary 3.6, $\chi(X) = \chi(X, 1) = 0$.

COROLLARY 3.7. (*Ref. 17*) *Stratified spaces with odd-dimensional strata have vanishing Euler characteristic.*

Eulerian manifolds are defined as compact spaces whose local links have constant Euler characteristic (18, 19). This implies the following corollary.

COROLLARY 3.8. *A stratified space X is a weighted manifold with the constant weight function $\omega = 1$ if and only if X is an Eulerian manifold.*

THEOREM 3.9. *Let X_1 and X_2 be stratified spaces with weight functions ω_1 and ω_2, respectively. Then $X_1 \times X_2$ is a stratified space with weight function $\omega_1 \times \omega_2$, defined by $\omega_1 \times \omega_2(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$, $(x_1, x_2) \in X_1 \times X_2$, and*

$$\partial(\omega_1 \times \omega_2) = (\partial\omega_1) \times \omega_2 + \omega_1 \times (\partial\omega_2) - \partial\omega_1 \times \partial\omega_2.$$
$$\chi(X_1 \times X_2, \omega_1 \times \omega_2) = \chi(X_1, \omega_1)\chi(X_2, \omega_2).$$

Moreover, $X_1 \times X_2$ is a weighted manifold with weight $\omega_1 \times \omega_2$ if and only if one of the three conditions is satisfied: (i) $\omega_1 = 0$ or $\omega_2 = 0$, (ii) $\partial\omega_1 = 0$ and $\partial\omega_2 = 0$, or (iii) $\partial\omega_1 = 2\omega_1$ and $\partial\omega_2 = 2\omega_2$.

THEOREM 3.10. *Let X_1 and X_2 be stratified spaces with weight functions ω_1 and ω_2, respectively. If $\partial(X_1, \omega_1) = X_0 = \partial(X_2, \omega_2)$ and ω_0 is a weight function on X_0, then $X_1 \cup_{X_0} X_2$ is a stratified space with weight*

$$\omega(x) = \begin{cases} \omega_1(x) & \text{for } x \in X_1 - X_0 \\ \omega_2(x) & \text{for } x \in X_2 - X_0 \\ \omega_0(x) & \text{for } x \in X_0, \end{cases}$$

and on $(X_1 - X_0) \cup (X_2 - X_0)$, $\partial\omega = 0$, on X_0,

$$\partial\omega = (\partial\omega_1 + \partial\omega_2) + (\partial_{X_0}\omega_1 + \partial_{X_0}\omega_2 - \partial_{X_0}\omega_0)$$
$$- 2(\omega_1 + \omega_2 - \omega_0).$$

Moreover, if $\omega_1 = \omega_2 = \omega_0$ on X_0 and $\partial\omega_1 = \partial\omega_2 = \omega_0$, then $X_1 \cup_{X_0} X_2$ is a weighted manifold with weight ω.

Under the conditions of Theorem 3.10, one has $\chi(X_1 \cup_{X_0} X_2, \omega) = \chi(X_1, \omega_1) + \chi(X_2, \omega_2) - \chi(X_0, \omega_0)$. If X_1 and X_2 are both even-dimensional, then $\chi(X_1 \cup_{X_0} X_2, \omega) = \chi(X_1, \omega_1) + \chi(X_2, \omega_2)$. If X_1 and X_2 are both odd-dimensional, then $\chi(X_0, \omega_0) = \chi(X_1, \omega_1) + \chi(X_2, \omega_2)$.

Now let X be given the intrinsic stratification filtrated as **3.1** and let $W_n(X)$ denote the R-module of weight functions on X_n. If X_n is empty, we assume that $W_n(X) = \{0\}$. Because $\partial_n \partial_{n+1} = 0$, we have the chain complex

$$\cdots \longrightarrow W_{n+1}(X) \xrightarrow{\partial_{n+1}} W_n(X) \xrightarrow{\partial_n} W_{n-1}(X) \longrightarrow \cdots$$

Define the *nth weight homology* as

$$WH_n(X) = \ker \partial_n / \mathrm{im}\partial_{n+1}.$$

The operator $\check{}$ is a chain automorphism on the chain complex $\{W_n(X) : n \ge 0\}$ and induces an involutive automorphism on $WH_n(X)$.

