The Ehrhart polynomial of the Birkhoff polytope [1]

MATTHIAS BECK AND DENNIS PIXTON

Abstract: The n^{th} Birkhoff polytope is the set of all doubly stochastic $n \times n$ matrices, that is, those matrices with nonnegative real coefficients in which every row and column sums to one. A wide open problem concerns the volumes of these polytopes, which have been known for $n \leq 8$. We present a new, complex-analytic way to compute the Ehrhart polynomial of the Birkhoff polytope, that is, the function counting the integer points in the dilated polytope. The leading term of the Ehrhart polynomial is—up to a trivial factor—the volume of the polytope, which is one reason why we are interested in this counting function. We implemented our methods in form of a computer program, which yielded the Ehrhart polynomial (and hence the volume) of the ninth Birkhoff polytope.

1 Introduction

One of the most intriguing objects of combinatorial geometry is the n^{th} *Birkhoff polytope*

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \,:\, x_{jk} \geq 0, \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\} ,$$

often described as the set of all $n \times n$ *doubly stochastic matrices*. $\mathcal{B}_n$ is a convex polytope with integer vertices. It possesses fascinating combinatorial properties [4, 5, 6, 8, 22] and relates to many mathematical areas [10, 14]. A long-standing open problem is the determination of the relative volume of $\mathcal{B}_n$, which has been known only up to $n = 8$ [7]. In this paper, we propose a new method of calculating this volume and use it to compute $\operatorname{vol} \mathcal{B}_9$.

One of the recent attempts to compute $\operatorname{vol} \mathcal{B}_n$ relies on the theory of counting functions for the integer points in polytopes. Ehrhart proved [11] that for a polytope $\mathcal{P} \subset \mathbb{R}^d$ with integral vertices, the number

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$$

is a polynomial in the positive integer variable t. He showed various other properties for this counting function (in fact, in the more general setting of $\mathcal{P}$ having rational vertices), of which we mention three here:

- The degree of $L_{\mathcal{P}}$ is the dimension of $\mathcal{P}$.
- The leading term of $L_{\mathcal{P}}$ is the relative volume of $\mathcal{P}$, normalized with respect to the sublattice of $\mathbb{Z}^d$ on the affine subspace spanned by $\mathcal{P}$.
- Since $L_{\mathcal{P}}$ is a polynomial, we can evaluate it at nonpositive integers. These evaluations yield

$$L_{\mathcal{P}}(0) = \chi(\mathcal{P}) \,, \tag{1}$$

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t) \,. \tag{2}$$

[1] Version of 2/22/2002

Here $\chi(\mathcal{P})$ denotes the Euler characteristic, $\mathcal{P}^\circ$ the relative interior of $\mathcal{P}$. The reciprocity law (2) was in its full generality proved by Macdonald [15].

The application of this theory to the Birkhoff polytope $\mathcal{B}_n$ incorporates the nice interpretation of the number of integral points in $t\mathcal{B}_n$ as the number of *semi-magic squares*, namely, square matrices whose nonnegative integral coefficients sum up to the same integer t along each row and column.

We will denote the Ehrhart polynomial of $\mathcal{B}_n$ by

$$H_n(t) := L_{\mathcal{B}_n}(t) \ .$$

It is not hard to see that $\dim \mathcal{B}_n = (n-1)^2$, hence H_n is a polynomial in t of degree $(n-1)^2$. The first two of these polynomials are trivial:

$$H_1(t) = 1 \qquad\qquad H_2(t) = t+1 \ ,$$

the first nontrivial case was computed by MacMahon [16] as

$$H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2} \ . \tag{3}$$

The structural properties of H_n were first studied in [12, 18, 19]. It is a nice exercise to deduce from (2) that

$$H_n(-n-t) = (-1)^{n-1}\ H_n(t) \tag{4}$$

and

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0 \ .$$

This allows the following strategy of computing H_n, and therefore, the volume of $\mathcal{B}_n$: compute the first $\binom{n-1}{2}$ values of H_n, use the above symmetry and trivial values of H_n, and calculate the polynomial H_n by interpolation. In fact, as far as we are aware of, the volume of $\mathcal{B}_8$ was computed using essentially this method, combined with some nice computational tricks [7].

We propose a new, completely different approach of computing H_n (and hence $\operatorname{vol}\mathcal{B}_n$). It is based on an analytic method by the first author of computing the Ehrhart polynomial of a polytope [2]. We will introduce the application of this method to the Birkhoff polytope in the following section.

Some recent refreshing approaches—of a more algebraic-geometric/topological flavor—to the problem of computing $\operatorname{vol}\mathcal{B}_n$ and H_n can be found in [1, 9, 21].

