# THE PARTIAL-FRACTIONS METHOD FOR COUNTING SOLUTIONS TO INTEGRAL LINEAR SYSTEMS

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*Dedicated to Lou Billera on the occasion of his sixtieth birthday.*

ABSTRACT. We present a new tool to compute the number $\phi_{\mathbf{A}}(\mathbf{b})$ of integer solutions to

$$\mathbf{x} \geq 0 , \qquad \mathbf{A}\,\mathbf{x} = \mathbf{b} ,$$

where the coefficients of $\mathbf{A}$ and $\mathbf{b}$ are integral. $\phi_{\mathbf{A}}(\mathbf{b})$ is often described as a *vector partition function.* Our methods use partial fraction expansions of Euler's generating function for $\phi_{\mathbf{A}}(\mathbf{b})$. Applications include Ehrhart (quasi-)polynomials counting integer points in dilated polytopes, their multivariate analogs (exemplified through transportation polytopes), and Littlewood-Richardson coefficients appearing in symmetric functions and tensor multiplicities.

## 1. EULER'S GENERATING FUNCTION

We are interested in computing the number of integer solutions of the linear system

$$\mathbf{x} \in \mathbb{R}^d_{\geq 0} , \qquad \mathbf{A}\,\mathbf{x} = \mathbf{b} ,$$

where $\mathbf{A}$ is an $(m \times d)$-integral matrix and $\mathbf{b} \in \mathbb{Z}^m$. We think of $\mathbf{A}$ as fixed and study the number of solutions $\phi_{\mathbf{A}}(\mathbf{b})$ as a function of $\mathbf{b}$. (Strictly speaking, this function is only defined for those $\mathbf{b}$ which lie in the nonnegative linear span of the columns of $\mathbf{A}$.)

The function $\phi_{\mathbf{A}}(\mathbf{b})$, often called a *vector partition function*, appears in a wealth of mathematical areas and beyond: Number Theory (partitions), Discrete Geometry (polyhedra), Commutative Algebra (Hilbert series), Algebraic Geometry (toric varieties), Representation Theory (tensor product multiplicities), Optimization (integer programming), as well as applications to Chemistry, Biology, Physics, Computer Science, and Economics.

Denote the columns of $\mathbf{A}$ by $\mathbf{c}_1, \dots, \mathbf{c}_d$. The following lemma goes back to at least Euler [17]:

**Lemma 1** (Euler). *$\phi_{\mathbf{A}}(\mathbf{b})$ equals the coefficient of $\mathbf{z}^{\mathbf{b}} := z_1^{b_1} \cdots z_m^{b_m}$ of the function*

$$f(\mathbf{z}) = \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})}$$

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*Date*: August 25, 2003.

2000 *Mathematics Subject Classification. Primary* 05A15, 52C07; *Secondary* 52C45, 17B20.

*Key words and phrases.* Vector partition function, partial fractions expansion, quasi-polynomial, lattice-point counting, rational convex polytope, Ehrhart theory, transportation polytope, Littlewood-Richardson coefficients, Kostant partition function, BZ-triangles.

This paper was written while the author was a postdoctoral fellow at the Mathematical Sciences Research Institute in Berkeley. The author thanks MSRI for its hospitality. Thanks also to Alex Feingold and Ira Gessel for fruitful discussions.

*expanded as a power series centered at* $\mathbf{z} = 0$.

*Proof.* Expand each factor of the right-hand side into a geometric series. □

Equivalently, the coefficient of $\mathbf{z}^{\mathbf{b}}$ in $f(\mathbf{z})$ equals the constant term in $\frac{f(\mathbf{z})}{\mathbf{z}^{\mathbf{b}}}$, denoted by const $\frac{f(\mathbf{z})}{\mathbf{z}^{\mathbf{b}}}$. So Euler's Lemma can be conveniently stated as

$$\phi_{\mathbf{A}}(\mathbf{b}) = \operatorname{const} \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}} \ .$$

In a series of articles [3, 5, 6, 7], we used complex integration of $\frac{f(\mathbf{z})}{\mathbf{z}^{\mathbf{b}}}$ to compute $\phi_{\mathbf{A}}(\mathbf{b})$ for special cases of $\mathbf{A}$. Similar techniques were applied in [25]. Here we expand $\frac{f(\mathbf{z})}{\mathbf{z}^{\mathbf{b}}}$ into partial fractions to compute its constant term, and hence $\phi_{\mathbf{A}}(\mathbf{b})$. This work constitutes in a sense a refinement of the complex-integration methods, with the advantage that it is more flexible and—more importantly—applicable for *any* integral linear system.

We will show applications of our methods to Ehrhart (quasi-)polynomials counting integer points in dilated polytopes, their multivariate analogs (exemplified through transportation polytopes), and Littlewood-Richardson coefficients appearing in symmetric functions and tensor multiplicities.