Roughly speaking, the weight homology is a measurement about the singularity of a space. For instance, if X is an n-manifold without boundary and $R = \mathbf{Z}$, then $WH_n(X) \simeq \mathbf{Z}_2$ and $WH_i(X) \simeq 0$ for $i \le n - 1$. If X is an n-manifold with boundary, then $WH_n(X) \simeq \mathbf{Z}$ and $WH_i(X) \simeq 0$ for $i \le n - 1$. Unlike the simplicial homology, the weight homology of a contractible space may not be trivial.

Example 3.11: Let $X \subset \mathbf{R}^d$ be a compact d-polyhedron and let $B^d(x, r)$ denote the ball of radius r centered at $x \in X$. Define the angle $\omega(x)$ by

$$\omega(x) = \lim_{r \to 0} \frac{V_d(X \cap B^d(x, r))}{V_d(B^d(x, r))}, \qquad \textbf{[3.3]}$$

where V_d is the volume, i.e., the Lebesgue measure on $\mathbf{R}^d$. If K is a regular cell decomposition of X (see ref. 20), then ω is constant on each open cell of K. So ω is a weight function on X with respect to the stratification K. Let us recall the geometric cone relation

$$\sum_{\sigma\leq\tau\leq\in K}(-1)^{\dim\tau}T(\tau,X)=\bar{T}(-\sigma,-X)=\bar{T}^{-}(\sigma,X) \quad \mathbf{[3.4]}$$

for $\sigma\in\hat{K}$ (theorem 1.1 in ref. 20). By integrating both sides of **3.4**, we obtain the angle-sum relation

$$\check{\omega}(\sigma)=\sum_{\sigma\leq\tau\in\hat{K}}(-1)^{\dim\tau+1}\omega(\tau)=(-1)^{d+1}\omega(\sigma). \quad \mathbf{[3.5]}$$

This means that $\partial\omega=\omega+(-1)^d\check{\omega}=0$. Thus X is a weighted manifold with the angle weight function ω.

4. Double Reciprocity Laws

In this section, we make use of **2.9** to generalize the reciprocity law of ref. 3. Let R be the polynomial ring $\mathbf{Z}[t,z]$ and let X be a d-dimensional compact topological space with a fixed stratification.

Let Δ be a stratified triangulation of X. We view Δ as a simplicial complex consisting of open simplices. For each open simplex $\sigma\in\hat{\Delta}$, define $\phi(\sigma)=z^{1+\dim\sigma}$. Then $\hat{\phi}(\sigma)=(1-z)^{1+\dim\sigma}$. Let ω be a weight function on X. Define the polynomial

$$P(X,\omega;t,z)=S(\hat{\Delta},\phi(\omega-t\partial\omega)). \quad \mathbf{[4.1]}$$

Then $S(\hat{\Delta},\hat{\phi}[\omega-(1-t)\partial\omega])=P(X,\omega;1-t,1-z)$.

THEOREM 4.1. *For any stratified d-dimensional space X with a weight function ω and a stratified triangulation Δ,*

$$P(X,\omega;t,z)+(-1)^dP(X,\omega;1-t,1-z)=0. \quad \mathbf{[4.2]}$$

If X is a manifold with boundary and $t=1/2$, then **4.2** is obtained in ref. 3 and is equivalent to corollary 7.2 in ref. 21. Actually, **4.2** is equivalent to the Dehn–Sommerville equation **2.5**. Whenever X is a Euclidean d-polyhedron in $\mathbf{R}^d$ and ω is the angle weight function given by **3.3**, we denote $S(\hat{\Delta},\phi\omega)$ by $\Omega(X;z)$ because $\partial\omega=0$. Then $S(\hat{\Delta},\hat{\phi}\omega)=\Omega(X;1-z)$. Thus $\Omega(X;z)=(-1)^{d+1}\Omega(X;1-z)$, which is equivalent to the angle-sum relation of ref. 20. In particular, if X is a d-polytope, the reciprocity law with the angle weight is equivalent to the angle-sum relations of (22, 23).

