Some Experimental Results on the Frobenius Problem [1]

Matthias Beck and Shelemyahu Zacks

Abstract. We study the *Frobenius problem*: given relatively prime positive integers $a_1, \ldots, a_d$, find the largest value of t (the *Frobenius number*) such that $\sum_{k=1}^{d} m_k a_k = t$ has no solution in nonnegative integers $m_1, \ldots, m_d$. Based on empirical data, we conjecture that except for some special cases the Frobenius number can be bounded from above by $\sqrt{a_1 a_2 a_3}^{5/4} - a_1 - a_2 - a_3$.

1 Introduction

Given positive integers $a_1, \ldots, a_d$ with $\gcd(a_1, \ldots, a_d) = 1$, we call an integer t *representable* if there exist nonnegative integers $m_1, \ldots, m_d$ such that

$$t = \sum_{j=1}^{d} m_j a_j \ .$$

In this paper, we discuss the *linear Diophantine problem of Frobenius*: namely, find the largest integer which is not representable. We call this largest integer the *Frobenius number* $g(a_1, \ldots, a_d)$; its study was initiated in the 19th century. For $d = 2$, it is well known (most probably at least since Sylvester [15]) that

$$g(a_1, a_2) = a_1 a_2 - a_1 - a_2 \tag{1}$$

For $d > 2$, all attempts for explicit formulas have proved elusive. Two excellent survey papers on the Frobenius problem are [1] and [14].

Our goal is to establish bounds for $g(a_1, \ldots, a_d)$. The literature on such bounds is vast—see, for example, [3, 5, 6, 7, 14, 16]. We focus on the first non-trivial case $d = 3$; any bound for this case yields a general bound, as one can easily see that $g(a_1, \ldots, a_d) \leq g(a_1, a_2, a_3)$. All upper bounds in the literature are proportional to the product of two of the a_k's. On the other hand, Davison proved in [6] the lower bound $g(a_1, a_2, a_3) \geq \sqrt{3 a_1 a_2 a_3} - a_1 - a_2 - a_3$. Experimental data (see Figure 3 below) shows that this bound is sharp in the sense that it is very often very close to $g(a_1, a_2, a_3)$. This motivates the question whether one can establish an *upper* bound proportional to $\sqrt{a_1 a_2 a_3}^p$ where $p < 4/3$. ($p = 4/3$ would be comparable to the known bounds.) In this paper we illustrate empirically, on the basis of more than three thousand randomly chosen points, that $g(a_1, a_2, a_3) \leq \sqrt{a_1 a_2 a_3}^{5/4} - a_1 - a_2 - a_3$.

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2 Some geometric-combinatorial ingredients

Another motivation for the search for an upper bound proportional to $\sqrt{a_1 a_2 a_3}^p$ comes from the following formula of [3], which is the basis for our study: Let a, b, c be pairwise relatively prime positive integers, and define

$$N_t(a,b,c) := \#\left\{(m_1,m_2,m_3) \in \mathbb{Z}^3 : \ m_k \geq 0, \ am_1 + bm_2 + cm_3 = t\right\} .$$

Then

$$N_t(a,b,c) = \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$
$$+\sigma_{-t}(b,c;a) + \sigma_{-t}(a,c;b) + \sigma_{-t}(a,b;c) \ , \tag{2}$$

where

$$\sigma_t(a,b;c) := \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^t}{(\lambda^a - 1)(\lambda^b - 1)}$$

is a *Fourier-Dedekind sum*. One interpretation of $N_t(a,b,c)$ is the number of partitions of t with parts in the set $\{a,b,c\}$. Geometrically, $N_t(a,b,c)$ enumerates integer points on the 2-dimensional polytope

$$\left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : \ x_k \geq 0, \ ax_1 + bx_2 + cx_3 = 1\right\} ,$$

dilated by t. The Frobenius problem hence asks for the largest integer dilate of this polytope that contains no integer point, in other words, the largest t for which $N_t(a,b,c) = 0$. It is also worth mentioning that the condition that a, b, and c are *pairwise* relatively prime is no restriction, due to Johnson's formula [8]: if $m = \gcd(a,b)$ then

$$g(a,b,c) = m \ g\left(\frac{a}{m}, \frac{b}{m}, c\right) + (m-1)c \ . \tag{3}$$