2 An integral counting integers

A convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is an intersection of halfspaces. This allows the compact description

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\mathbf{x} \le \mathbf{b}\right\} \ ,$$

for some $(m \times d)$-matrix $\mathbf{A}$ and m-dimensional vector $\mathbf{b}$. Here the inequality is understood componentwise. In fact, we may convert these inequalities into equalities by introducing 'slack variables.'

If $\mathcal{P}$ has rational vertices (those polytopes are called *rational*), we can choose $\mathbf{A}$ and $\mathbf{b}$ in such a way that all their entries are integers. In summary, we may assume that a convex rational polytope $\mathcal{P}$ is given by

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \ \mathbf{A}\mathbf{x} = \mathbf{b} \right\} , \tag{5}$$

where $\mathbf{A} \in M_{m \times d}(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^m$. (If we are interested in counting the integer points in $\mathcal{P}$, we may assume that $\mathcal{P}$ is in the nonnegative orthant, i.e., the points in $\mathcal{P}$ have nonnegative coordinates, as translation by an integer vector does not change the lattice-point count.) The following straightforward theorem can be found in [2]. We use the standard multivariate notation $\mathbf{v}^{\mathbf{w}} := v_1^{w_1} \cdots v_n^{w_n}$.

Theorem 1 [2, Theorem 8] *Suppose the convex rational polytope $\mathcal{P}$ is given by* (5)*, and denote the columns of $\mathbf{A}$ by $\mathbf{c}_1, \ldots, \mathbf{c}_d$. Then*

$$L_{\mathcal{P}}(t) = \frac{1}{(2\pi i)^m} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_m|=\epsilon_m} \frac{z_1^{-tb_1-1} \cdots z_m^{-tb_m-1}}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})} \, d\mathbf{z} \ .$$

Here $0 < \epsilon_1, \ldots, \epsilon_m < 1$ are different numbers such that we can expand all the $\frac{1}{1-\mathbf{z}^{\mathbf{c}_k}}$ into power series about 0.

We can view the Birkhoff polytope $\mathcal{B}_n$ as given in the form of (5), where

$$\mathbf{A} = \begin{pmatrix} 1 & \cdots & 1 & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & \cdots & 1 \\ 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix}$$

is an $(2n \times n^2)$-matrix and $\mathbf{b} = (1, \ldots, 1) \in \mathbb{Z}^{2n}$. Hence Theorem 1 gives for this special case

$$H_n(t) = \frac{1}{(2\pi i)^{2n}} \int \cdots \int \frac{(z_1 \cdots z_{2n})^{-t-1}}{(1-z_1 z_{n+1})(1-z_1 z_{n+2}) \cdots (1-z_n z_{2n})} \, d\mathbf{z} \ .$$

Here it is understood that each integral is over a circle with radius < 1 centered at 0; all appearing radii should be different. We can separate, say, the last n variables and obtain

$$H_n(t) = \frac{1}{(2\pi i)^n} \int \cdots \int (z_1 \cdots z_n)^{-t-1} \left(\frac{1}{2\pi i} \int \frac{z^{-t-1}}{(1-z_1 z) \cdots (1-z_n z)} \, dz \right)^n \, dz_n \cdots dz_1 \ .$$

The innermost integral is easy to compute: It is equal to the residue at 0 of

$$\frac{1}{z^{t+1}(1-z_1 z) \cdots (1-z_n z)}$$

and, by the residue theorem, equal to the negative of the sum of the residues at $\frac{1}{z_1}, \ldots, \frac{1}{z_n}$. (Note that here we use the fact that $t > 0$.) The residues at these simple poles are easily computed, and we obtain the starting point for our computations.

Theorem 2 *For any pairwise different* $0 < \epsilon_1, \ldots, \epsilon_n < 1$,

$$H_n(t) = \frac{1}{(2\pi i)^n} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} (z_1 \cdots z_n)^{-t-1} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j \neq k}(z_k - z_j)} \right)^n dz_n \cdots dz_1 \ .$$

Remark. It is not hard to conclude from the form of the integrand that H_n is indeed a polynomial in t. In fact, one can make this property more apparent by noticing that the expression in parenthesis is actually a polynomial; namely

$$H_n(t) = \frac{1}{(2\pi i)^n} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} (z_1 \cdots z_n)^{-t-1} \left(\sum_{m_1+\cdots+m_n=t} z_1^{m_1} \cdots z_n^{m_n} \right)^n dz_n \cdots dz_1 \ ,$$

where the sum is over all ordered partitions of t. This formula can also be proved "more directly" combinatorially.[2]

3 Small n do not require a computer

We will now illustrate the computation of H_n (and hence $\operatorname{vol} \mathcal{B}_n$) by means of Theorem 2 for $n = 3$ and 4. These calculations "by hand" give an idea what computational tricks one might use in tackling larger n with the aid of a computer.