Let X be a d-polyhedron with a cubical decomposition $\mathcal{C}$, i.e., every open cell σ of $\mathcal{C}$ is isomorphic to a relatively open cube. For each $\sigma\in\hat{\mathcal{C}}$, let $\phi(\sigma)=x^{\dim\sigma}$ for $\sigma>\hat{0}$ and let $\phi(\hat{0})=0$. Each $\hat{\sigma}$ contains exact $\binom{\dim\sigma}{k}2^{\dim\sigma-k}$ open k-faces if $\sigma>\hat{0}$. Then $\hat{\phi}(\hat{0})=0$ and $\hat{\phi}(\sigma)=-(2-z)^{\dim\sigma}$. For a weight function ω on X, define the polynomial

$$Q(X,\omega;t,z)=S(\hat{\mathcal{C}},\phi(\omega-t\partial\omega)). \quad \mathbf{[4.3]}$$

Then $S(\hat{\mathcal{C}},\hat{\phi}[\omega-(1-t)\partial\omega])=-Q(X,\omega;1-t,2-z)$.

THEOREM 4.2. *For any d-dimensional stratified space X with a weight function ω and a cubical decomposition $\mathcal{C}$,*

$$Q(X,\omega;t,z)=(-1)^dQ(X,\omega;1-t,2-z). \quad \mathbf{[4.4]}$$

The reciprocity law **4.4** is equivalent to the cubical form of the Dehn–Sommerville equation. Whenever X is a Euclidean d-polyhedron and ω is the angle weight function, we denote $S(\hat{\mathcal{C}},\phi\omega)$ by $\Xi(X;z)$ because $\partial\omega=0$, and then $\Xi(X;z)=(-1)^d\Xi(X;2-z)$, which is equivalent to the cubical form of angle-sum relations. In particular, for a polytope, it is equivalent to the cubical form of the angle-sum relations of ref. 22.

Now let X be a polyhedral complex of $\mathbf{R}^m$ with lattice vertices, i.e., all 0-dimensional simplices have integral coordinates. Let X be decomposed into a disjoint union of relatively open lattice simplices. For each relatively open cell σ of X, denote by $\bar{\sigma}$ the closure of σ. Let $h:\mathbf{R}^m\longrightarrow\mathbf{R}$ be a homogeneous polynomial function of degree k. For a positive integer n, define $i(\sigma,h;n)=\sum_{x\in n\sigma\cap\mathbf{Z}^m}h(x)$ and $i(\bar{\sigma},h;n)=\sum_{x\in n\bar{\sigma}\cap\mathbf{Z}^m}h(x)$. Pukhlikov and Khovanskii (24) showed that $i(\sigma,h;n)$ and $i(\bar{\sigma},h;n)$ are polynomial functions of n of degree $k+\dim\sigma$; Brion and Vergne (25) showed that $i(\sigma,h;n)$ and $i(\bar{\sigma},h;n)$ satisfy the reciprocity law $i(\bar{\sigma},h;n)=(-1)^{k+\dim\sigma}i(\sigma,h;-n)$. If $h=1$, then $k=0$ and it reduces to the reciprocity law of Ehrhart. With each relatively open cell σ of X, we associate a polynomial $\phi(\sigma,h)=(-1)^{1+\dim\sigma}i(\sigma,h;z)$. Then $\hat{\phi}(\sigma,h)=\sum_{\tau\leq\sigma}(-1)^{1+\dim\tau}\phi(\tau,h)=i(\bar{\sigma},h;z)=(-1)^{k+\dim\sigma}i(\sigma;-z)$. Let ω be a weight function on X. Define the polynomial

$$L(X,\omega,h;t,z)=S(\hat{X},\phi(\omega-t\partial\omega)). \quad \mathbf{[4.5]}$$

Then $S(\hat{X},\hat{\phi}[\omega-(1-t)\partial\omega])=(-1)^{k+1}L(X,\omega,h;1-t,-z)$. The polynomial $L(X,\omega,h;t,z)$ is independent of lattice cell decompositions of X and $L(X,\omega,h;t,n)=\sum_{\sigma\in X}(\omega-t\partial\omega)(\sigma)i(\sigma,h;n)$ counts the number of lattice points of nX with weight $(\omega-t\partial\omega)h$. If we write $\partial\omega$ explicitly, then

$$L(X,\omega,h;t,n)=\sum_{\sigma\in\Delta}\omega(\sigma)[(1-t)i(\sigma,h;n)+t(-1)^di(\bar{\sigma},h;n)].$$