## 2. Vector partition functions

The nature of the counting function $\phi_{\mathbf{A}}(\mathbf{b})$ is given by the following theorem. A *quasi-polynomial* is a finite sum $Q(\mathbf{b}) = \sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{b})\, \mathbf{b}^{\mathbf{n}}$ with coefficients $c_{\mathbf{n}}$ that are functions of $\mathbf{b}$ which are periodic in every component of $\mathbf{b}$. The *degree* of $Q$ is the degree of the largest power $\mathbf{b}^{\mathbf{n}}$ appearing in $Q$. A matrix is *unimodular* if every square submatrix has determinant $\pm 1$.

**Theorem 2** (Sturmfels [30]). *The function* $\phi_{\mathbf{A}}(\mathbf{b})$ *is a piecewise-defined quasi-polynomial in* $\mathbf{b}$ *of degree* $d - \operatorname{rank}(\mathbf{A})$. *The regions of* $\mathbb{R}^m$ *in which* $\phi_{\mathbf{A}}(\mathbf{b})$ *is a single quasi-polynomial are polyhedral, that is, they are defined by linear constraints. If* $\mathbf{A}$ *is unimodular then* $\phi_{\mathbf{A}}$ *is a piecewise-defined polynomial.*

The computation of both $\phi_{\mathbf{A}}(\mathbf{b})$ and the chamber complex consisting of the regions of quasipolynomiality give rise to challenging problems. The most powerful technique for computing $\phi_{\mathbf{A}}(\mathbf{b})$ which we are aware of is due to Brion and Vergne [9]. (The methods described here are much more elementary.) The chamber complex is still much of a mystery. A promising approach can be found in [10]

We finish this section with a *reciprocity theorem.* Let $\phi^{\circ}_{\mathbf{A}}(\mathbf{b})$ count the integer solutions of

$$\mathbf{x} > 0 \ , \qquad \mathbf{A}\,\mathbf{x} = \mathbf{b} \ .$$

**Theorem 3** ([4]). $\phi_{\mathbf{A}}(-\mathbf{b}) = (-1)^{d - \operatorname{rank} \mathbf{A}} \phi^{\circ}_{\mathbf{A}}(\mathbf{b})$ .

This identity gives rise to a symmetry property of $\phi_{\mathbf{A}}(\mathbf{b})$. Let $r_k$ denote the sum of the entries in the $k^{\text{th}}$ row of $\mathbf{A}$, and let $\mathbf{r} = (r_1, \ldots, r_m)$. Then the integer solutions of

$$\mathbf{x} > 0 \ , \qquad \mathbf{A}\,\mathbf{x} = \mathbf{b}$$

are in bijection (via $x_k \mapsto x_k - 1$) with the integer solutions of

$$\mathbf{x} \geq 0 \ , \qquad \mathbf{A}\,\mathbf{x} = \mathbf{b} - \mathbf{r} \ ,$$

and hence $\phi^\circ_{\mathbf{A}}(\mathbf{b}) = \phi_{\mathbf{A}}(\mathbf{b} - \mathbf{r})$. This yields:

**Corollary 4.** $\phi_{\mathbf{A}}(\mathbf{b}) = (-1)^{d-\operatorname{rank}\mathbf{A}} \phi_{\mathbf{A}}(-\mathbf{b} - \mathbf{r})$ .

## 3. The partial-fractions method

This section describes the idea behind our computations. Recall that our goal is to compute

$$\phi_{\mathbf{A}}(\mathbf{b}) = \operatorname{const} \frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}} \ .$$

We start by expanding

$$\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}} \ .$$

into partial fractions in one of the components of $\mathbf{z}$, say $z_1$:

$$\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}} = \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1 - \mathbf{z}^{\mathbf{c}_k}} + \sum_{j=1}^{b_1} \frac{B_j(\mathbf{z})}{z_1^j} \ .$$

Here $A_k$ and $B_j$ are polynomials in $z_1$, rational functions in $z_2, \dots, z_m$, and exponential in $b_1$. (We are tacitly assuming that there are no multiple poles, which generally holds unless $m = 1$, a case which can be handled easily.) The two sums on the right-hand side correspond to the analytic and the meromorphic part with respect to $z_1 = 0$. The latter does not contribute to the $z_1$-constant term, whence

$$\phi_{\mathbf{A}}(\mathbf{b}) = \operatorname{const}_{z_2,\dots,z_m} \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \operatorname{const}_{z_1} \sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1 - \mathbf{z}^{\mathbf{c}_k}}$$

$$= \operatorname{const} \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \sum_{k=1}^{d} \frac{A_k(0, z_2, \dots, z_m, b_1)}{1 - (0, z_2, \dots, z_m)^{\mathbf{c}_k}} \ . \qquad (1)$$

The effect of one partial fraction expansion is to eliminate one of the variables of the generating function, at the cost of replacing one rational function by a sum of such. The following idea is now evident.