In [3], formulas analogous to (2) for $d > 3$ are given. In our case ($d = 3$), a straightforward calculation shows

$$\sigma_t(a,b;c) = -\frac{1}{4c} \sum_{k=1}^{c-1} \frac{\mathrm{e}^{\frac{\pi i k}{c}(-2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}} \ .$$

In fact, $\sigma_t(a,b;c)$ is a *Dedekind-Rademacher sum* [12], as shown in [3]. Hence we can rewrite (2) as

$$N_t(a,b,c) = \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$
$$-\frac{1}{4a} \sum_{k=1}^{a-1} \frac{\mathrm{e}^{\frac{\pi i k}{a}(2t+b+c)}}{\sin\frac{\pi k b}{a} \sin\frac{\pi k c}{a}} - \frac{1}{4b} \sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{\pi i k}{b}(2t+a+c)}}{\sin\frac{\pi k a}{b} \sin\frac{\pi k c}{b}} - \frac{1}{4c} \sum_{k=1}^{c-1} \frac{\mathrm{e}^{\frac{\pi i k}{c}(2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}} \tag{4}$$

If we write the "periodic part" of $N_t(a,b,c)$ as

$$P_t(a,b,c) := \frac{1}{4a} \sum_{k=1}^{a-1} \frac{\mathrm{e}^{\frac{\pi i k}{a}(2t+b+c)}}{\sin\frac{\pi k b}{a} \sin\frac{\pi k c}{a}} + \frac{1}{4b} \sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{\pi i k}{b}(2t+a+c)}}{\sin\frac{\pi k a}{b} \sin\frac{\pi k c}{b}} + \frac{1}{4c} \sum_{k=1}^{c-1} \frac{\mathrm{e}^{\frac{\pi i k}{c}(2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}} \ ,$$

(4) becomes

$$N_t(a,b,c) = \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) - P_t(a,b,c) \; .$$

If we can bound $P_t(a,b,c)$ from above by, say, B then the roots of $N_t(a,b,c)$—and hence $g(a,b,c)$—can be bounded from above:

$$\begin{aligned} g(a,b,c) &\leq abc\left(-\tfrac{1}{2}\left(\tfrac{1}{ab}+\tfrac{1}{ac}+\tfrac{1}{bc}\right) + \sqrt{\tfrac{1}{4}\left(\tfrac{1}{ab}+\tfrac{1}{ac}+\tfrac{1}{bc}\right)^2 - \tfrac{2}{abc}\left(\tfrac{1}{12}\left(\tfrac{3}{a}+\tfrac{3}{b}+\tfrac{3}{c}+\tfrac{a}{bc}+\tfrac{b}{ac}+\tfrac{c}{ab}\right) - B\right)}\right) \\ &= -\tfrac{1}{2}(a+b+c) + \sqrt{\tfrac{1}{4}(abc)^2\left(\tfrac{1}{ab}+\tfrac{1}{ac}+\tfrac{1}{bc}\right)^2 - \tfrac{1}{6}abc\left(\tfrac{3}{a}+\tfrac{3}{b}+\tfrac{3}{c}+\tfrac{a}{bc}+\tfrac{b}{ac}+\tfrac{c}{ab}\right) + 2B\,abc} \\ &= \sqrt{2B\,abc + \tfrac{1}{12}\left(a^2+b^2+c^2\right)} - \tfrac{1}{2}(a+b+c) \end{aligned}$$

From this computation, the question of the existence of an upper bound for $g(a,b,c)$ proportional to $\sqrt{abc}^p$ comes up naturally. Unfortunately, it is not clear how to bound the periodic part $P_t(a,b,c)$ effectively. An almost trivial bound for $P_t(a,b,c)$ yielded in [3] the inequality

$$g(a,b,c) \leq \frac{1}{2}\left(\sqrt{abc\,(a+b+c)} - a - b - c\right) \; ,$$

which is of comparable size to the other upper bounds for $g(a,b,c)$ in the literature. However, we believe one can obtain bounds of smaller magnitude.

3 Special cases

On the path to such "better" bounds, we first have to exclude some cases which definitely yield Frobenius numbers of size a_k^2. One of these cases are triples (a,b,c) such that c is representable by a and b: by (1), we obtain in this case $g(a,b,c) = ab - a - b$.