By the theorem,

$$H_3(t) = \frac{1}{(2\pi i)^3} \int (z_1 z_2 z_3)^{-t-1} \left(\frac{z_1^{t+2}}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2^{t+2}}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3^{t+2}}{(z_3 - z_1)(z_3 - z_2)} \right)^3 d\mathbf{z}.$$

We have to order the radii of the integration paths for each variable; we choose $0 < \epsilon_3 < \epsilon_2 < \epsilon_1 < 1$. We heavily use this fact after multiply out the cubic: integrating, for example, the term

$$\frac{z_1^{-t-1} z_2^{-t-1} z_3^{2t+5}}{(z_3 - z_2)^3 (z_3 - z_1)^3}$$

with respect to z_3 gives 0, as this function is analytic at the z_3-origin and $|z_1|, |z_2| > \epsilon_3$. After exploiting this observation for all the terms stemming from the cubic, the only integrals surviving are

$$\frac{1}{(2\pi i)^3} \int \frac{z_1^{2t+5} z_2^{-t-1} z_3^{-t-1}}{(z_1 - z_2)^3 (z_1 - z_3)^3} \, d\mathbf{z}$$

and

$$-\frac{3}{(2\pi i)^3} \int \frac{z_1^{t+3} z_2 z_3^{-t-1}}{(z_1 - z_2)^3 (z_1 - z_3)^2 (z_2 - z_3)} \, d\mathbf{z} \ .$$

[2]The authors thank Sinai Robins and Frank Sottile for their help in the proof of this equivalence and its combinatorial interpretation.

The first integral factors and yields, again by residue calculus,

$$\frac{1}{(2\pi i)^3}\int \frac{z_1^{2t+5}z_2^{-t-1}z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3}\,d\mathbf{z} = \frac{1}{(2\pi i)^3}\int z_1^{2t+5}\left(\int \frac{z^{-t-1}}{(z_1-z)^3(z_1-z)^3}\,dz\right)^2 dz_1 = \binom{t+2}{2}^2 .$$

For the second integral, it is most efficient to integrate with respect to z_2 first:

$$-\frac{3}{(2\pi i)^3}\int \frac{z_1^{t+3}z_2z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^2(z_2-z_3)}\,d\mathbf{z} = -\frac{3}{(2\pi i)^2}\int \frac{z_1^{t+3}z_3^{-t}}{(z_1-z_3)^5}\,dz_3\,dz_1 = -3\binom{t+3}{4} .$$

Adding up the last two lines gives finally

$$H_3(t) = \binom{t+2}{2}^2 - 3\binom{t+3}{4} = \frac{1}{8}t^4+\frac{3}{4}t^3+\frac{15}{8}t^2+\frac{9}{4}t+1 ,$$

which is equal to (3). To obtain the volume, the leading term of H_3 has to be multiplied by the relative volume of the fundamental domain of the sublattice of $\mathbb{Z}^9$ spanned by $\mathcal{B}_3$, which is 9; hence

$$\operatorname{vol}\mathcal{B}_3 = \frac{9}{8} .$$

In general, it is not hard to convince oneself that the relative volume of the fundamental domain of the sublattice of $\mathbb{Z}^{n^2}$ spanned by $\mathcal{B}_n$ is n^{n-1}.

The number of integrals we have to evaluate to compute H_4 is only slightly higher. By Theorem 2,

$$H_4(t) = \frac{1}{(2\pi i)^4}\int_{|z_1|=\epsilon_1}\int_{|z_2|=\epsilon_2}\int_{|z_3|=\epsilon_3}\int_{|z_4|=\epsilon_4}(z_1z_2z_3z_4)^{-t-1}\left(\sum_{k=1}^{4}\frac{z_k^{t+3}}{\prod_{j\neq k}(z_k-z_j)}\right)^4 d\mathbf{z} .$$

Again we have a choice of ordering the radii; we use $0<\epsilon_4<\epsilon_3<\epsilon_2<\epsilon_1<1$. After multiplying out the quartic, we have to calculate five integrals; their evaluation—again straightforward by means of the residue theorem—is as follows. As before, we can 'save' computation effort by choosing a particular order with which we integrate.