**Algorithm.** *Apply (1) repeatedly to eliminate $z_1$, then $z_2$, etc., up to $z_m$.*

As a first remark, we note that this algorithm computes $\phi_{\mathbf{A}}(\mathbf{b})$ *as a function of* $\mathbf{b}$, that is, it allows symbolic computation. Secondly, it is not very hard to deduce Sturmfels's Theorem 2 from this algorithm. What might be more important, however, is the fact that the constraints which define the regions of quasipolynomiality of $\phi_{\mathbf{A}}$ are obtained "on the go" as one computes $\phi_{\mathbf{A}}$: When expanding into partial fractions, one has to check where the poles of a rational function are. The components of $\mathbf{b}$ will appear (linearly) in the exponents of these rational functions, and hence one will automatically have to split the computation into cases which give rise to different regions in the chamber complex of quasipolynomiality.

Our algorithm is best illustrated by going through an actual example.

## 4. An illustrating example

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, and write $\mathbf{b} = (a, b)$, so $\phi_{\mathbf{A}}(a, b)$ counts the integer solutions of

$$x_1, x_2, x_3, x_4 \geq 0 , \qquad \begin{array}{rcl} x_1 + 2x_2 + x_3 & = & a \\ x_1 + x_2 + x_4 & = & b \, . \end{array}$$

By Euler's Lemma 1,

$$\phi_{\mathbf{A}}(a, b) = \operatorname{const} \frac{1}{(1 - zw)(1 - z^2w)(1 - z)(1 - w) z^a w^b} \, .$$

We first expand into partial fractions with respect to $w$:

$$\frac{1}{(1 - zw)(1 - z^2w)(1 - w) w^b} = -\frac{\frac{z^{b+1}}{(1-z)^2}}{1 - zw} + \frac{\frac{z^{2b+3}}{(1-z)(1-z^2)}}{1 - z^2 w} + \frac{\frac{1}{(1-z)(1-z^2)}}{1 - w} + \sum_{k=1}^{b} \frac{\cdots}{w^k} \, .$$

Taking constant terms gives

$$\begin{aligned} \phi_{\mathbf{A}}(a, b) &= \operatorname{const}_z \frac{1}{(1 - z) z^a} \operatorname{const}_w \frac{1}{(1 - zw)(1 - z^2 w)(1 - w) w^b} \\ &= \operatorname{const} \frac{1}{(1 - z) z^a} \left( -\frac{z^{b+1}}{(1 - z)^2} + \frac{z^{2b+3}}{(1 - z)(1 - z^2)} + \frac{1}{(1 - z)(1 - z^2)} \right) \\ &= \operatorname{const} \left( -\frac{z^{b-a+1}}{(1 - z)^3} + \frac{z^{2b-a+3}}{(1 - z)^2 (1 - z^2)} + \frac{1}{(1 - z)^2 (1 - z^2) z^a} \right) \end{aligned} \tag{2}$$

(At this point, we could interpret each of the three constant terms as counting integer solutions to new linear systems. For example, the last term gives the number of integer points $(x, y) \geq 0$ satisfying $2x + y \leq a$. To keep a general flavor, we continue with our general algorithm.)

We compute the constant term of each of the three terms separately. For starters,

$$\operatorname{const} \frac{z^{b-a+1}}{(1 - z)^3} = 0$$

if $b - a + 1 > 0$, equivalently (since $a$ and $b$ are integers) $b \geq a$. If $b < a$, we use

$$\frac{1}{(1 - z)^3} = \sum_{k \geq 0} \binom{k + 2}{2} z^k \, ,$$

which gives

$$\operatorname{const} \frac{1}{(1 - z)^3 z^{a-b-1}} = \binom{a - b + 1}{2} = \frac{(a - b)^2}{2} + \frac{a - b}{2} \, .$$

For the second term in (2),

$$\operatorname{const} \frac{z^{2b-a+3}}{(1 - z)^2 (1 - z^2)} = 0$$

if $2b - a + 3 > 0$. If $a \geq 2b + 3$ we expand into partial fractions again:

$$\begin{aligned}
\text{const}\, \frac{z^{2b-a+3}}{(1-z)^2(1-z^2)} &= \text{const}\left( \frac{1/2}{(1-z)^3} + \frac{\frac{a-2b-3}{2} + \frac{1}{4}}{(1-z)^2} + \frac{\frac{(a-2b-3)^2}{4} + \frac{a-2b-3}{2} + \frac{1}{8}}{1-z} + \frac{(-1)^{a+1}/8}{1+z} \right) \\
&= \frac{(a-2b)^2}{4} + \frac{2b-a}{2} + \frac{1+(-1)^{a+1}}{8} \,.
\end{aligned}$$

The computation for the last term in (2) is almost identical. (Note that this term always contributes.)