A less trivial example for such special cases was given by Lewin [10], who studied the Frobenius number of *almost arithmetic sequences*: If $m,n > 0$, $\gcd(a,n) = 1$, and $d \leq a$, then

$$g\left(a, ma+n, ma+2n, \ldots, ma+(d-1)\,n\right) = \left(m\left\lfloor\frac{a-2}{d-1}\right\rfloor + m - 1\right)a + (a-1)\,n \; .$$

Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. For arithmetic sequences ($m = 1$), this formula goes back to Roberts [13], for consecutive numbers ($m = n = 1$) it is due to Brauer [4]. For the special case $d = 3$, we obtain

$$g(a, ma+n, ma+2n) = \left(m\left\lfloor\frac{a}{2}\right\rfloor - 1\right)a + (a-1)\,n \; .$$

As a function in a, $b := ma+n$, $c := ma+2n$, this Frobenius number grows proportionally to ab, which means an upper bound proportional to $\sqrt{abc}^p$ with $p < 4/3$ can not be achieved. Hence in our computations and conjectures about upper bounds for $g(a,b,c)$, we will exclude the cases of one of the numbers being representable by the other two, and almost arithmetic sequences. Finally, as noted above, thanks to (3) we may assume without loss of generality that a, b, and c are pairwise coprime. The triples (a,b,c) that are not excluded will be called *admissible* below.

4 Computations

In the present section we discuss the computation of the Frobenius number. For convenience we computed the number $f(a,b,c) = g(a,b,c) + a + b + c$. It is not hard to see that $f(a,b,c)$ is the largest integer that can not be represented by a linear combination of a, b, and c with *positive* integer coefficients. The respective counting function

$$\overline{N}_t(a,b,c) := \#\left\{(m_1,m_2,m_3) \in \mathbb{Z}^3 : \ m_k > 0, \ am_1 + bm_2 + cm_3 = t\right\} .$$

can also be found in [3] and is closely related to $N_t(a,b,c)$:

$$\begin{aligned}\overline{N}_t(a,b,c) = {} & \frac{t^2}{2abc} - \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) \\ & - \frac{1}{4a}\sum_{k=1}^{a-1}\frac{\mathrm{e}^{\frac{\pi i k}{a}(-2t+b+c)}}{\sin\frac{\pi k b}{a}\sin\frac{\pi k c}{a}} - \frac{1}{4b}\sum_{k=1}^{b-1}\frac{\mathrm{e}^{\frac{\pi i k}{b}(-2t+a+c)}}{\sin\frac{\pi k a}{b}\sin\frac{\pi k c}{b}} - \frac{1}{4c}\sum_{k=1}^{c-1}\frac{\mathrm{e}^{\frac{\pi i k}{c}(-2t+a+b)}}{\sin\frac{\pi k a}{c}\sin\frac{\pi k b}{c}} .\end{aligned}$$

The following illustrates our algorithm.

```
STEP 0: Initiate the intervals I1, I2, I3 for the selection of the
        arguments a,b,c;

STEP 1: Draw at random integers a,b,c from I1, I2, I3 respectively;

STEP 2: Test a,b,c for coprimality and for almost arithmetic sequences;

STEP 3: IF (a,b,c are not pairwise coprime) or IF (a,b,c are almost arithmetic)
        {discard a,b,c and GOTO STEP 1}
        ELSE {SET delta <- min(a,b,c); GOTO STEP 4};

STEP 4: Compute z=sqrt(a*b*c),
        SET mb <- INT(sqrt(3)*z)+delta;
        SET t <- mb;

STEP 5: Compute NB(t,a,b,c);

STEP 6: IF (NB(t,a,b,c)>0) {SET t <- t-1, and GOTO STEP 5}
        ELSE {GOTO STEP 7};

STEP 7: SET f <- t;

STEP 8: IF(mb-f < delta) {SET
        mb <- mb+delta
        t <- mb
        GOTO STEP 5}
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```
        ELSE {GOTO STEP 9};

STEP 9: PRINT f(a,b,c) <- f;
        STOP.
```

For example, for $a = 7, b = 13, c = 30$ the program yields the Frobenius number $f(7, 13, 30) = 96$, or $g(7, 13, 30) = 46$. This program was tested against arguments which yield known results, and found to be correct.