$$\begin{aligned}
&\frac{1}{(2\pi i)^4}\int \frac{z_1^{3t+11}z_2^{-t-1}z_3^{-t-1}z_4^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3(z_1-z_4)^3}\,d\mathbf{z} = \frac{1}{(2\pi i)^4}\int z_1^{3t+11}\left(\int \frac{z^{-t-1}}{(z_1-z)^3}\,dz\right)^3 dz_1\\
&\quad = \binom{t+3}{3}^3 ,\\
&-\frac{4}{(2\pi i)^4}\int \frac{z_1^{2t+8}z_2^2z_3^{-t-1}z_4^{-t-1}}{(z_1-z_2)^4(z_1-z_3)^3(z_1-z_4)^3(z_2-z_3)(z_2-z_4)}\,d\mathbf{z}\\
&\quad = -\frac{4}{(2\pi i)^4}\int \frac{z_1^{2t+8}z_2^2}{(z_1-z_2)^4}\left(\int \frac{z^{-t-1}}{(z_1-z)^3(z_2-z)}\,dz\right)^2 dz_1\,dz_2\\
&\quad = \frac{4}{(2\pi i)^2}\int \frac{z_1^{2t+8}z_2^{-t+1}}{(z_1-z_2)^7}\left(2\binom{t+2}{2}\frac{z_1^{-t-3}}{z_1-z_2}+2(t+1)\frac{z_1^{-t-2}}{(z_1-z_2)^2}+2\frac{z_1^{-t-1}}{(z_1-z_2)^3}-\frac{z_2^{-t-1}}{(z_1-z_2)^3}\right)d\mathbf{z}\\
&\quad = 8\binom{t+2}{2}\binom{t+5}{7}+8(t+1)\binom{t+6}{8}+8\binom{t+7}{9}-4\binom{2t+8}{9} ,
\end{aligned}$$

$$
\begin{aligned}
&\frac{4}{(2\pi i)^1}\int \frac{z_1^{2t+8}z_2^{-t-1}z_3^2z_4^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^4(z_1-z_4)^3(z_2-z_3)(z_3-z_4)}\,d\mathbf{z}\\
&\quad=\frac{4}{(2\pi i)^3}\int \frac{z_1^{2t+8}z_2^{-t+1}z_4^{-t-1}}{(z_1-z_2)^3(z_1-z_4)^7(z_2-z_4)}\,d\mathbf{z}\\
&\quad=\frac{4}{(2\pi i)^2}\int \frac{z_1^{2t+8}z_4^{-t-1}}{(z_1-z_4)^7}\left(\binom{t+2}{2}\frac{z_1^{-t-3}}{z_1-z_4}+(t+1)\frac{z_1^{-t-2}}{(z_1-z_4)^2}+\frac{z_1^{-t-1}}{(z_1-z_4)^3}\right)d\mathbf{z}\\
&\quad=4\left(\binom{t+2}{2}\binom{t+5}{7}+(t+1)\binom{t+6}{8}+\binom{t+7}{9}\right),\\
&\frac{6}{(2\pi i)^1}\int \frac{z_1^{t+5}z_2^{t+5}z_3^{-t-1}z_4^{-t-1}}{(z_1-z_2)^1(z_1-z_3)^2(z_1-z_4)^2(z_2-z_3)^2(z_2-z_4)^2}\,d\mathbf{z}\\
&\quad=\frac{6}{(2\pi i)^4}\int \frac{z_1^{t+5}z_2^{t+5}}{(z_1-z_2)^4}\left(\int \frac{z^{-t-1}}{(z_1-z)^2(z_2-z)^2}\,dz\right)^2 dz_1\,dz_2\\
&\quad=\frac{6}{(2\pi i)^2}\int \frac{z_1^{t+5}z_2^{t+5}}{(z_1-z_2)^4}\left((t+1)^2\frac{z_2^{-2t-4}}{(z_1-z_2)^4}-4(t+1)\frac{z_2^{-2t-3}}{(z_1-z_2)^5}+4\frac{z_2^{-2t-2}}{(z_1-z_2)^6}\right)d\mathbf{z}\\
&\quad=6(t+1)^2\binom{t+5}{7}-24(t+1)\binom{t+5}{8}+24\binom{t+5}{9},\\
&-\frac{12}{(2\pi i)^4}\int \frac{z_1^{t+5}z_2^2z_3^2z_4^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3(z_1-z_4)^2(z_2-z_3)^2(z_2-z_4)(z_3-z_4)}\,d\mathbf{z}\\
&\quad=-\frac{12}{(2\pi i)^3}\int \frac{z_1^{t+5}z_2^2z_4^{-t+1}}{(z_1-z_2)^3(z_1-z_4)^5(z_2-z_4)^3}\,d\mathbf{z}\\
&\quad=-\frac{12}{(2\pi i)^2}\int \frac{z_1^{t+5}z_4^{-t+1}}{(z_1-z_4)^5}\left(\frac{1}{(z_1-z_4)^3}+6\frac{z_4}{(z_1-z_4)^4}+6\frac{z_4^2}{(z_1-z_4)^5}\right)d\mathbf{z}\\
&\quad=-12\binom{t+5}{7}-72\binom{t+5}{8}-72\binom{t+5}{9}.
\end{aligned}
$$