THEOREM 4.3. *Let X be a bounded lattice d-polyhedron with a weight function ω. If h is a homogeneous polynomial function of degree k, then*

$$L(X,\omega,h;t,z)=(-1)^{k+d}L(X,\omega,h;1-t,-z). \quad \mathbf{[4.6]}$$

Eq. **4.6** generalizes the reciprocity law of ref. 25. For $h=1$, it generalizes Ehrhart's reciprocity law for lattice polytopes as well as Macdonald's extension to lattice manifolds with boundary, and in this case we write $L(X,\omega;t,z)=L(X,\omega,1;t,z)$. For $t=1/2$, we write $L(X,\omega,h;z)=L(X,\omega,h;1/2,z)$. For $\omega=1$, we write $L(X,h;z)=L(X,\omega,h;z)$. If ω is the angle weight function, we define $A(X,h;z)=S(X,\phi\omega)$ because $\partial\omega=0$.

COROLLARY 4.4.

$$A(X,h;z)=(-1)^{k+d}A(X,h;-z). \quad \mathbf{[4.7]}$$

If X is a rational polyhedron, the reciprocity laws **4.6** and **4.7** are still valid if z is replaced by an integer. However, $L(X,\omega,h;t,n)$ is no longer a polynomial of the positive integer variable n but a quasi-polynomial ref. 13.

5. Volume Formulas

Let us recall some well-known facts about the polynomial $i(\sigma;z)$ for an open d-simplex σ of $\mathbf{R}^m$: the constant term of $i(\sigma;z)$ is $(-1)^d$ (the Euler characteristic of σ) and the leading coefficient of $i(\sigma;z)$ is the d-dimensional *relative volume* $v(\sigma)$ of σ. The lattice points of the affine span $\langle\sigma\rangle$ of σ form an abelian group of rank d, i.e., $\langle\sigma\rangle\cap\mathbf{Z}^m$ is isomorphic to $\mathbf{Z}^d$. Hence there exists an invertible affine linear transformation $T:\langle\sigma\rangle\longrightarrow\mathbf{R}^d$ satisfying $T(\langle\sigma\rangle\cap\mathbf{Z}^m)=\mathbf{Z}^d$. The relative volume of σ is just the d-dimensional volume of the image $T(\sigma)$. If $\dim\sigma=m$, then $v(\sigma)$ is the same as the volume $V(\sigma)$ of σ. Thus for a bounded lattice d-polyhedron X with a weight function ω, the polynomial $L(X,\omega;z)$ has the constant term equal to $\chi(X,\omega-\frac{1}{2}\partial\omega)$ and the coefficient of x^d equal to the relative volume of X with weight $\omega-\frac{1}{2}\partial\omega$. Write $L(X,\omega;z)$ as

$$L(X,\omega;z)=a_0+a_1z+\cdots+a_dz^d.$$

Then $a_0 = \chi(X, \omega - \frac{1}{2}\partial\omega) = \sum_{\sigma \in X}(-1)^{\dim \sigma}(\omega - \frac{1}{2}\partial\omega)(\sigma)$ and $a_d = v(X, \omega - \frac{1}{2}\partial\omega) = \sum_{\sigma \in X} v(\sigma)(\omega - \frac{1}{2}\partial\omega)(\sigma)$, where X is assumed to be decomposed into a disjoint union of open simplices σ. Notice that $\partial\omega$ is zero on open d-simplices, so $v(X, \omega - \frac{1}{2}\partial\omega) = v(X, \omega)$. If $\dim X = m$, then $v(X)$ is the same as the volume $V(X)$ of X.

THEOREM 5.1. *For a bounded lattice d-polyhedron X and a weight function ω,*

$$d!v(X, \omega) = \sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i}L(X, \omega; i). \qquad \textbf{[5.1]}$$

More generally, if $m_0, m_1, \cdots, m_d$ are any distinct nonnegative integers, then

$$v(X, \omega) = \sum_{i=0}^{d}\frac{L(X, \omega; m_i)}{\pi'_{d+1}(m_i)}, \qquad \textbf{[5.2]}$$

where $\pi'_{d+1}(z)$ is the derivative with respect to z of the polynomial $\pi_{d+1}(z) = \prod_{i=0}^{d}(z - m_i)$.