Our program is to choose at random arguments a, b, c in a certain range (in our case $[1, 750]$), and test the triplets for admissibility. For admissible triplets a, b, c we compute the Frobenius number $f(a, b, c)$ based on the straightforward observation that, once we have $a = \min(a, b, c)$ consecutive integers which are representable, we know that every integer beyond that interval is representable as well. We start searching for roots of $\overline{N}_t(a, b, c)$ at the lower bound $\sqrt{3abc}$. If a root is found at an integer f, we repeat this search until we found an interval of a integers t with $\overline{N}_t(a, b, c) > 0$, that is, an interval of a representable integers. At this stopping point, the integer f is the sought-after Frobenius number $f(a, b, c)$.

We have created a QUICK BASIC program[2], following the above algorithm. The program proved to be quite efficient, since most of the values of $f(a, b, c)$ were found to be close to the lower bound $\sqrt{3abc}$, as shown in the analysis below. The Dedekind-Rademacher sums appearing in (4) can be computed very efficiently because they satisfy a reciprocity law [12], which allows us to calculate their values similar in spirit to the Euclidean algorithm. This implies that for a given t, $\overline{N}_t(a, b, c)$ can be computed with our rather simple algorithm in $O(\log(c))$ time, assuming that $c = \max(a, b, c)$. Hence if $f(a, b, c)$ is close to the lower bound $\sqrt{3abc}$—which, again, happens in the vast majority of cases—, we obtain $f(a, b, c)$ in $O(a \log(c))$ time. On the other hand, we can of course not assume that $f(a, b, c)$ is close to $\sqrt{3abc}$; still we get 'at worst' a computation time of $O(ab \log(c))$. What makes this analysis even more appealing is that it applies to the general case of the Frobenius problem. As mentioned above, there is an analog for (2) and (4) for $d > 3$ [3], which again is a lattice-point count in a polytope and as such is known (for fixed d) to be computable in $O(p(\log a_1, \dots, \log a_d))$ time for some polynomial p [2]. With an analogous algorithm for the general case, we would hence be able to compute $f(a_1, \dots, a_d)$ in $O(a_1 a_2\, p(\log a_1, \dots, \log a_d))$ time, where $a_1 < a_2 < \dots < a_d$. As in the three-variable case—in fact, even more so—, most Frobenius numbers will be situated very close to the lower bound $\sqrt{3a_1a_2a_3}$, which means that in the vast majority of cases we can expect a computation time of $O(a_1\, p(\log a_1, \dots, \log a_d))$.

The computational complexity of the Frobenius problem is very interesting and still gives rise to ongoing studies. Davison [6] provided an algorithm for the three-variable case ($a < b < c$) which runs in $O(\log b)$ time. The general case is still open. While Kannan [9] proved that there is a polynomial-time algorithm (polynomial in $\log a_1, \dots, \log a_d$) to find $g(a_1, \dots, a_d)$ for fixed d, no such algorithm is known for $d > 3$. The fastest general algorithm which we are aware of is due to Nijenhuis [11] and runs in $O(d\, a \log a)$ time, where $a = \min(a_1, \dots, a_d)$. Hence, while our primitive algorithm is not competitive for the three-variable case of the Frobenius problem, it might be worthwhile to develop it further in the general case.

[2]Our program can be downloaded at www.math.binghamton.edu/matthias/frobcomp.html

With our program we generated at random almost four thousand triplets of integers. The results which we present are based on 3806 admissible triplets. Our main question is the relation of the Frobenius number $f = f(a, b, c)$ to $z := \sqrt{abc}$. The following is a statistical description of the ratios $R := f/z$.

Descriptive Statistics

Variable	N	Mean	Median	StDev	Min	Max	Q_1	Q_3
R	3806	2.422	2.129	0.837	1.623	10.555	1.942	2.614

Q_1 and Q_3 are the first and third quartiles, respectively. We see in the above table that 50% of the cases have a ratio smaller than 2.13, and 75% have ratio smaller than 2.61. In the following figures we present a box-plot and a histogram of the variable R.