Adding them all up gives

$$
\begin{aligned}
H_4(t)&=\binom{t+3}{3}^3+6\left(2t^2+5t+1\right)\binom{t+5}{7}-24\,(t+4)\binom{t+5}{8}+12(t+1)\binom{t+6}{8}\\
&\qquad-4\binom{2t+8}{9}-48\binom{t+5}{9}+12\binom{t+7}{9}\\
&=\frac{11}{11340}\,t^9+\frac{11}{630}\,t^8+\frac{19}{135}\,t^7+\frac{2}{3}\,t^6+\frac{1109}{540}\,t^5+\frac{43}{10}\,t^4+\frac{35117}{5670}\,t^3+\frac{379}{63}\,t^2+\frac{65}{18}\,t+1
\end{aligned}
$$

and hence

$$
\operatorname{vol}\mathcal{B}_4=4^3\cdot\frac{11}{11340}=\frac{176}{2835}\,.
$$

4 Larger n do

As we have seen in the examples, after multiplying out the integrand of Theorem 2, many of the terms do not contribute to the integral. The following proposition will provide us with a general statement to that extend.

For a rational function f in n variables z_j we use the notation $d_r(f)$ for the degree of f in the variables $z_1, \ldots, z_r$.

Proposition 3 *Suppose $p_1, \ldots, p_n$ are integers, q_{jk} are nonnegative integers ($1 \le j < k \le n$), $1 > \epsilon_1 > \cdots > \epsilon_n > 0$, and*

$$f(z_1, \ldots, z_n) = \frac{\prod_{1 \le j \le n} z_j^{p_j}}{\prod_{1 \le j < k \le n} (z_j - z_k)^{q_{jk}}} \ .$$

If $1 \le r \le n$ and $d_r(f) < -r$ then

$$\int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} f(\mathbf{z})\, d\mathbf{z} = 0 \ .$$

Proof. This is clear if $r = 1$ since f, considered as a function of z_1, is analytic outside the circle $|z_1| = \epsilon_1$ and has zero residue at infinity since its degree is less than -1.

We continue by induction, so suppose $r > 1$ and the proposition is true for smaller r. Suppose that $d_r(f) < -r$; we may assume $d_{r-1} \ge -(r-1)$. Note that $d_r(f) \ge d_{r-1}(f) + d$ where d is the degree of f in the single variable z_r (the discrepancy is the sum of the exponents q_{jr} for $j < r$). Hence $d \le d_r(f) - d_{r-1}(f) < -1$. We consider f as a function of z_r and apply the residue theorem to the region outside the circle $|z_r| = \epsilon_r$. As above f has zero residue at infinity, so we only need to consider the residues at the poles z_j for $j < r$. Evaluating these residues converts the integral of f into a (possibly huge) linear combination of integrals of functions of the same form as f, but in the $n-1$ variables $z_1, \ldots, z_{r-1}, z_{r+1}, \ldots, z_n$. If g is any one of these functions then we easily calculate $d_{r-1}(g) = d_r(f) + 1 < -(r-1)$, and, by the induction hypothesis, the integral of g is zero. □

From this proposition we obtain the starting point for our 'algorithm.'

Corollary 4 *For $1 > \epsilon_1 > \cdots > \epsilon_n > 0$ and $t \ge 0$, $H_n(t) =$*

$$\frac{1}{(2\pi i)^n} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} (z_1 \cdots z_n)^{-t-1} \sum_{m_1+\cdots+m_n=n}^{*} \binom{n}{m_1, \ldots, m_n} \prod_{k=1}^{n} \left(\frac{z_k^{t+n-1}}{\prod_{j \ne k} (z_k - z_j)} \right)^{m_k} d\mathbf{z} \ ,$$

where $\sum^*$ denotes that we only sum over those n-tuples of non-negative integers satisfying $m_1 + \cdots + m_n = n$ and $m_1 + \cdots + m_r > r$ if $1 \le r < n$.