$$\begin{aligned}
\text{const}\, \frac{1}{(1-z)^2(1-z^2)z^a} &= \text{const}\left( \frac{1/2}{(1-z)^3} + \frac{\frac{a}{2} + \frac{1}{4}}{(1-z)^2} + \frac{\frac{a^2}{4} + \frac{a}{2} + \frac{1}{8}}{1-z} + \frac{(-1)^a/8}{1+z} \right) \\
&= \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} \,.
\end{aligned}$$

Summing up all terms in (2) gives finally

$$\begin{aligned}
\phi_{\mathbf{A}}(a,b) &= \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \leq b, \\ -\frac{(a-b)^2}{2} - \frac{a-b}{2} + \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a > b > \frac{a-3}{2}, \\ -\frac{(a-b)^2}{2} - \frac{a-b}{2} + \frac{(a-2b)^2}{4} + \frac{2b-a}{2} + \frac{1+(-1)^{a+1}}{8} + \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } b \leq \frac{a-3}{2} \end{cases} \\
&= \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \leq b, \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } a > b > \frac{a-3}{2}, \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \leq \frac{a-3}{2}. \end{cases}
\end{aligned}$$

It is a fun exercise to show that $\phi_{\mathbf{A}}(a,b) = \phi_{\mathbf{A}}(-a-4, -b-3)$, as promised by Corollary 4.

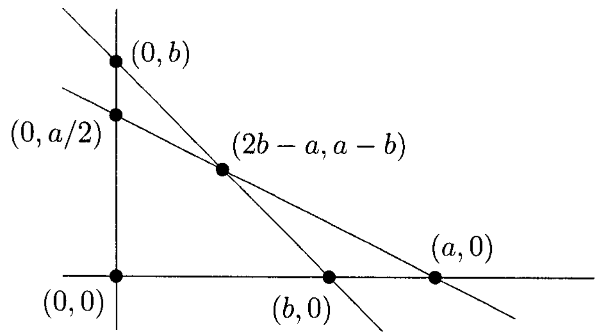
We pause for a moment to introduce the geometry behind our linear system. By thinking of $x_3$ and $x_4$ as slack variables, we can see that

$$x_1, x_2, x_3, x_4 \geq 0 \,, \qquad \begin{array}{rcl} x_1 + 2x_2 + x_3 &=& a \\ x_1 + x_2 + x_4 &=& b \,. \end{array}$$

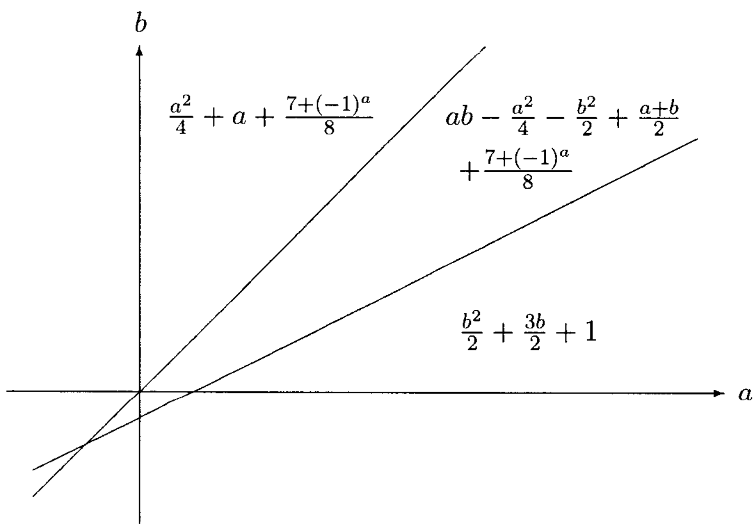
is equivalent to

$$x_1, x_2 \geq 0 \,, \qquad \begin{array}{rcl} x_1 + 2x_2 &\leq& a \\ x_1 + x_2 &\leq& b \,. \end{array}$$

Depending on the relationship between $a$ and $b$, the geometric figure described by this linear system is a quadrilateral or triangle.




The inequalities which define the different cases for $\phi_{\mathbf{A}}(a,b)$ determine the chamber complex in the parameter space, and in this example also the combinatorial type of the polytope: in the first ($a \le b$) and last ($b \le \frac{a-3}{2}$) case, it is a triangle, and in the second case ($a > b > \frac{a-3}{2}$) a quadrilateral. In fact, the geometry of the triangle in the first case only depends on $a$, which is reflected in $\phi_{\mathbf{A}}(a,b)$, and similarly the last case shows only dependency on $b$. In this last case, the triangle actually has integer vertices (the linear system is unimodular), whence we get a polynomial. In the first two cases the vertices of the polytope are half-integral, which is reflected in the period-2 quasi-polynomials. The following picture illustrates the three chambers.