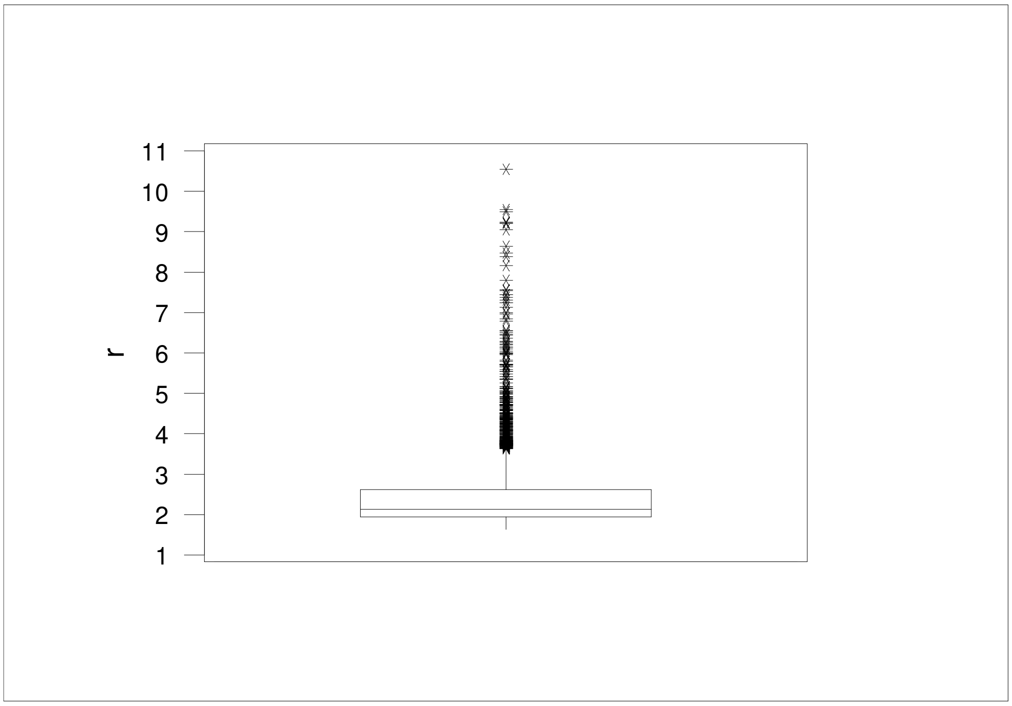

Figure 1: Box-plot

In the box-plot, the bottom line of the box corresponds to the first quartile Q_1.The top line of the box corresponds to the third quartile Q_3. There are 503 points above the value of $R = 3$. Only 21 points, which are listed below, have a value of R greater than 7.