Proof: Express $L(X, \omega; z)/\pi_{d+1}(z)$ as a sum of the partial fractions

$$\frac{L(X, \omega; z)}{\pi_{d+1}(z)} = \sum_{i=0}^{d}\frac{L(X, \omega; m_i)}{\pi'_{d+1}(m_i)} \cdot \frac{1}{z - m_i}.$$

Multiply both sides by $\pi_{d+1}(z)$ and compare the coefficients of x^d on the left and right. We obtain **5.2**. In particular, by setting $m_i = i$, **5.1** follows. □

The reciprocity law $L(X, \omega; z) = (-1)^d L(X, \omega; -z)$ implies that the coefficient of x^{d-1} is zero. We then have Theorem 5.2.

THEOREM 5.2. *Let X be a bounded lattice d-polyhedron with a weight function ω. Then*

$$(d-1)(d!)v(X, \omega) = 2\sum_{i=0}^{d-1}(-1)^{d-i-1}\binom{d-1}{i}L(X, \omega; i). \qquad \textbf{[5.3]}$$

More generally, if $m_1, m_2, \cdots, m_d$ are distinct nonnegative integers, then

$$v(X, \omega)\sum_{i=1}^{d} m_i = \sum_{i=1}^{d}\frac{L(X, \omega, m_i)}{\pi'_d(m_i)}, \qquad \textbf{[5.4]}$$

where $\pi'_d(z)$ is the derivative with respect to z of the polynomial $\pi_d(z) = \prod_{i=1}^{d}(z - m_i)$.

Proof: Express $L(X, \omega, m_i)/\pi'_d(m_i)$ as a sum of the partial fractions

$$\frac{L(X, \omega; z)}{\pi_d(z)} = v(X, \omega) + \sum_{i=1}^{d}\frac{L(X, \omega; m_i)}{\pi'_d(m_i)} \cdot \frac{1}{z - m_i}.$$

Multiply both sides by $\pi_d(z)$ and compare the coefficient of x^{d-1} on the left and right. We obtain **5.4**. In particular, by setting $m_i = i - 1$, **5.3** is obtained. □

Finally, $L(X, \omega; z) = (-1)^d L(X, \omega; -z)$ implies that the coefficients of $z^{d-1}, z^{d-3}, z^{d-5}, \cdots$ are all zero. Then for even $d = 2n$,

$$L(X, \omega; z) = a_0 + a_2 z^2 + a_4 z^4 + \cdots + a_{2n} z^{2n};$$

for odd $d = 2n + 1$,

$$L(X, \omega; z) = z(a_1 + a_5 z^2 + \cdots + a_{2n+1} z^{2n}).$$

THEOREM 5.3. *Let X be a bounded lattice d-polyhedron with a weight function ω. For even d = 2n, let $m_0, m_1, \cdots, m_n$ be the distinct nonnegative integers, then*

$$v(X, \omega) = \sum_{i=0}^{n}\frac{L(X, \omega; m_i)}{\pi'_{n+1}(m_i^2)}, \qquad \textbf{[5.5]}$$

where $\pi'_{n+1}(z)$ is the derivative with respect to z of the polynomial $\pi_{n+1}(z) = \prod_{i=0}^{n}(z - m_i^2)$. For odd d = 2n + 1, let $m_0, m_1, \cdots, m_n$ be distinct positive integers, then

$$v(X, \omega) = \sum_{i=0}^{n}\frac{L(X, \omega; m_i)}{m_i \pi'_{n+1}(m_i^2)}, \qquad \textbf{[5.6]}$$

where $\pi'_{n+1}(z)$ is the derivative with respect to z of $\pi_{n+1}(z) = \prod_{i=0}^{n}(z - m_i^2)$.