Remark. The condition on $m_1, \ldots, m_n$ can be visualized through lattice paths from $(0,0)$ to (n,n) using the steps $(1,m_1),(1,m_2),\ldots,(1,m_n)$. The condition means that these paths stay strictly above the diagonal (except at the start and end).[3]

Proof. By Theorem 2,

$$H_n(t) = \frac{1}{(2\pi i)^n} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} (z_1 \cdots z_n)^{-t-1} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j\neq k}(z_k - z_j)} \right)^n d\mathbf{z}$$

$$= \frac{1}{(2\pi i)^n} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_n|=\epsilon_n} (z_1 \cdots z_n)^{-t-1} \sum_{m_1+\cdots+m_n=n} \binom{n}{m_1,\ldots,m_n} \prod_{k=1}^{n} \left(\frac{z_k^{t+n-1}}{\prod_{j\neq k}(z_k - z_j)} \right)^{m_k} d\mathbf{z} \ .$$

We select a partition $m_1 + \cdots + m_n = n$ and rewrite the corresponding integrand in the language of Proposition 3:

$$p_j = (n-1)m_j + t(m_j - 1) - 1 \ , \qquad q_{jk} = m_j + m_k \ .$$

Now suppose $1 \leq r < n$. The degree of the denominator in $z_1, \ldots, z_r$ is

$$\sum_{j=1}^{r} \sum_{k=j+1}^{n} (m_j + m_k) = (n-1) \sum_{j=1}^{r} m_j + r \sum_{j=r+1}^{n} m_j = (n-1) \sum_{j=1}^{r} m_j + nr - r \sum_{j=1}^{r} m_j$$

$$= (n-1) \sum_{j=1}^{r} m_j + nr - r \sum_{j=1}^{r} (m_j - 1) - r^2 \ .$$

We subtract this from $\sum_{j=1}^{r} p_j$ to get

$$d_r(f) = (t+r) \sum_{j=1}^{r} (m_j - 1) - r(n-r) - r \ .$$

Here t is non-negative and $r(n-r)$ is positive, so Proposition 3 implies that the integral is zero unless $\sum_{j=1}^{r}(m_j - 1) > 0$. □

Theoretically. Corollary 4 tells us what we have to do to compute H_n. For practical purposes, its statement is almost worthless. For once, the number of terms in the sum equals the n^{th} Catalan number

$$\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \ ,$$

which grows exponentially with n. Another slippery point is the evaluation of each integral. As we have seen in the examples, and as can be easily seen for the general case, we can compute each integral step by step one variable at a time. However, this means at each step we 'convert' a rational function into a sum of rational functions (of one variable less) by means of the residue theorem. Again, this means that the number of (single-variable) integrals we have to compute grows immensely as n increases. In fact, if we just 'feed' the statement of Corollary 4 into a computer and tell it to integrate each summand, say, starting with z_1, then z_2, and so on, the computation time explodes once one tries $n = 7$ or 8. We feel that computationally this is as involved as calculating

[3]The authors thank Lou Billera for pointing out this lattice-path interpretation.

a sufficient number of values of H_n and then interpolating this polynomial. However, complex analysis allows us some shortcuts which turn out to speed up the computation by a huge factor and which make Corollary 4 valuable, even from a computational perspective. These 'tricks' all showed up already in the examples and include

1. realizing when a function is analytic at the z_k-origin,
2. choosing the most efficient order of variables to integrate, and
3. factoring the integral if some of the variables appear in a symmetric fashion.

Finally, if we are only interested in the volume of $\mathcal{B}_n$, we may also be

4. dispensing a particular integral if it does not contribute to the leading term of H_n.

It is worth noting that every of these computational 'speed-ups' decreases the total computation time substantially. By applying them to Corollary 4, we implemented a C-program for the specific functions we have to integrate to compute H_n.[4] We were able to verify all previous values ($n \leq 8$) and to compute $\operatorname{vol}\mathcal{B}_9$ and H_9. The 65 coefficients of this polynomial can be found at `www.math.binghamton.edu/dennis/Birkhoff`. The following table gives the volumes together with the respective computing time (on a 1GHz PC running under Linux). Note that computing the full polynomial H_n takes longer, as we cannot make use of shortcut #4. In fact, the computation of H_9 took more than 300 days. (We divided the process onto several computers.)

n	$\operatorname{vol}\mathcal{B}_n$	time
1	1	$< .01$ sec
2	2	$< .01$ sec
3	$\frac{9}{8}$	$< .01$ sec
4	$\frac{176}{2835}$	$< .01$ sec
5	$\frac{23590375}{167382319104}$	$< .01$ sec
6	$\frac{9700106723}{1319281996032 \cdot 10^6}$	.18 sec
7	$\frac{77436678274508929033}{13730296368223523839986892 8 \cdot 10^8}$	15 sec
8	$\frac{5562533838576105333259507434329}{12589036260095477950081480942693339803308928 \cdot 10^{10}}$	54 min
9	$\frac{55949812970279602224689568637276605247549669 1}{92692623409952636498965146712806984296051951329202419606108477153345536 \cdot 10^{14}}$	317 hr

[4]The program can be found at `www.math.binghamton.edu/dennis/Birkhoff`.