## 5. Ehrhart quasi-polynomials

A *convex polytope* $\mathcal{P}$ in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. Alternatively (and this correspondence is nontrivial [31]), one can define $\mathcal{P}$ as the bounded intersection of affine halfspaces. A polytope is *rational* if all of its vertices have rational coordinates. (A *vertex* of $\mathcal{P}$ is a point $\mathbf{v} \in \mathcal{P}$ for which there is a hyperplane $H$ such that $\{v\} = \mathcal{P} \cap H$.) We denote by $\mathcal{P}^\circ$ the relative interior of $\mathcal{P}$. For a positive integer $t$, let $L_{\mathcal{P}}(t)$ denote the number of integer points ("lattice points") in the dilated polytope $t\mathcal{P} = \{tx : x \in \mathcal{P}\}$. The fundamental result about the structure of $L_{\mathcal{P}}$ is as follows.

**Theorem 5** (Ehrhart [16]). *If $\mathcal{P}$ is a convex rational polytope, then the functions $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in $t$ whose degree is the dimension of $\mathcal{P}$. If $\mathcal{P}$ has integer vertices, then $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ are polynomials.*

Ehrhart conjectured and partially proved the following *reciprocity law*, which was proved by Macdonald [26].

**Theorem 6** (Ehrhart-Macdonald). $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

The computation of Ehrhart quasi-polynomials is only slightly easier than that of $\phi_{\mathbf{A}}$. Recent work includes [1, 9, 11, 12, 13, 15, 20, 21, 22, 28].

Suppose the convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is given by an intersection of halfspaces, that is,

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\} ,$$

for some $(m \times d)$-matrix $\mathbf{A}$ and $m$-dimensional vector $\mathbf{b}$. We may convert these inequalities into equalities by introducing slack variables. If $\mathcal{P}$ has rational vertices, we can choose $\mathbf{A}$ and $\mathbf{b}$ in such a way that all their entries are integers. In summary, we may assume that a convex rational polytope $\mathcal{P}$ is given by

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\} ,$$

where $\mathbf{A} \in M_{m\times d}(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^m$. (If we are interested in counting the integer points in $\mathcal{P}$, we may assume that $\mathcal{P}$ is in the nonnegative orthant, i.e., the points in $\mathcal{P}$ have nonnegative coordinates, as translation by an integer vector does not change the lattice-point count.)

The connection to vector partition functions is now evident. Since $t\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = t\,\mathbf{b} \right\}$, we obtain $L_{\mathcal{P}}(t) = \phi_{\mathbf{A}}(t\mathbf{b})$ as a special evaluation of $\phi_{\mathbf{A}}$. Note that $t\mathbf{b}, t = 1, 2, \ldots$ lie in the same chamber of quasipolynomiality of $\phi_{\mathbf{A}}$. Ehrhart's Theorem 5 is therefore a special case of Sturmfels's Theorem 2, and Theorem 6 is a special case of Theorem 3.

As an example, the quadrilateral $\mathcal{Q}$ described by

$$x, y \ge 0 , \qquad \begin{array}{rcl} x + 2y & \le & 5 \\ x + y & \le & 4 \end{array}$$

(a special case of the polygons appearing in Section 4) with vertices $(0,0), (4,0), (3,1), (0,5/2)$ has the Ehrhart quasi-polynomial

$$L_{\mathcal{Q}}(t) = \phi_{\mathbf{A}}(5t, 4t) = \frac{23}{4}\,t^2 + \frac{9}{2}\,t + \frac{7 + (-1)^t}{8} .$$

Ehrhart quasi-polynomials are easier to compute, since one does not need to derive/know the chamber complex of quasipolynomiality of $\phi_{\mathbf{A}}$. Our algorithm, naturally, works just as well for $L_{\mathcal{P}}$.

## 6. Transportation polytopes

For positive integers $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that $a_1 + \cdots + a_m = b_1 + \cdots + b_n$, define the *transportation polytope*

$$\mathcal{T}(a_1, \ldots, a_m, b_1, \ldots, b_n) = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \in \mathbb{R}^{mn}_{\geq 0} : \begin{array}{l} \sum_k x_{jk} = a_j \text{ for all } 1 \leq j \leq m, \\ \sum_j x_{jk} = b_k \text{ for all } 1 \leq k \leq n \end{array} \right\}.$$

These polytopes (which also go by the name of *contingency tables*) were motivated by problems in Statistics and Operations Research. For combinatorial properties of transportation polytopes see, for example, [14, 23]. A famous subclass of transportation polytopes are the *Birkhoff polytopes*, for which $m = n$ and $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 1$.

We are interested in the number of integer points in $\mathcal{T}(a_1, \ldots, a_m, b_1, \ldots, b_n)$ as a function in $a_1, \ldots, a_m, b_1, \ldots, b_n$. This is a vector partition function $\phi_{\mathbf{A}}(\mathbf{b})$ where

$$\mathbf{A} = \begin{pmatrix} 1 & \cdots & 1 & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & \cdots & 1 \\ 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix}$$

and $\mathbf{b} = (a_1, \ldots, a_m, b_1, \ldots, b_n)$. In fact, $\mathbf{A}$ does not have full rank, so we may omit one of the rows.