In **5.5**, set $m_i = i$. Then $\pi'_{n+1}(i^2) = (-1)^{n-i}(n-i)!(n+i)!/2$ for $1 \le i \le n$ and $\pi'_{n+1}(0) = (-1)^n n! n!$. In **5.6**, set $m_i = i + 1$. Then $\pi'_{n+1}(j^2) = (-1)^{n+1-j}(n + 1 - j)!(n + 1 + j)!/2j^2$ for $1 \le j \le n + 1$. We thus have Corollary 5.4.

COROLLARY 5.4. *Let X be a bounded lattice d-polyhedron with a weight function ω. Then for even d = 2n,*

$$d!v(X, \omega) = 2\sum_{i=1}^{n}(-1)^{n-i}\binom{d}{n-i}L(X, \omega; i) + (-1)^n\binom{d}{n}L(X, \omega; 0);$$

for odd d = 2n + 1,

$$(d+1)!v(X, \omega) = 2\sum_{i=1}^{n+1}(-1)^{n+1-i}\binom{d+1}{n+1-i}iL(X, \omega; i).$$

If X is a bounded lattice d-polyhedron of $\mathbf{R}^d$ and ω is the angle weight function, then $\omega(\sigma) = 1$ for each open d-simplex σ. Thus the leading coefficient of $A(X, \omega; z)$ is the volume $V(X)$ of X. All of the volume formulas in this section about $L(X, \omega; z)$ are true for $A(X; z)$ by replacing $v(X, \omega)$ with $V(X)$ and L with A.

One can consider lattice polyhedra of an arbitrary lattice $\mathbf{L}$ instead of the integral lattice $\mathbf{Z}^m$. All of the formulas about the integral lattice are obviously true for any $\mathbf{L}$-lattice. However, for the volume formulas, $v(X, \omega)$ needs to be replaced by $v(X, \omega)/\det(\mathbf{L})$, where $\det(\mathbf{L})$ is the relative volume of the unit parallelogram of $\mathbf{L}$.

Now we take $\omega = 1$, then $v(X, \omega) = v(X)$. For even $d = 2n$, $(\omega - \frac{1}{2}\partial\omega)(x) = \omega(x) - \frac{1}{2}\chi(\mathrm{lk}(x, X))$, so $\partial\omega(x) = \chi(\mathrm{lk}(x, X))$. Because $\partial\partial\omega = 0$, $(\bar{X}, \partial\omega)$ is an odd-dimensional weighted manifold. Thus $\chi(\bar{X}, \partial\omega) = \int_{\bar{X}} \chi(\mathrm{lk}(x, X))\mathrm{d}\chi(x) = 0$. For odd $d = 2n + 1$, $(\omega - \frac{1}{2}\partial\omega)(x) = \frac{1}{2}\chi(\mathrm{lk}(x, X))$. Corollary 5.4 becomes Corollary 5.5.

COROLLARY 5.5. *Let X be a bounded lattice d-polyhedron of $\mathbf{R}^m$. Then for even d = 2n,*

$$d!v(X) = \sum_{i=1}^{n}(-1)^{n-i}\binom{d}{n-i}\Bigg[2\#(X \cap i^{-1}\mathbf{Z}^m) - \sum_{x \in \bar{X} \cap i^{-1}\mathbf{Z}^m}\chi(\mathrm{lk}(x, X))\Bigg] + (-1)^n\binom{d}{n}\chi(X);$$

for odd d = 2n + 1,

$$(d+1)!v(X) = \sum_{i=1}^{n+1}(-1)^{n+1-i}\binom{d+1}{n+1-i}i\sum_{x \in \bar{X} \cap i^{-1}\mathbf{Z}^m}\chi(\mathrm{lk}(x, X)).$$

Theorem 5.1 and Theorem 5.2 are generalizations of Macdonald's results (26) for lattice polyhedral manifolds with boundary to arbitrary lattice polyhedra. The special cases of Corollary 5.5 for $d = m = 2$ and 3 are unified generalizations of various versions of Pick's theorem and the volume formulas of Reeve (27, 28). Finally, our weight function may be related to McMullen's recent paper (29) on weights of polytopes.

I thank the referees for pointing out a few related references and for other comments. This work was supported by RGC Competitive Earmarked Research Grants under HKUST595/94P and HKUST707/96P.

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