5 An outlook towards transportation polytopes

The Birkhoff polytopes are special cases of transportation polytopes, which are defined below. The study of this class of polytopes, which are naturally at least as fascinating as the Birkhoff polytopes, was motivated by problems in linear programming; for combinatorial properties see, for example, [10, 13]. The goal of this section is to show how our methods can be applied in this more general setting.

Fix positive real numbers $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. Let $\mathbf{a} = (a_1, \ldots, a_m), \mathbf{b} = (b_1, \ldots, b_n)$.

$$\mathcal{T}_{\mathbf{a},\mathbf{b}} = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \in \mathbb{R}^{mn} : \; x_{jk} \geq 0, \; \begin{array}{l} \sum_k x_{jk} = a_j \text{ for all } 1 \leq j \leq m, \\ \sum_j x_{jk} = b_k \text{ for all } 1 \leq k \leq n \end{array} \right\}$$

is the set of *solutions to the transportation problem with parameters* $\mathbf{a}, \mathbf{b}$. It is a convex $(m-1)(n-1)$-dimensional polytope in $\mathbb{R}^{mn}$—hence we refer to the $\mathcal{T}_{\mathbf{a},\mathbf{b}}$ as *transportation polytopes.* Note that $\mathcal{B}_n = \mathcal{T}_{\mathbf{1},\mathbf{1}}$ where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. We will be exclusively interested in the case when $a_1, \ldots, a_m, b_1, \ldots, b_n$ are integers. One reason for this is that the vertices of $\mathcal{T}_{\mathbf{a},\mathbf{b}}$ are then integral. Let

$$T(\mathbf{a}, \mathbf{b}) = T(a_1, \ldots, a_m, b_1, \ldots, b_n) = \#\left(\mathcal{T}_{\mathbf{a},\mathbf{b}} \cap \mathbb{Z}^{mn}\right)$$

denote the number of integer points in the transportation polytope with integral parameters $\mathbf{a}, \mathbf{b}$. Geometrically, each of these parameters determines the position of a hyperplane bounding the polytope $\mathcal{T}_{\mathbf{a},\mathbf{b}}$. It is well known (see, for example, [20]) that $T(\mathbf{a}, \mathbf{b})$ is a piecewise-defined polynomial in $a_1, \ldots, a_m, b_1, \ldots, b_n$, namely, it is a (different) polynomial for each region in the $(m+n)$-dimensional space of the parameters $a_1, \ldots, a_m, b_1, \ldots, b_n$ in which the combinatorial type (that is, the face structure) of $\mathcal{T}_{\mathbf{a},\mathbf{b}}$ does not change.

In fact, one can derive results for the counting function $T(\mathbf{a}, \mathbf{b})$ which are 'higher-dimensional' extensions of (4), by application of the main theorem in [3], which in turn is a generalization of the Ehrhart-Macdonald reciprocity theorem (2). To state this theorem, we need to define the following integer-point counting function. Suppose $\mathcal{P}$ is a rational polytope given in the form (5); denote by $\mathcal{P}(\mathbf{t}) = \{\mathbf{x} \in \mathbb{R}^d : \; \mathbf{A}\mathbf{x} = \mathbf{t}\}$ a polytope which we obtain from $\mathcal{P} = \mathcal{P}(\mathbf{b})$ by translating (some of) its bounding hyperplanes. (Classical Ehrhart dilation is the special case $\mathbf{t} = t\mathbf{b}$.) Let

$$\mathcal{L}_{\mathcal{P}}(\mathbf{t}) = \#\left(\mathcal{P}(\mathbf{t}) \cap \mathbb{Z}^d\right) .$$

[3, Theorem 4] states that $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{P}^\circ}$ are piecewise defined (multivariable) quasipolynomials satisfying

$$\mathcal{L}_{\mathcal{P}}(-\mathbf{t}) = (-1)^{\dim \mathcal{P}} \mathcal{L}_{\mathcal{P}^\circ}(\mathbf{t}) .$$