As an example, we show how to compute $\phi_{\mathbf{A}}$ for the first non-trivial case, namely $m = n = 3$. After reducing $\mathbf{A}$ to full rank, we have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = (a_1, a_2, b_1, b_2, b_3)$$

(note that $a_3 = b_1 + b_2 + b_3 - a_1 - a_2$). Euler's Lemma 1 states that the number of integer points in $\mathcal{T}(a_1, a_2, a_3, b_1, b_2, b_3)$ equals

$$\phi_{\mathbf{A}}(a_1, a_2, b_1, b_2, b_3) = \\ \operatorname{const} \frac{1}{(1-z_1w_1)(1-z_1w_2)(1-z_1w_3)(1-z_2w_1)(1-z_2w_2)(1-z_2w_3)(1-w_1)(1-w_2)(1-w_3) z_1^{a_1} z_2^{a_2} w_1^{b_1} w_2^{b_2} w_3^{b_3}} .$$

Partial fractions with respect to $w_1$ give

$$\text{const}\,\frac{1}{(1-z_1w_1)(1-z_2w_1)(1-w_1)w_1^{b_1}} =$$
$$\text{const}\left(\frac{z_1^{b_1+2}}{(z_1-z_2)(z_1-1)}+\frac{z_2^{b_1+2}}{(z_2-z_1)(z_2-1)}+\frac{1}{(z_1-1)(z_2-1)}\right)$$

and after obtaining very similarly looking expressions for the partial fractions expansions with respect to $w_2$ and $w_3$ we obtain

$$\begin{aligned}\phi_{\mathbf{A}}(a_1,a_2,b_1,b_2,b_3) = \text{const}\,\frac{1}{z_1^{a_1}z_2^{a_2}}&\left(\frac{z_1^{b_1+2}}{(z_1-z_2)(z_1-1)}+\frac{z_2^{b_1+2}}{(z_2-z_1)(z_2-1)}+\frac{1}{(z_1-1)(z_2-1)}\right)\\ \times&\left(\frac{z_1^{b_2+2}}{(z_1-z_2)(z_1-1)}+\frac{z_2^{b_2+2}}{(z_2-z_1)(z_2-1)}+\frac{1}{(z_1-1)(z_2-1)}\right)\\ \times&\left(\frac{z_1^{b_3+2}}{(z_1-z_2)(z_1-1)}+\frac{z_2^{b_3+2}}{(z_2-z_1)(z_2-1)}+\frac{1}{(z_1-1)(z_2-1)}\right)\end{aligned}$$

Which of the 27 terms on the right contribute to $\phi_{\mathbf{A}}(a_1,a_2,b_1,b_2,b_3)$ now depends on the relationship between $a_1,a_2,b_1,b_2,b_3$. One term always contributes, namely

$$\text{const}\,\frac{1}{z_1^{a_1}z_2^{a_2}(z_1-1)^3(z_2-1)^3} = \binom{a_1+2}{2}\binom{a_2+2}{2}.$$

If $b_k > a_j - 2$ for all $k$ and $j$, this is $\phi_{\mathbf{A}}$. Otherwise, there is more work to be done. As an example for computing another term, let us derive (assuming $a_1 \geq b_1+2$)

$$\text{const}\,\frac{z_1^{b_1+2}}{z_1^{a_1}z_2^{a_2}(z_1-z_2)(z_1-1)^3(z_2-1)^2}\,.$$

Partial fractions with respect to $z_1$ gives

$$\begin{aligned}&\text{const}\,\frac{1}{z_1^{a_1-b_1-2}(z_1-z_2)(z_1-1)^3}\\ &=\frac{z_2^{b_1-a_1+1}}{(1-z_2)^3}+\frac{1}{z_2-1}+\frac{(a_1-b_1-2)(z_2-1)-1}{(z_2-1)^2}+\frac{\frac{1}{2}(a_1-b_1-2)^2(z_2-1)^2+\frac{3}{2}(a_1-b_1-2)+1}{(z_2-1)^3}\\ &=\frac{z_2^{b_1-a_1+1}-\frac{3}{2}(a_1-b_1)+2}{(1-z_2)^3}-\frac{1}{(z_2-1)^2}+\frac{\frac{1}{2}(a_1-b_1)^2-a_1+b_1+1}{z_2-1}\,.\end{aligned}$$

Finally, taking $z_2$-constant term gives

$$\begin{aligned}&\text{const}\,\frac{z_1^{b_1+2}}{z_1^{a_1}z_2^{a_2}(z_1-z_2)(z_1-1)^3(z_2-1)^2}\\ &=\text{const}\,\frac{1}{z_2^{a_2}(z_2-1)^2}\left(\frac{z_2^{b_1-a_1+1}-\frac{3}{2}(a_1-b_1)+2}{(1-z_2)^3}-\frac{1}{(z_2-1)^2}+\frac{\frac{1}{2}(a_1-b_1)^2-a_1+b_1+1}{z_2-1}\right)\\ &=\binom{a_1+a_2-b_1+3}{4}+\left(2+\tfrac{3}{2}(b_1-a_1)\right)\binom{a_2+4}{4}-\binom{a_2+3}{3}-\left(\tfrac{1}{2}(a_1-b_1)^2-a_1+b_1+1\right)\binom{a_2+2}{2}\end{aligned}$$

The first term appears only when $a_1 + a_2 - b_1 - 1 \geq 0$. This example shows that the computation of both the counting function and the chamber complex of polynomiality of $\phi_{\mathbf{A}}$ is involved, on the other hand, it should be relatively easily implementable on a computer.