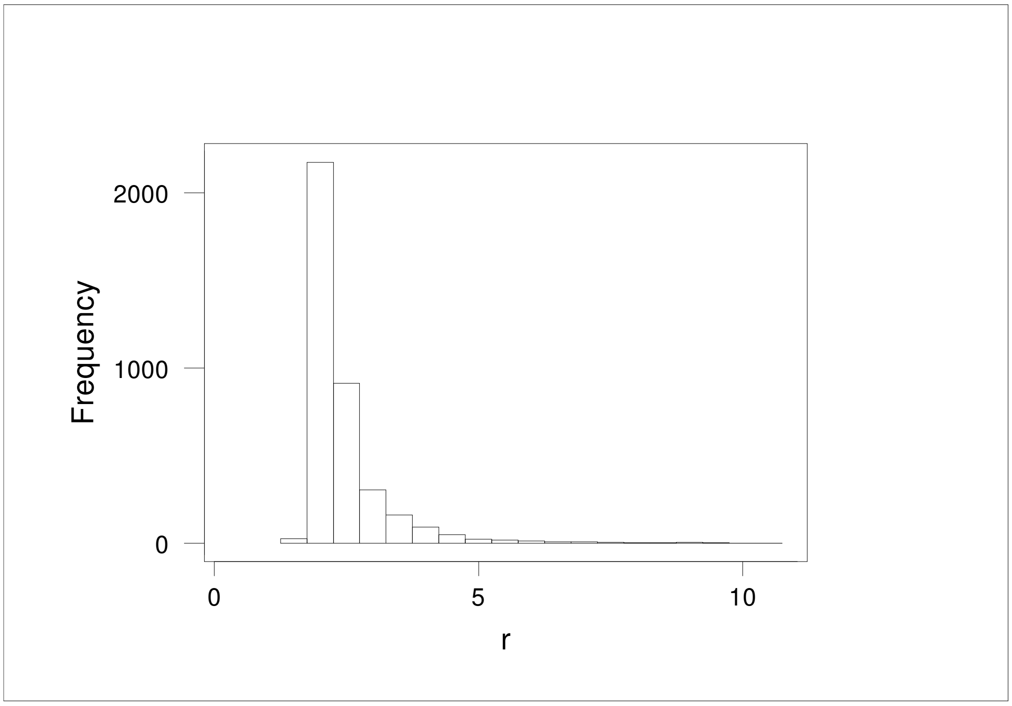

Figure 2: Histogram

a	b	c	$f(a,b,c)$	$z=\sqrt{abc}$	$z^{5/4}$	$R=f/z$
256	333	461	44700	6268.9	55782	7.1304
256	367	495	49392	6819.5	61972	7.2427
377	451	488	66021	9109.0	88989	7.2479
373	595	666	88845	12157.7	127662	7.3077
353	405	509	62908	8530.5	81982	7.3745
268	371	505	52733	7086.0	65013	7.4419
467	554	657	97078	13037.6	139314	7.4460
364	421	535	68309	9054.6	88325	7.5441
433	667	702	107688	14238.9	155541	7.5630
469	479	484	81317	10427.4	105371	7.7984
506	515	533	96167	11785.3	122794	8.1599
275	401	424	57366	6837.9	62180	8.3894
554	559	569	112492	13274.5	142486	8.4743
557	627	662	131461	15205.1	168845	8.6458
605	661	689	150258	16599.2	188413	9.0521
322	419	547	79094	8590.7	82706	9.2069
622	655	721	157955	17138.9	196101	9.2162
635	661	713	159890	17299.5	198399	9.2425
354	505	557	94789	9978.7	99734	9.4991
356	479	589	95774	10021.9	100274	9.5565
422	521	745	135090	12798.3	136126	10.5553

In Figure 3 we presents all the points (z, f). Notice that all the points are above the straight line $\sqrt{3}z$, which illustrates Davison's lower bound. The upper bound is, however, convex. It is included in the figure as the graph $z^{5/4}$.

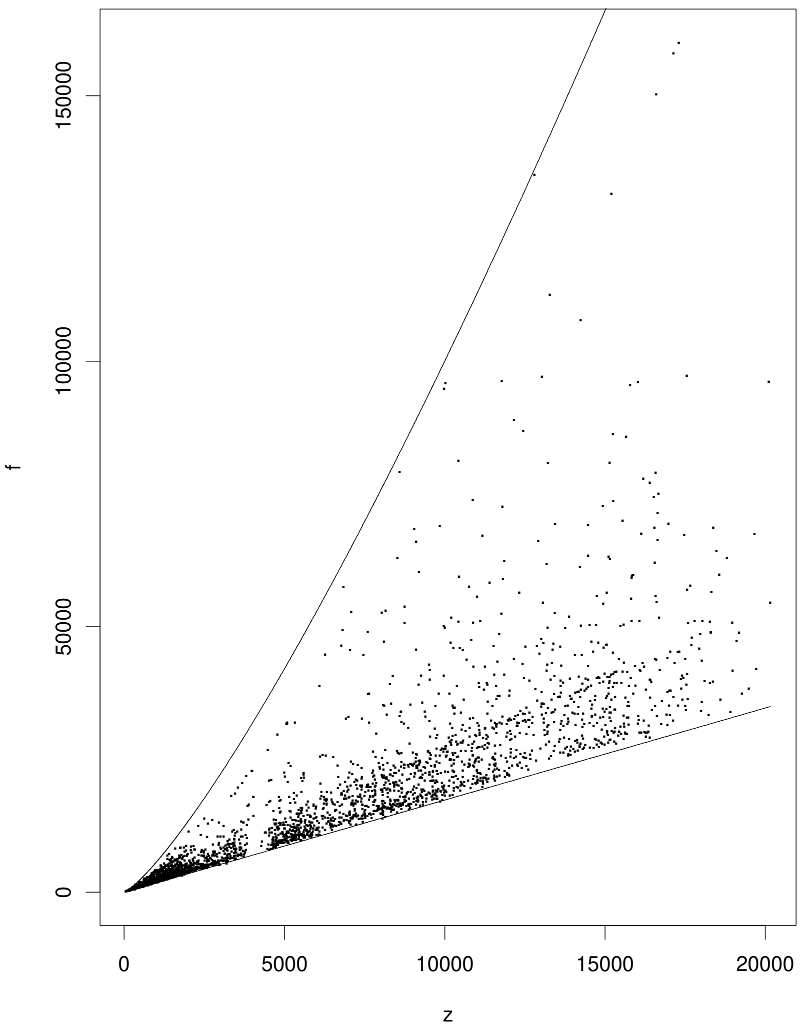

Figure 3: $f = f(a, b, c)$ as a function of $z = \sqrt{abc}$

The 'gap' around $z = 4500$ stems from two different runs of our program; in the first one we only experimented with small values of a, b, c.