As easy as we deduced before (4) from (2), this reciprocity theorem yields a symmetry result for the transporation counting function. Denote the d-dimensional vector all of whose entries are one by $\mathbf{1}_d$. Then

$$T(-\mathbf{a} - n\mathbf{1}_m, -\mathbf{b} - m\mathbf{1}_n) = (-1)^{mn-1} \, T(\mathbf{a}, \mathbf{b}) .$$

We finally turn to the problem of writing $T(\mathbf{a},\mathbf{b})$ in form of an integral. As with $\mathcal{B}_n$, we can view $\mathcal{T}_{\mathbf{a},\mathbf{b}}$ as given in the form of (5) and apply Theorem 1 to obtain

$$T(\mathbf{a},\mathbf{b}) = \frac{1}{(2\pi i)^{m+n}} \int \frac{z_1^{-a_1-1}\cdots z_m^{-a_m-1} w_1^{-b_1-1}\cdots w_n^{-b_n-1}}{\prod_{\substack{1\le j\le m\\ 1\le k\le n}} (1-z_j w_k)}\, d\mathbf{z}\, d\mathbf{w}\ .$$

Again the integral with respect to each (one-dimensional) variable is over a circle with radius < 1 centered at 0, and all appearing radii are different. As in the Birkhoff case, we can separate, say, the $\mathbf{w}$-variables and obtain

$$T(\mathbf{a},\mathbf{b}) = \frac{1}{(2\pi i)^{m+n}} \int z_1^{-a_1-1}\cdots z_m^{-a_m-1} \prod_{k=1}^{n} \int \frac{w_k^{-b_k-1}}{\prod_{j=1}^{m}(1-z_j w_k)}\, dw_k\, d\mathbf{z}\ .$$

And as with the Birkhoff polytope, the innermost integrals are easy to compute by means of the residue theorem. This yields the following statement which generalizes Theorem 2.

Theorem 5 *For any pairwise different* $0 < \epsilon_1, \ldots, \epsilon_m < 1$,

$$T(\mathbf{a},\mathbf{b}) = \frac{1}{(2\pi i)^m} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_m|=\epsilon_m} z_1^{-a_1-1}\cdots z_m^{-a_m-1} \prod_{k=1}^{n}\sum_{j=1}^{m} \frac{z_j^{b_k+m-1}}{\prod_{l\neq j}(z_j-z_l)}\, d\mathbf{z}\ .$$

Remark. It is not hard to conclude from the form of the integrand that $T(\mathbf{a},\mathbf{b})$ is indeed a piecewise-defined polynomial in $a_1,\ldots,a_m,b_1,\ldots,b_n$.

Theoretically we could now use this theorem to produce formulas for $T(\mathbf{a},\mathbf{b})$ just as we did for $H_n(t)$. There is one major difference: $T(\mathbf{a},\mathbf{b})$ is only a *piecewise-defined* polynomial. In fact, we can see this from the form of the integral in Theorem 5: whether we will get a nonzero contribution at a certain step in the computation depends heavily on the relationship between $a_1,\ldots,a_m,b_1,\ldots,b_n$.

The fact that the counting function $T(\mathbf{a},\mathbf{b})$ is of a somewhat more delicate nature naturally has computational consequences. We feel that providing any general results on this function would go beyond the scope of this article and will hopefully be subject of a future project. On the other hand, we adjusted our algorithm to examples of values of $T(\mathbf{a},\mathbf{b})$ for three (fixed) pairs $(\mathbf{a},\mathbf{b})$, which have been previously computed by Mount [17] and DeLoera and Sturmfels [9]. The first example is

$$T((3046,5173,6116,10928),(182,778,3635,9558,11110)) =$$
$$2319643659612889757482961153193875 3$$

Its computation took 20/10 minutes with Mount's/DeLoera–Sturmfels's algorithm, respectively [9, 17]. We computed this number in 0.2 seconds, based on Theorem 5. A similar phenomenon happens with

$$T((338106,574203,678876,1213008),(20202,142746,410755,1007773,1222717)) =$$
$$3160528209301169094598220490521497877480049630580229972623 97$$

and

$$T((30201, 59791, 70017, 41731, 58270), (81016, 68993, 47000, 43001, 20000)) =$$
$$2464053826815198108639701803342226405075725113340175811250949 5633028$$

which took 35 minutes/10 days with the DeLoera–Sturmfels algorithm [9], 0.3/2.9 seconds with ours.

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Department of Mathematical Sciences
State University of New York
Binghamton, NY 13902-6000
matthias@math.binghamton.edu
dennis@math.binghamton.edu