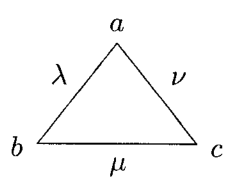
## 7. Littlewood-Richardson coefficients

The Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$ appear in various mathematical context, for example, as tensor-product multiplicities of irreducible representations in $GL_k\mathbb{C}$ [19], or as structure constants of the ring of symmetric polynomials in the basis of Schur polynomials [27, 29], similarly as structure constants of the cohomology ring of the Grassmannian in the basis of Schubert classes [18]. The Littlewood-Richardson coefficients are vector partition functions, in these settings often called *Kostant partition functions.*

Combinatorial models which interpret Littlewood-Richardson coefficients as integer points in polyhedra include the Littlewood-Richardson rule [29], Berenstein-Zelevinsky triangles [8], and Knutson-Tao honeycombs [24].

We give the elementary example of BZ-triangles corresponding to the simple Lie algebra $sl_2\mathbb{C}$ (of type $A_1$) to illustrate how our methods apply. Granted, the multiplicities for $sl_2\mathbb{C}$ can be easily computed by hand (and much faster than what we are about to do). However, our methods are general, and it is our hope that they will lead to formulas or algorithms for the higher-dimensional cases, which are much more complicated.

The BZ triangle in the case of $sl_2\mathbb{C}$ gives the following constraints:




$$\begin{aligned} a + b &= \lambda \\ b + c &= \mu \\ c + a &= \nu \end{aligned}$$

The BZ-theory states that $c_{\lambda\mu}^{\nu}$ equals the number of nonnegative integer solutions $(a, b, c)$ of this system. In the language of this paper, $c_{\lambda\mu}^{\nu} = \phi_{\mathbf{A}}(\mathbf{b})$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = (\lambda, \mu, \nu) \ .$$

By Euler's Lemma 1,

$$\phi_{\mathbf{A}}(\lambda, \mu, \nu) = \operatorname{const} \frac{1}{(1 - xy)\,(1 - xz)\,(1 - yz)\, x^{\lambda} y^{\mu} z^{\nu}} \ .$$

We start with the partial fractions expansion with respect to $x$:

$$\begin{aligned} \operatorname{const} \frac{1}{(1 - xy)\,(1 - xz)\, x^{\lambda}} &= \operatorname{const} \left( \frac{y^{\lambda+1}/(y - z)}{1 - xy} + \frac{z^{\lambda+1}/(z - y)}{1 - xz} \right) \\ &= \frac{y^{\lambda+1}}{y - z} + \frac{z^{\lambda+1}}{z - y} \ , \end{aligned}$$

whence

$$\phi_{\mathbf{A}}(\lambda,\mu,\nu) = \operatorname{const} \frac{y^{\lambda+1} - z^{\lambda+1}}{(y-z)(1-yz)y^{\mu}z^{\nu}} \,.$$

(Similar to the previous examples, we can expand the function on the right-hand side to see that the constant term counts integer solutions $(j,k)$ to the following system:

$$0 \le x_1 \le \lambda,\ x_2 \ge 0, \qquad \begin{array}{rcl} x_1 + x_2 &=& \mu \\ \lambda - x_1 + x_2 &=& \nu \,. \end{array}$$

Again the number of nontrivial constraints is reduced by one. In this case, the dimension also got reduced. We return to our algorithm again in favor of keeping this computation "general".)

By splitting up the sum,

$$\begin{aligned} \phi_{\mathbf{A}}(\lambda,\mu,\nu) &= \operatorname{const} \frac{y^{\lambda+1} - z^{\lambda+1}}{(y-z)(1-yz)y^{\mu}z^{\nu}} \\ &= \operatorname{const} \frac{1}{(y-z)(1-yz)y^{\mu-\lambda-1}z^{\nu}} - \operatorname{const} \frac{1}{(y-z)(1-yz)y^{\mu}z^{\nu-\lambda-1}} \,, \end{aligned} \tag{3}$$

we can see that we have to consider cases, in view of the expansion in $y$, $\lambda \ge \mu$ and $\lambda < \mu$. (In this example, this can be avoided due to the symmetry of the linear system. Again, we go through this computation to give a flavor of the general case.) At this stage, it is also convenient to put an order on the size of the variables; we choose $|y| < |z|$. (This will introduce an asymmetry which is not present in the linear system.)