5 Conjectures and closing remarks

Randomly chosen admissible arguments tend to yield a Frobenius number f smaller than the expected number (mean) which is estimated to be $2.422z$. The distribution of $R = f/z$ is very skewed (positive asymmetry) as seen in the Figure 1. Since 3806 random points yielded $f < z^{5/4}$, or $g(a,b,c) < \sqrt{abc}^{5/4} - a - b - c$, the probability that a future randomly chosen admissible triplet with $z = \sqrt{abc} < 20000$ will yield $f \geq z^{5/4}$ is smaller than $1/3806 < 0.0003$.

In general, our data suggests that one can obtain an upper bound of smaller magnitude than what the above cited results state. Again, the upper bounds in the literature are comparable to an upper bound proportional to $\sqrt{abc}^{4/3}$. We believe the following is true.

Conjecture 1 *There exists an upper bound for (a,b,c) proportional to $\sqrt{abc}^{p}$ where $p < \frac{4}{3}$, valid for all admissible triplets (a,b,c).*

In fact, our data suggests something more precise.

Conjecture 2 *For all admissible triplets (a,b,c),*

$$g(a,b,c) \leq \sqrt{abc}^{5/4} - a - b - c \ .$$

According to the above remark, the current credibility coefficient of this conjecture for admissible triples (a,b,c) such that $\sqrt{abc} < 20000$ is 0.9997.

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References

[1] J. L. Ramirez Alfonsin, *The diophantine Frobenius problem*, Report No. 00893, Forschungsinstitut für diskrete Mathematik, Universität Bonn, 2000.

[2] Alexander I. Barvinok, *Computing the Ehrhart polynomial of a convex lattice polytope*, Discrete Comput. Geom. **12** (1994), no. 1, 35–48. MR 95e:52015

[3] Matthias Beck, Ricardo Diaz, and Sinai Robins, *The Frobenius problem, rational polytopes, and Fourier–Dedekind sums*, J. Number Theory (to appear 2002).

[4] Alfred Brauer, *On a problem of partitions*, Amer. J. Math. **64** (1942), 299–312. MR 3,270d

[5] Alfred Brauer and James E. Shockley, *On a problem of Frobenius*, J. Reine Angew. Math. **211** (1962), 215–220. MR 26 #6113

[6] J. L. Davison, *On the linear Diophantine problem of Frobenius*, J. Number Theory **48** (1994), no. 3, 353–363. MR 95j:11033

[7] P. Erdős and R. L. Graham, *On a linear diophantine problem of Frobenius*, Acta Arith. **21** (1972), 399–408. MR 47 #127

[8] S. M. Johnson, *A linear diophantine problem*, Canad. J. Math. **12** (1960), 390–398. MR 22 #12074

[9] Ravi Kannan, *Lattice translates of a polytope and the Frobenius problem*, Combinatorica **12** (1992), no. 2, 161–177. MR 93k:52015

[10] Mordechai Lewin, *An algorithm for a solution of a problem of Frobenius*, J. Reine Angew. Math. **276** (1975), 68–82. MR 52 #259

[11] Albert Nijenhuis, *A minimal-path algorithm for the "money changing problem"*, Amer. Math. Monthly **86** (1979), no. 10, 832–835. MR 82m:68070a

[12] H. Rademacher, *Some remarks on certain generalized Dedekind sums*, Acta Arith. **9** (1964), 97–105. MR 29 #1172

[13] J. B. Roberts, *Note on linear forms*, Proc. Amer. Math. Soc. **7** (1956), 465–469. MR 19,1038d

[14] Ernst S. Selmer, *On the linear Diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294** (1977), 1–17. MR 56 #246

[15] J. J. Sylvester, *Mathematical questions with their solutions*, Educational Times **41** (1884), 171–178.

[16] Yehoshua Vitek, *Bounds for a linear diophantine problem of Frobenius*, J. London Math. Soc. (2) **10** (1975), 79–85. MR 51 #10218

Department of Mathematical Sciences
State University of New York
Binghamton, NY 13902-6000

matthias@math.binghamton.edu
shelly@math.binghamton.edu