For the second term in (3), we use the partial fractions expansion

$$\frac{1}{(y-z)(1-yz)y^{\mu}} = \frac{\frac{1}{(1-z^2)z^{\mu}}}{y-z} + \frac{\frac{z^{\mu+1}}{1-z^2}}{1-yz} + \sum_{k=1}^{\mu} \frac{\cdots}{y^k} \tag{4}$$

to obtain

$$\begin{aligned} -\operatorname{const} \frac{1}{(y-z)(1-yz)y^{\mu}z^{\nu-\lambda-1}} &= -\operatorname{const} \frac{1}{z^{\nu-\lambda-1}} \left( \frac{1}{-z(1-z^2)z^{\mu}} + \frac{z^{\mu+1}}{1-z^2} \right) \\ &= \operatorname{const} \left( \frac{z^{\lambda-\mu-\nu}}{1-z^2} - \frac{z^{\lambda+\mu-\nu+2}}{1-z^2} \right) \end{aligned} \tag{5}$$

If $\lambda \ge \mu$, then the first term of (3) has $y$-constant term zero, as

$$\frac{1}{(y-z)(1-yz)y^{\mu-\lambda-1}z^{\nu}} = \frac{y^{\lambda-\mu+1}}{(y/z-1)(1-yz)z^{\nu+1}} \,.$$

If $\lambda < \mu$, then both terms in (3) survive. For the first one we use (4) with $\mu$ replaced by $\mu-\lambda-1$ to obtain

$$\begin{aligned} \operatorname{const} \frac{1}{(y-z)(1-yz)y^{\mu-\lambda-1}z^{\nu}} &= \operatorname{const} \frac{1}{z^{\nu}} \left( \frac{1}{-z(1-z^2)z^{\mu-\lambda-1}} + \frac{z^{\mu-\lambda}}{1-z^2} \right) \\ &= \operatorname{const} \left( -\frac{z^{\lambda-\mu-\nu}}{1-z^2} + \frac{z^{\mu-\lambda-\nu}}{1-z^2} \right) \end{aligned}$$

Hence we obtain in this case

$$\phi_{\mathbf{A}}(\lambda,\mu,\nu) = \text{const}\left(\frac{z^{\mu-\lambda-\nu}}{1-z^2} - \frac{z^{\lambda+\mu-\nu+2}}{1-z^2}\right)$$

We can therefore combine the two cases $\lambda \geq \mu$ and $\lambda < \mu$:

$$\phi_{\mathbf{A}}(\lambda,\mu,\nu) = \text{const}\left(\frac{z^{|\lambda-\mu|-\nu}}{1-z^2} - \frac{z^{\lambda+\mu-\nu+2}}{1-z^2}\right).$$

By expanding the right-hand side

$$\frac{z^{|\lambda-\mu|-\nu} - z^{\lambda+\mu-\nu+2}}{1-z^2} = z^{|\lambda-\mu|-\nu}\sum_{k\geq 0} z^{2k} - z^{\lambda+\mu-\nu+2}\sum_{k\geq 0} z^{2k} = \sum_{|\lambda-\mu|-\nu\leq 2k<\lambda+\mu-\nu+2} z^{2k}$$

we can now easily verify that the constant term (and hence $\phi_{\mathbf{A}}(\lambda,\mu,\nu)$) is one if $\lambda+\mu+\nu$ is even and $|\lambda-\mu| \leq \nu \leq \lambda+\mu$ and zero otherwise; this confirms a classical result in Representation Theory.

Geometrically, we studied a zero-dimensional rational polytope. We computed a quasi-polynomial of degree 0 with period 2, and the walls of the chamber complex of quasipolynomiality are parts of the hyperplanes given by $\lambda-\mu=\nu$, $\mu-\lambda=\nu$, and $\nu=\lambda+\mu$.

## 8. Concluding remarks

Many open problems and questions remain untouched. Most importantly, we hope that our ideas will give rise to practical implementations of computing vector partition functions in their various disguises in form of a computer software.

Not unrelated is the question of computational complexity. It is known that vector partition functions can be computed in polynomial time if $d$ is fixed [2]. We have not analyzed the complexity of our algorithm. This seems interesting in the light that our algorithm depends on the number of contraints, geometrically corresponding to the facets (codimension-1 faces) of the polytope. Barvinok's algorithm, in contrast, depends on the number of vertices of the polytope.

Another venue which should be explored is the following: After each step in our algorithm, one could reinterpret the constant term of each summand as counting integer solutions of a linear system. It might be interesting to simplify each of these linear systems (as far as this is possible without changing the number of integer solutions), and then proceed with the algorithm.

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polynomial in z of degree n-1

$$\frac{1}{1 - z^n \cdot C} \cdot f(z) = \frac{A(z)}{1 - z^n \cdot C} + \cdots$$

depends on other variables